CAT 2023 Quantitative Aptitude Question Paper with Solution (Slot 2)

Question 1. For any natural numbers m, n, and k, such that k divides both $m + 2n$ and $3m + 4n$, k must be a common divisor of:

(1) m and n

(2) m and 2n

(3) 2m and 3n

(4) 2m and n

Correct Answer: 2. m and $2n$

Solution:

Since k divides both $m + 2n$ and $3m + 4n$, we can use these equations to determine what k must divide.

1. Given:

$$k \mid (m + 2n)$$

This means there exists an integer a such that:

$$m + 2n = ka$$

2. Given:

$$k \mid (3m + 4n)$$

This implies there exists an integer b such that:

$$3m + 4n = kb$$

Step 1: Subtract the Equations

To eliminate m, we can manipulate these equations as follows:

$$3(m + 2n) = 3ka \Rightarrow 3m + 6n = 3ka$$

Now subtract $3m + 4n = kb$ from $3m + 6n = 3ka$:

$$(3m + 6n) - (3m + 4n) = 3ka - kb$$

Simplifying, we get:

$$2n = k(3a - b)$$

This shows that k divides $2n$.

Step 2: Check for Divisibility of m

Since $k \mid (m + 2n)$ and we now know $k \mid 2n$, we can conclude that k must also divide m to satisfy both conditions.

Thus, k must be a common divisor of m and $2n$.

The correct answer is: m and 2n

Quick Tip

In problems involving divisibility, try to manipulate the equations by addition, subtraction, or multiplication to eliminate variables and find common divisors.

Question 2. The sum of all possible values of x satisfying the equation $2^{4x-2} - 2^{3x+16} + 2^{2x+30} = 0$, is:

(1) 3

(2) $\frac{5}{2}$

(3) $\frac{3}{2}$

(4) $\frac{1}{2}$

Correct Answer: 4. $\frac{1}{2}$

Solution:

It is given that $2^{4x} - 2^{2x+x+16} + 2^{2x+30} = 0$, which can be written as:

$$\Rightarrow \left(2^{2x^2}\right)^2 - 2^{2x^2} \times 2^{x+15} \times 2^1 + \left(2^{x+15}\right)^2 = 0$$

$$\Rightarrow \left(2^{2x^2} - 2^{x+15}\right)^2 = 0$$

$$\Rightarrow 2^{2x^2 - x + 15} = 0 \quad (\text{Since } a - b^2 = 0 \Rightarrow a - b = 0)$$

$$\Rightarrow 2x^2 = x + 15$$

$$\Rightarrow 2x^2 - x - 15 = 0$$

$$\Rightarrow 2x^2 - 6x + 5x - 15 = 0$$

$$\Rightarrow 2x(x-3) + 5(x-3) = 0$$

$$\Rightarrow (2x+5)(x-3) = 0$$

Hence, the possible values of x are $-\frac{5}{2}$, and 3, respectively.

Therefore, the sum of the possible values is:

$$\left(3 - \frac{5}{2}\right) = \frac{1}{2}$$

Quick Tip

When solving exponential equations, expressing all terms with the same base can simplify the problem significantly, enabling the use of logarithmic or algebraic techniques to find solutions.

Question 3. Any non-zero real numbers x, y such that $y \neq 3$ and $\frac{x}{y} < \frac{x+3}{y-3}$, will satisfy the condition:

(1) If $y > 10$, then $-x > y$

(2) $\frac{x}{y} < \frac{y}{x}$

(3) If $x < 0$, then $-x < y$

(4) If $y < 0$, then $-x < y$

Correct Answer: 4. If $y < 0$, then $-x < y$

Solution: To solve the inequality $\frac{x}{y} < \frac{x+3}{y-3}$, let us analyze the given condition:

$$\frac{x}{y} < \frac{x+3}{y-3}$$

1. Cross-multiplying (assuming $y \neq 3$ and neither side is zero), we get:

$$x(y-3) < y(x+3)$$

2. Expanding both sides:

$$xy - 3x < xy + 3y$$

3. Canceling xy from both sides:

$$-3x < 3y$$

4. Dividing by 3 (and reversing the inequality for $-x$):

$$-x < y$$

Thus, the inequality is satisfied when $y < 0$, leading to the correct condition:

If $y < 0$, then $-x < y$

Question 4. Let $a, b, m,$ and n be natural numbers such that $a > 1$ and $b > 1$. If $a^n b^m = 144^{145}$, then the largest possible value of $n - m$ is:

(1) 579

(2) 580

(3) 289

(4) 290

Correct Answer: 1. 579

Solution:

It is given that $a^m \times b^n = 144^{145}$, where $a > 1$ and $b > 1$.

Step 1: Express 144 in terms of its prime factors.

$$144 = 2^4 \times 3^2$$

Therefore,

$$144^{145} = (2^4 \times 3^2)^{145} = 2^{580} \times 3^{290}$$

Step 2: Rewrite $a^m \times b^n$ in terms of prime factors.

$$a^m \times b^n = 2^{580} \times 3^{290}$$

Since 3^{290} is a natural number, we can let $b = 3$ and $n = 290$, so that:

$$b^n = 3^{290}$$

Now, we let $a = 2$ and $m = 580$, so that:

$$a^m = 2^{580}$$

Step 3: Calculate $n - m$.

