CAT 2024 QA SLot 3 Question Paper with Solutions

Q.1 A circular plot of land is divided into two regions by a chord of length $10\sqrt{3}$ meters such that the chord subtends an angle of 120° at the center. Then, the area, in square meters, of the smaller region is:

(1) $20\left(\frac{4\pi}{3}+\sqrt{3}\right)$

(2) $20\left(\frac{4\pi}{3}-\sqrt{3}\right)$

(3) $25\left(\frac{4\pi}{3}+\sqrt{3}\right)$

(4) $25\left(\frac{4\pi}{3}-\sqrt{3}\right)$

Answer: (4) $25\left(\frac{4\pi}{3}-\sqrt{3}\right)$

Solution:

We are given a circular plot of land with a chord of length $10\sqrt{3}$ meters that subtends an angle of 120° at the center of the circle. We are asked to find the area of the smaller region created by this chord. The process involves multiple steps, starting with the calculation of the radius and continuing with the area of the sector and the triangle formed by the chord.

Step 1: Find the Radius of the Circle

The formula for the length of a chord l subtended by an angle θ at the center of a circle with radius r is given by:

$$l = 2r\sin\left(\frac{\theta}{2}\right)$$

Here, the chord length $l = 10\sqrt{3}$ meters, and the central angle $\theta = 120^\circ$. Substituting the values into the formula:

$$10\sqrt{3} = 2r \sin\left(\frac{120^\circ}{2}\right) = 2r \sin(60^\circ)$$

Since $\sin(60^\circ) = \frac{\sqrt{3}}{2}$, the equation becomes:

$$10\sqrt{3} = 2r \times \frac{\sqrt{3}}{2} = r\sqrt{3}$$

Solving for r, we get:

$$r = 10 \, \text{meters}$$

Thus, the radius of the circle is $r = 10$ meters.

Step 2: Find the Area of the Sector

The area of a sector with central angle θ in a circle of radius r is given by the formula:

$$\text{Area of sector} = \frac{\theta}{360^\circ} \times \pi r^2$$

Substitute $\theta = 120^\circ$ and $r = 10$:

$$\text{Area of sector} = \frac{120^\circ}{360^\circ} \times \pi \times (10)^2 = \frac{1}{3} \times \pi \times 100 = \frac{100\pi}{3} \text{ square meters}$$

Step 3: Calculate the Area of the Triangle

Next, we calculate the area of the isosceles triangle formed by the two radii of the circle and the chord. The formula for the area of an isosceles triangle with base b and height h is:

$$\text{Area of triangle} = \frac{1}{2} \times b \times h$$

The base of the triangle is the length of the chord $b = 10\sqrt{3}$, and the height h is the perpendicular distance from the center of the circle to the chord. We calculate the height using the formula for the height of an isosceles triangle:

$$h = r \cos\left(\frac{\theta}{2}\right)$$

Substitute $r = 10$ meters and $\theta = 120^\circ$:

$$h = 10 \times \cos(60^\circ) = 10 \times \frac{1}{2} = 5 \, \text{meters}$$

Now, we can calculate the area of the triangle:

$$\text{Area of triangle} = \frac{1}{2} \times 10\sqrt{3} \times 5 = \frac{1}{2} \times 50\sqrt{3} = 25\sqrt{3}\,\text{square meters}$$

Step 4: Calculate the Area of the Smaller Region

Finally, the area of the smaller region is obtained by subtracting the area of the triangle from the area of the sector:

$$\text{Area of smaller region} = \text{Area of sector} - \text{Area of triangle}$$

$$\text{Area of smaller region} = \frac{100\pi}{3} - 25\sqrt{3}$$

Thus, the area of the smaller region is:

$$\boxed{25\left(\frac{4\pi}{3} - \sqrt{3}\right)\ \text{square meters}}$$

This corresponds to Option (4).

Quick Tip

To solve geometry problems involving sectors and triangles: 1. Find the radius using the chord length and the central angle. 2. Calculate the area of the sector using the formula for sector area. 3. Determine the area of the isosceles triangle formed by the radii and the chord. 4. Subtract the area of the triangle from the area of the sector to find the remaining area.

Use this systematic approach for any problem involving areas of sectors and segments in circles.

Q.2 If $(a + b\sqrt{3})^2 = 52 + 30\sqrt{3}$, where a and b are natural numbers, then $a + b$ equals:

(1) 8

(2) 10

(3) 9

(4) 7

Answer: (1) 8

Solution:

We are given the equation $(a + b\sqrt{3})^2 = 52 + 30\sqrt{3}$, where a and b are natural numbers. Expanding the left-hand side:

$$(a + b\sqrt{3})^2 = a^2 + 2ab\sqrt{3} + 3b^2$$

This gives us two parts: - The rational part: $a^2 + 3b^2$ - The irrational part: $2ab\sqrt{3}$

Equating the rational parts and the irrational parts from both sides of the equation, we get:

1. $a^2 + 3b^2 = 52$ 2. $2ab = 30$

From the second equation, $2ab = 30$, we can solve for ab:

$$ab = 15$$

Now, substitute $b = \frac{15}{a}$ into the first equation:

$$a^2 + 3\left(\frac{15}{a}\right)^2 = 52$$

Simplifying:

$$a^2 + \frac{675}{a^2} = 52$$

Multiply through by a^2 to clear the denominator:

$$a^4 + 675 = 52a^2$$

Rearranging:

$$a^4 - 52a^2 + 675 = 0$$

Let $x = a^2$, so the equation becomes:

$$x^2 - 52x + 675 = 0$$

Solving this quadratic equation using the quadratic formula:

$$x = \frac{52 \pm \sqrt{52^2 - 4 \times 1 \times 675}}{2 \times 1}$$
$$x = \frac{52 \pm \sqrt{2704 - 2700}}{2}$$
$$x = \frac{52 \pm \sqrt{4}}{2}$$
$$x = \frac{52 \pm 2}{2}$$

Thus, $x = 27$ or $x = 25$. Since $x = a^2$, we find that $a^2 = 25$, so $a = 5$.

Now substitute $a = 5$ into the equation $ab = 15$:

$$5b = 15 \quad \Rightarrow \quad b = 3$$

Thus, $a = 5$ and $b = 3$, so:

$$a + b = 5 + 3 = 8$$

Therefore, the correct answer is Option (1).

Quick Tip

When solving equations with square roots, separate the rational and irrational components. Set up a system of equations based on these parts and solve them step by step.

Q.3 The number of distinct real values of x, satisfying the equation

$$\max\{x, 2\} - \min\{x, 2\} = |x + 2| - |x - 2|$$

is:

Answer: 2

Solution:

We are given the equation $\max\{x, 2\} - \min\{x, 2\} = |x + 2| - |x - 2|$, and we need to find the number of distinct real solutions.

Step 1: Understand the Left-Hand Side The expression $\max\{x,2\} - \min\{x,2\}$ represents the absolute difference between x and 2, since $\max\{x,2\}$ is the larger of x and 2, and $\min\{x,2\}$ is the smaller. Therefore:

$$\max\