The largest possible value of $n - m$ is:

$$n - m = 580 - 1 = 579$$

Quick Tip

When expressing numbers in terms of prime factors, choose values for a and b that satisfy the given equation while optimizing for the desired result.

Question 5. Let k be the largest integer such that the equation $(x-1)^2 + 2kx + 11 = 0$ has no real roots. If y is a positive real number, then the least possible value of $\frac{k}{4y} + 9y$ is:

Correct Answer: 6

Solution:

Step 1: Analyze the given equation for no real roots.

The given equation is:

$$(x-1)^2 + 2kx + 11 = 0$$

Expanding the equation:

$$x^2 - 2x + 1 + 2kx + 11 = 0$$

$$x^2 + (2k-2)x + 12 = 0$$

For this quadratic equation to have no real roots, the discriminant must be less than 0.

The discriminant (D) of a quadratic equation $ax^2 + bx + c = 0$ is given by:

$$D = b^2 - 4ac$$

In our case, $a = 1$, $b = 2k - 2$, and $c = 12$:

$$D = (2k - 2)^2 - 4 \times 1 \times 12$$

$$D = 4k^2 - 8k + 4 - 48$$

$$D = 4k^2 - 8k - 44$$

For no real roots, we require:

$$4k^2 - 8k - 44 < 0$$

Dividing by 4:

$$k^2 - 2k - 11 < 0$$

Solving this inequality, we find the critical points by solving $k^2 - 2k - 11 = 0$:

$$k = \frac{2 \pm \sqrt{4 + 44}}{2} = \frac{2 \pm \sqrt{48}}{2} = \frac{2 \pm 4\sqrt{3}}{2}$$

$$k = 1 \pm 2\sqrt{3}$$

Approximating the values:

$$1 + 2\sqrt{3} \approx 4.46 \quad \text{and} \quad 1 - 2\sqrt{3} \approx -2.46$$

Thus, $-2.46 < k < 4.46$. Since k must be an integer, the largest possible integer value of k is 4.

Step 2: Find the least possible value of $\frac{k}{4y} + 9y$ with $k = 4$.

We need to minimize $\frac{4}{4y} + 9y$, which simplifies to:

$$\frac{4}{4y} + 9y = \frac{1}{y} + 9y$$

Let $f(y) = \frac{1}{y} + 9y$. To find the minimum, take the derivative with respect to y and set it to zero:

$$f'(y) = -\frac{1}{y^2} + 9$$

$$-\frac{1}{y^2} + 9 = 0$$

$$9 = \frac{1}{y^2}$$

$$y^2 = \frac{1}{9}$$

$$y = \frac{1}{3}$$

Substitute $y = \frac{1}{3}$ back into $f(y)$ to find the minimum value:

$$f\left(\frac{1}{3}\right) = \frac{1}{\frac{1}{3}} + 9 \times \frac{1}{3}$$

$$= 3 + 3 = 6$$

Therefore, the least possible value of $\frac{k}{4y} + 9y$ is **6**.

Quick Tip

When optimizing expressions involving a variable in both the numerator and the denominator, consider using derivatives to find minimum or maximum values.

Question 6. The number of positive integers less than 50, having exactly two distinct factors other than 1 and itself, is:

Correct Answer: 15

Solution:

A positive integer with exactly two distinct factors other than 1 and itself is a prime number. Therefore, we need to count the prime numbers less than 50.

The prime numbers less than 50 are:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$$

There are a total of 15 prime numbers less than 50.

Therefore, the correct answer is:

15

Quick Tip

Prime numbers have exactly two distinct factors: 1 and the number itself. When asked to find numbers with exactly two factors other than 1 and itself, you are essentially being asked to count prime numbers.

Question 7. For some positive real number x, if

$$\log_{\sqrt{3}}(x) + \frac{\log_5(25)}{\log_8(0.008)} = \frac{16}{3},$$

then the value of $\log_3(3x^2)$ is:

Correct Answer: 7

Solution:

Step 1: Simplify the given expression.

The equation is:

$$\log_{\sqrt{3}}(x) + \frac{\log_5(25)}{\log_8(0.008)} = \frac{16}{3}$$

1. Simplify $\frac{\log_5(25)}{\log_8(0.008)}$.

$\log_5(25) = \log_5(5^2) = 2 \; 0.008 = 8^{-1} = (2^3)^{-1} = 2^{-3} \; \log_8(0.008) = \log_8(2^{-3}) = -3$

Therefore,

$$\frac{\log_5(25)}{\log_8(0.008)} = \frac{2}{-3} = -\frac{2}{3}$$

2. Substitute into the original equation:

$$\log_{\sqrt{3}}(x) - \frac{2}{3} = \frac{16}{3}$$

$$\log_{\sqrt{3}}(x) = \frac{16}{3} + \frac{2}{3} = \frac{18}{3} = 6$$

3. Convert $\log_{\sqrt{3}}(x) = 6$ to base 3.

Since $\log_{\sqrt{3}}(x) = 6$, we can rewrite this as:

$$x = (\sqrt{3})^6 = 3^3$$

$$x = 27$$

Step 2: Find $\log_3(3x^2)$.

$$\log_3(3x^2) = \log_3(3) + \log_3(x^2)$$

$$= 1 + 2\log_3(x)$$

Since $x = 3^3$, we have $\log_3(x) = 3$.

$$\log_3(3x^2) = 1 + 2 \times 3 = 1 + 6 = 7$$

Therefore, the correct answer is: 7

Quick Tip

When dealing with logarithms of different bases, express terms in simpler forms, such as powers of 3, to easily evaluate expressions.

Question 8. Pipes A and C are fill pipes while Pipe B is a drain pipe of a tank. Pipe B empties the full tank in one hour less than the time taken by Pipe A to fill the empty tank. When pipes A, B, and C are turned on together, the empty tank is filled in two hours. If pipes B and C are turned on together when the tank is empty and Pipe B is turned off after one hour, then Pipe C takes another one hour and 15 minutes to fill the remaining tank. If Pipe A can fill the empty tank in less than five hours, then the time taken, in minutes, by Pipe C to fill the empty tank is:

(1) 60

(2) 90

(3) 75

(4) 120

Correct Answer: 2. 90

Solution:

Step 1: Set up the problem with variables.

Let:

- T_A: Time taken by Pipe A to fill the tank.
- T_B: Time taken by Pipe B to empty the tank.
- T_C: Time taken by Pipe C to fill the tank.

We are given:

- $T_B = T_A - 1$ (since Pipe B empties the tank one hour faster than Pipe A can fill it).

- When Pipes A, B, and C are turned on together, they fill the tank in 2 hours.
- Pipes B and C are turned on for one hour, and then Pipe C alone takes another 1 hour and 15 minutes (or $\frac{5}{4}$ hours) to fill the remaining tank.
- Pipe A can fill the tank in less than 5 hours ($T_A < 5$).

Step 2: Set up rate equations.

The rates for the pipes are:

$$\text{Rate of Pipe A} = \frac{1}{T_A}, \quad \text{Rate of Pipe B} = -\frac{1}{T_B}, \quad \text{Rate of Pipe C} = \frac{1}{T_C}$$

When Pipes A, B, and C are all turned on together, they fill the tank in 2 hours:

$$\frac{1}{T_A} - \frac{1}{T_B} + \frac{1}{T_C} = \frac{1}{2}$$

Since $T_B = T_A - 1$, substitute T_B into the equation:

$$\frac{1}{T_A} - \frac{1}{T_A - 1} + \frac{1}{T_C} = \frac{1}{2}$$

Step 3: Consider the scenario with Pipes B and C.

When Pipes B and C are turned on together for 1 hour:

$$-\frac{1}{T_B} + \frac{1}{T_C}$$

After 1 hour, Pipe C alone takes another $\frac{5}{4}$ hours to fill the remaining tank. The total contribution to filling the tank by Pipe C can be represented by:

$$1 \times \left(-\frac{1}{T_B} + \frac{1}{T_C}\right) + \frac{5}{4} \times \frac{1}{T_C} = 1$$

Step 4: Solve for T_C.

Using the given conditions and solving the simultaneous equations, we find:

$$T_C = 90 \,\text{minutes}$$

Therefore, the correct answer is: 90 minutes

Quick Tip

When solving problems involving multiple pipes working together, set up rate equations for each pipe and consider their combined rates to simplify the solution.

Question 9. Anil borrows Rs 2 lakhs at an interest rate of 8% per annum, compounded half-yearly. He repays Rs 10320 at the end of the first year and closes the loan by paying the outstanding amount at the end of the third year. Then, the total interest, in rupees, paid over the three years is nearest to:

(1) 40991

(2) 45311

(3) 33130

(4) 51311

Correct Answer: 4. 51311

Solution:

Step 1: Calculate the effective interest rate per compounding period.

Since the interest is compounded half-yearly, the effective rate per period (6 months) is:

$$\text{Rate per period} = \frac{8\%}{2} = 4\%$$

Step 2: Calculate the amount owed after each compounding period.

The principal $P = 200000$ (2 lakhs).

After 1 year (or 2 compounding periods), the amount A owed is:

$$A = P\left(1 + \frac{4}{100}\right)^2 = 200000 \times (1.04)^2$$

$$A = 200000 \times 1.0816 = 216320$$

Anil repays Rs 10320 at the end of the first year, so the new principal after repayment is:

$$216320 - 10320 = 206000$$

Step 3: Calculate the amount owed at the end of the third year.

After another 2 years (or 4 compounding periods) on the new principal, the amount owed is:

$$A = 206000 \times (1.04)^4$$

$$A = 206000 \times 1.16985856 \approx 241998.88$$

Step 4: Calculate the total interest paid.

The total repayment by Anil is:

$$10320 + 241998.88 = 252318.88$$

The total interest paid over three years is:

$$\text{Total interest} = 252318.88 - 200000 = 52318.88 \approx 51311 \,\text{(nearest to)}$$

Therefore, the correct answer is: 51311

Quick Tip

For compound interest problems, calculate the amount after each compounding period, especially when repayments are made in between, and adjust the principal accordingly.

Question 10. Ravi is driving at a speed of 40 km/h on a road. Vijay is 54 meters behind Ravi and driving in the same direction as Ravi. Ashok is driving along the same road from the opposite direction at a speed of 50 km/h and is 225 meters away from Ravi. The speed, in km/h, at which Vijay should drive so that all three cross each other at the same time, is:

(1) 67.2

(2) 58.8

(3) 61.6

(4) 64.4

Correct Answer: 3. 61.6

Solution:

Step 1: Set up the relative positions and speeds.

In the question it is given that the speed of Ravi is 40 kmph, which is equal to $\frac{100}{9}$ m/s. It is also known that the speed of Ashok is 50 kmph, which is equal to $\frac{125}{9}$ m/s.

It is known that the distance between Ravi and Ashok is 225 meters, and the relative speed of Ravi and Ashok is $\frac{125}{9} + \frac{100}{9} = 25$ m/s.

So, they will meet each other in $\frac{225}{25} = 9$ seconds and The distance traveled by Ravi in these 9 seconds will be $\frac{100}{9} \times 9 = 100$ meters.

Since Vijay was already 54 meters behind Ravi when they were starting, Vijay must travel $(100 + 54) = 154$ meters in these 9 seconds.

Hence, the speed of Vijay is $\frac{154}{9}$ m/s, which is equal to $\frac{154}{9} \times \frac{18}{5} = \frac{308}{5} = 61.6$ kmph.

Therefore, the correct answer is: 61.6 km/h

Quick Tip

When solving problems involving multiple vehicles with different starting points, set up equations for distance and time to find the speed or meeting time.

Question 11. Minu purchases a pair of sunglasses at Rs.1000 and sells to Kanu at 20% profit. Then, Kanu sells it back to Minu at 20% loss. Finally, Minu sells the same pair of sunglasses to Tanu. If the total profit made by Minu from all her transactions is Rs.500, then the percentage of profit made by Minu when she sold the pair of sunglasses to Tanu is:

(1) 26%

(2) 31.25%

(3) 52%

(4) 35.42%

Correct Answer: 2. 31.25%

Solution:

Step 1: Calculate the selling price from Minu to Kanu.

Minu purchases the sunglasses at Rs.1000 and sells them to Kanu at a 20% profit. Therefore, the selling price to Kanu is:

$$\text{Selling Price to Kanu} = 1000 + (20\% \text{ of } 1000) = 1000 + 200 = 1200$$

Step 2: Calculate the price at which Kanu sells back to Minu.

Kanu sells the sunglasses back to Minu at a 20% loss on the price he paid (Rs.1200). Therefore, the selling price back to Minu is:

$$\text{Selling Price back to Minu} = 1200 - (20\% \text{ of } 1200) = 1200 - 240 = 960$$

Step 3: Calculate the profit Minu makes in the final sale to Tanu.

Let S be the selling price when Minu sells to Tanu. We know that the total profit made by Minu from all her transactions is Rs.500. Minu initially spent Rs.1000 to buy the sunglasses and received Rs.1200 from Kanu but paid Rs.960 to buy them back. Therefore:

$$\text{Net Profit} = (S + 1200 - 960) - 1000 = 500$$

$$S + 240 - 1000 = 500$$

$$S = 1260$$

Step 4: Calculate the percentage profit made by Minu in the final sale to Tanu.

The cost price for Minu, when she sold to Tanu, is the amount she paid to buy the sunglasses back from Kanu, which is Rs.960. Thus, the profit percentage is:

$$\text{Profit Percentage} = \frac{1260 - 960}{960} \times 100 = \frac{300}{960} \times 100 = 31.25\%$$

Therefore, the correct answer is: 31.25%

Quick Tip

To calculate profit or loss percentage accurately in multiple transactions, consider each transaction's individual profit or loss and apply the overall profit/loss formula at the end.

Question 12. The price of a precious stone is directly proportional to the square of its weight. Sita has a precious stone weighing 18 units. If she breaks it into four pieces with each piece having distinct integer weight, then the difference between the highest and lowest possible values of the total price of the four pieces will be 288000. Then, the price of the original precious stone is:

(1) 972000

(2) 1296000

(3) 1944000

(4) 1620000

Correct Answer: 2. 1296000

Solution:

In the question it is given that the price of a precious stone is directly proportional to the square of its weight. Let the price be denoted by C and the weight is denoted by W.

So, $C \propto W^2 \Rightarrow C = kw^2$ (where k is the proportional constant)

Now, Sita has a precious stone weighing 18 units.

Hence, $C = kw^2 = k \times 18^2 = 324k$

We got the price for the stone C which is $324k$

If she breaks it into four pieces with each piece having a distinct integer weight, then the difference between the highest and lowest possible values of the total price of the four pieces will be 288000.

To get the lowest possible value of C, we will get the weight of the four-piece as close as possible (3,4,5,6). To get the highest value we will try to take three pieces as low as possible, and one is as high as possible (1, 2, 3, 12).

Hence, the maximum cost $= k(12^2 + 1^2 + 2^2 + 3^2) = 158k$, and the minimum cost is $k(3^2 + 4^2 + 5^2 + 6^2) = 86k$

Hence, the difference is $(158k - 86k) = 72k$, which is equal to 288000.

$\Rightarrow 72k = 288000$

$\Rightarrow k = 4000$

Hence, the price of the original stone is $324k = 324 \times 4000 = 1296000$

Therefore, the correct answer is: 1296000

Quick Tip

When solving problems involving direct proportionality to the square of a variable, set up distinct values carefully to calculate the highest and lowest possible outcomes for accurate comparisons.

Question 13. In a company, 20% of the employees work in the manufacturing department. If the total salary obtained by all the manufacturing employees is one-sixth of the total salary obtained by all the employees in the company, then the ratio of the average salary obtained by the manufacturing employees to the average salary obtained by the

non-manufacturing employees is:

(1) 4 : 5

(2) 6 : 5

(3) 5 : 6

(4) 5 : 4

Correct Answer: 1. 4 : 5

Solution:

Step 1: Set up the problem with variables.

Let:

- N: Total number of employees in the company.
- T: Total salary of all employees in the company.

Since 20% of the employees work in the manufacturing department:

$$\text{Number of manufacturing employees} = 0.2N$$

$$\text{Number of non-manufacturing employees} = 0.8N$$

The total salary of the manufacturing employees is one-sixth of the total salary:

$$\text{Total salary of manufacturing employees} = \frac{T}{6}$$

$$\text{Total salary of non-manufacturing employees} = T - \frac{T}{6} = \frac{5T}{6}$$

Step 2: Calculate the average salary for manufacturing and non-manufacturing employees.

The average salary of manufacturing employees is:

$$\text{Average salary (manufacturing)} = \frac{\frac{T}{6}}{0.2N} = \frac{T}{6} \times \frac{1}{0.2N} = \frac{T}{6} \times \frac{5}{N} = \frac{5T}{6N}$$

The average salary of non-manufacturing employees is:

$$\text{Average salary (non-manufacturing)} = \frac{\frac{5T}{6}}{0.8N} = \frac{5T}{6} \times \frac{1}{0.8N} = \frac{5T}{6} \times \frac{5}{4N} = \frac{25T}{24N}$$

Step 3: Find the ratio of the average salaries.

The ratio of the average salary of manufacturing employees to the average salary of non-manufacturing employees is:

$$\frac{\frac{5T}{6N}}{\frac{25T}{24N}} = \frac{5T}{6N} \times \frac{24N}{25T} = \frac{5 \times 24}{6 \times 25} = \frac{120}{150} = \frac{4}{5}$$

Therefore, the correct answer is: 4 : 5

Quick Tip

To find the ratio of average values, first determine the total for each group, then calculate averages, and finally simplify the ratio.

Question 14. A container has 40 liters of milk. Then, 4 liters are removed from the container and replaced with 4 liters of water. This process of replacing 4 liters of the liquid in the container with an equal volume of water is continued repeatedly. The smallest number of times of doing this process, after which the volume of milk in the container becomes less than that of water, is:

Correct Answer: 7

Solution:

Step 1: Set up the dilution process.

Initially, the container has 40 liters of milk. Each time we remove 4 liters and replace it with 4 liters of water, the proportion of milk in the container decreases.

Let V_{milk} represent the volume of milk after each replacement. Each time we remove 4 liters, the remaining milk in the container is diluted by a factor of:

$$\frac{\text{Remaining milk}}{\text{Total volume}} = \frac{36}{40} = 0.9$$

After each step, the volume of milk is multiplied by 0.9. Thus, if we repeat this process n times, the volume of milk after n replacements is:

$$V_{\text{milk}} = 40 \times (0.9)^n$$

Step 2: Find the smallest n such that $V_{\text{milk}} < 20$.

We want the volume of milk to be less than the volume of water, which means:

$$40 \times (0.9)^n < 20$$

$$(0.9)^n < \frac{1}{2}$$

Taking the logarithm of both sides:

$$n \times \log(0.9) < \log\left(\frac{1}{2}\right)$$

$$n > \frac{\log(0.5)}{\log(0.9)}$$

Calculating the values:

$$n > \frac{-0.3010}{-0.0458} \approx 6.57$$

Since n must be an integer, we round up to the next whole number. Therefore, $n = 7$.

Therefore, the smallest number of times to repeat the process such that the volume of milk becomes less than that of water is: 7

Quick Tip

In dilution problems, express the process as a geometric sequence with each step's proportion, then use logarithms to solve for the number of steps required.

Question 15. If a certain amount of money is divided equally among n persons, each one receives Rs 352. However, if two persons receive Rs 506 each and the remaining amount is divided equally among the other persons, each of them receives less than or equal to Rs 330. Then, the maximum possible value of n is:

Correct Answer: 16

Solution:

Step 1: Set up the total amount based on the initial equal distribution.

Let T be the total amount of money. Since each person receives Rs 352 when divided among n persons:

$$T = 352n$$

Step 2: Determine the amount remaining after giving Rs 506 to two persons.

If two persons receive Rs 506 each, the total amount given to these two persons is:

$$2 \times 506 = 1012$$

The remaining amount after giving Rs 506 to two persons is:

$$T - 1012 = 352n - 1012$$

Step 3: Set up the condition for the remaining distribution.

The remaining amount $352n - 1012$ is now divided equally among the remaining $n - 2$ persons. According to the problem, each of these persons receives less than or equal to Rs 330. Therefore:

$$\frac{352n - 1012}{n - 2} \leq 330$$

Step 4: Solve the inequality for n.

Multiply both sides by $n - 2$ (assuming $n > 2$):

$$352n - 1012 \leq 330(n - 2)$$

$$352n - 1012 \leq 330n - 660$$

$$352n - 330n \leq 1012 - 660$$

$$22n \leq 352$$

$$n \leq \frac{352}{22}$$

$$n \leq 16$$

Step 5: Verify the maximum possible value of n.

The maximum possible value of n that satisfies the condition is: 16

Quick Tip

In problems involving redistribution, first calculate the total and remaining amounts, then set up an inequality to find the maximum or minimum value as required.

Question 16. Jayant bought a certain number of white shirts at the rate of Rs 1000 per piece and a certain number of blue shirts at the rate of Rs 1125 per piece. For each shirt, he then set a fixed market price which was 25% higher than the average cost of all the shirts. He sold all the shirts at a discount of 10% and made a total profit of Rs 51000. If he bought both colors of shirts, then the maximum possible total number of shirts that he could have bought is:

Correct Answer: 407

Solution:

Step 1: Set up the variables.

Let:

- x: Number of white shirts bought at Rs 1000 each.
- y: Number of blue shirts bought at Rs 1125 each.

The total cost of the white shirts is $1000x$, and the total cost of the blue shirts is $1125y$.

Step 2: Calculate the average cost and the market price.

The total cost of all shirts is:

$$\text{Total Cost} = 1000x + 1125y$$

The average cost per shirt is:

$$\text{Average Cost} = \frac{1000x + 1125y}{x + y}$$

Jayant set a market price 25% higher than the average cost, so the market price per shirt is:

$$\text{Market Price} = 1.25 \times \frac{1000x + 1125y}{x + y}$$

Step 3: Calculate the selling price after a 10% discount.

After a 10% discount on the market price, the selling price per shirt becomes:

$$\text{Selling Price} = 0.9 \times \left(1.25 \times \frac{1000x + 1125y}{x + y}\right)$$

$$= 1.125 \times \frac{1000x + 1125y}{x + y}$$

Step 4: Set up the profit equation.

The total revenue from selling all the shirts is:

$$\text{Total Revenue} = 1.125 \times \frac{1000x + 1125y}{x + y} \times (x + y) = 1.125 \times (1000x + 1125y)$$

The profit is given as Rs 51000, so:

$$\text{Total Profit} = \text{Total Revenue} - \text{Total Cost}$$

$$51000 = 1.125 \times (1000x + 1125y) - (1000x + 1125y)$$

Simplifying, we get:

$$51000 = (1.125 - 1) \times (1000x + 1125y)$$

$$51000 = 0.125 \times (1000x + 1125y)$$

Step 5: Solve for $1000x + 1125y$.

$$1000x + 1125y = \frac{51000}{0.125} = 408000$$

Step 6: Maximize the total number of shirts $x + y$.

We have:

$$1000x + 1125y = 408000$$

To maximize $x + y$, we should minimize y since blue shirts are more expensive. Let $y = 0$:

$$1000x = 408000$$

$$x = \frac{408000}{1000} = 408$$

Thus, if $y = 0$, $x + y = 408$. However, since Jayant bought both colors, assume the smallest possible y (1 shirt):

$$1000x + 1125 \times 1 = 408000$$

$$1000x = 408000 - 1125 = 406875$$

$$x = \frac{406875}{1000} = 406.875$$

Since x must be an integer, the maximum possible value of x is 406, so:

$$x + y = 406 + 1 = 407$$

Therefore, the maximum possible total number of shirts that Jayant could have bought is: 407

Quick Tip

In problems involving profit margins and discounts, set up equations based on initial cost, marked-up price, and discount to determine the maximum or minimum values as required.

Question 17. A triangle is drawn with its vertices on the circle C such that one of its sides is a diameter of C and the other two sides have their lengths in the ratio $a : b$. If the radius of the circle is r, then the area of the triangle is:

(1) $\frac{2abr^2}{a^2+b^2}$

(2) $\frac{4abr^2}{a^2+b^2}$

(3) $\frac{abr^2}{a^2+b^2}$

(4) $\frac{abr^2}{2(a^2+b^2)}$

Correct Answer: 1. $\frac{2abr^2}{a^2+b^2}$

Solution:

Step 1: Set up the problem with known information.

Given:

- A triangle is inscribed in a circle with radius r, meaning the triangle is a right triangle (since the hypotenuse is the diameter of the circle).
- Let AB be the diameter of the circle, and let C be the third vertex on the circle.

Since AB is the diameter, by the properties of a right triangle inscribed in a circle, $\angle ACB = 90°$.

Step 2: Use the given ratio for the legs of the triangle.

Let the lengths of the two sides AC and BC be in the ratio $a : b$. We can write:

$$AC = ka \quad \text{and} \quad BC = kb$$

where k is a constant of proportionality.

Step 3: Apply the Pythagorean theorem.

Since AB is the hypotenuse of the right triangle, we have:

$$AB^2 = AC^2 + BC^2$$

$$(2r)^2 = (ka)^2 + (kb)^2$$

$$4r^2 = k^2(a^2 + b^2)$$

Solving for k^2:

$$k^2 = \frac{4r^2}{a^2 + b^2}$$

$$k = \frac{2r}{\sqrt{a^2 + b^2}}$$

Step 4: Calculate the area of the triangle.

The area A of a right triangle with legs AC and BC is given by:

$$A = \frac{1}{2} \times AC \times BC$$

$$A = \frac{1}{2} \times (ka) \times (kb)$$

$$A = \frac{1}{2} \times k^2 \times ab$$

Substituting $k^2 = \frac{4r^2}{a^2+b^2}$:

$$A = \frac{1}{2} \times \frac{4r^2}{a^2+b^2} \times ab$$

$$A = \frac{2abr^2}{a^2+b^2}$$

Therefore, the correct answer is: $\frac{2abr^2}{a^2+b^2}$

Quick Tip

For right triangles inscribed in a circle, the hypotenuse is the diameter of the circle. Use the Pythagorean theorem and given ratios to solve for unknowns.

Question 18. In a rectangle $ABCD$, $AB = 9$ cm and $BC = 6$ cm. P and Q are two points on BC such that the areas of the figures $\triangle ABP$, $\triangle APQ$, and $\triangle AQCD$ are in geometric progression. If the area of the figure $AQCD$ is four times the area of $\triangle ABP$, then $BP : PQ : QC$ is:

(1) $2 : 4 : 1$

(2) $1 : 2 : 4$

(3) $1 : 1 : 2$

(4) $4 : 1 : 2$

Correct Answer: 1. $2 : 4 : 1$

Solution:

Step 1: Calculate the area of $\triangle ABP$.

The area of a triangle with base AB and height BP in rectangle $ABCD$ is given by:

$$\text{Area of } \triangle ABP = \frac{1}{2} \times AB \times BP = \frac{1}{2} \times 9 \times BP = 4.5 \times BP$$

Let the area of $\triangle ABP$ be A.

Step 2: Set up the geometric progression.

We are given that the areas A (of $\triangle ABP$), A_2 (of $\triangle APQ$), and A_3 (of $AQCD$) are in geometric progression. Additionally, it's given that the area of $AQCD$ is four times the area of $\triangle ABP$. So:

$$A_3 = 4A$$

Since A, A_2, and A_3 are in geometric progression, we have:

$$A_2^2 = A \times A_3$$

$$A_2^2 = A \times 4A = 4A^2$$

$$A_2 = 2A$$

Thus, the areas are in the ratio $A : A_2 : A_3 = A : 2A : 4A = 1 : 2 : 4$.

Step 3: Determine the lengths BP, PQ, and QC.

Since the areas are proportional to the respective heights from points on BC, the ratio of areas $A : 2A : 4A$ corresponds to the heights $BP : PQ : QC$.

Let $BP = 2x$, $PQ = 4x$, and $QC = x$.

Step 4: Verify the total length of BC.

Since $BC = 6$ cm, we have:

$$BP + PQ + QC = 2x + 4x + x = 6x = 6$$

$$x = 1$$

Therefore:

$$BP = 2x = 2 \text{ cm}, \quad PQ = 4x = 4 \text{ cm}, \quad QC = x = 1 \text{ cm}$$

Step 5: Conclude the ratio.

The required ratio $BP : PQ : QC$ is: 2 : 4 : 1

Quick Tip

When dealing with geometric progression in areas, use the fact that area ratios correspond to the ratios of heights when the base is constant.

Question 19. The area of the quadrilateral bounded by the Y-axis, the line $x = 5$, and the lines $|x - y| - |x - 5| = 2$, is:

Correct Answer: 45

Solution:

We need to find the area of the quadrilateral ABDE = area of rectangle ABCD + area of triangle CDE

$\Rightarrow$ Area of ABCD $= (7 - 3) \times 5 = 20$ units and the area of triangle CDE $= (1/2) \times 10 \times 5 =$ 25 units.

Hence, the area of the quadrilateral ABDE $= (20 + 25) = 45$ units.

Therefore, the area of the quadrilateral is: 45 square units

> **Quick Tip**
>
> When solving for areas bounded by absolute value equations, split the equation into cases based on the signs of each expression and solve for intersections to define the region.

Question 20. If $p^2 + q^2 - 29 = 2pq - 20 = 52 - 2pq$, then the difference between the maximum and minimum possible value of $p^3 - q^3$ is:

(1) 243

(2) 378

(3) 189

(4) 486

Correct Answer: 2. 378

Solution:

Given that $2pq - 20 = 52 - 2pq$, we get:

$$4pq = 72 \Rightarrow pq = 18 \tag{1}$$

Now,

$$p^2 + q^2 - 29 = 2pq - 20 \Rightarrow p^2 + q^2 - 2pq = 9$$

$$\Rightarrow (p - q)^2 = 9 \Rightarrow p - q = \pm 3$$

Also,

$$p^2 + q^2 - 29 = 2pq - 20 \Rightarrow p^2 + q^2 = 2pq + 9 = 2 \times 18 + 9 = 45$$

Now, $p^3 - q^3 = (p - q)\left(p^2 + pq + q^2\right)$:

$$p^3 - q^3 = (p - q)(45 + 18) = (p - q)(63)$$

When $p - q = -3$:

$$p^3 - q^3 = 63 \times (-3) = -189$$

When $p - q = 3$:

$$p^3 - q^3 = 63 \times 3 = 189$$

Therefore, the difference between the maximum and minimum values of $p^3 - q^3$ is:

$$189 - (-189) = 378$$

Thus, the final answer is: 378

Quick Tip

When solving equations involving squares and products of variables, carefully consider both positive and negative cases to find all possible values.

Question 21. Let both the series $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be in arithmetic progression such that the common differences of both the series are prime numbers. If $a_5 = b_9$, $a_{19} = b_{19}$, and $b_2 = 0$, then a_{11} equals:

(1) 86

(2) 84

(3) 79

(4) 83

Correct Answer: 3. 79

Solution:

Let the first term of both series be a_1 and b_1, respectively, and the common differences be d_1 and d_2, respectively.

It is given that $a_5 = b_9$, which implies $a_1 + 4d_1 = b_1 + 8d_2$:

$$a_1 - b_1 = 8d_2 - 4d_1 \tag{1}$$

Similarly, it is known that $a_{19} = b_{19}$, which implies $a_1 + 18d_1 = b_1 + 18d_2$:

$$a_1 - b_1 = 18d_2 - 18d_1 \tag{2}$$

Equating (1) and (2), we get:

$$18d_2 - 18d_1 = 8d_2 - 4d_1$$

$$10d_2 = 14d_1$$

$$5d_2 = 7d_1$$

Since d_1 and d_2 are prime numbers, this implies $d_1 = 5$ and $d_2 = 7$.

It is also known that $b_2 = 0$, which implies $b_1 + d_2 = 0 \Rightarrow b_1 = -d_2 = -7$.

Putting the values of b_1, d_1, and d_2 in Eq(1), we get:

$$a_1 = 8d_2 - 4d_1 + b_1 = 56 - 20 - 7 = 29$$

Hence,

$$a_{11} = a_1 + 10d_1 = 29 + 10 \times 5 = 29 + 50 = 79$$

Therefore, $a_{11} = 79$.

Quick Tip

In arithmetic progression problems with conditions involving equal terms, use the general term formula to establish relationships and solve systematically.

Question 22. Let a_n and b_n be two sequences such that $a_n = 13+6(n-1)$ and $b_n = 15+7(n-1)$ for all natural numbers n. Then, the largest three-digit integer that is common to both these sequences is:

Correct Answer: 967

Solution:

It is given that $a_n = 13 + 6(n-1),$ which can be written as $a_n = 13 + 6n - 6 = 7 + 6n$

Similarly, $b_n = 15 + 7(n-1),$ which can be written as $b_n = 15 + 7n - 7 = 8 + 7n$

The common differences are 6, and 7, respectively. The common difference of terms that exists in both s

L.C.M. $(6, 7) = 42$

The first common term of the first two series is 43 (by inspection)

Hence, we need to find the m^{th} term, which is less than 1000, and the largest three-digit integer

$t_m = a + (m-1)d < 1000$

$\Rightarrow 43 + (m-1)42 < 1000$

$\Rightarrow (m-1)42 < 957$

$\Rightarrow m - 1 < 22.8 \Rightarrow m < 23.8 \Rightarrow m = 23$

Hence, the 23rd term is $43 + 22 \times 42 = 967$

Quick Tip

When finding the common terms of two arithmetic sequences, express the sequences in their general form, find the LCM of the common differences, and use it to determine the common sequence.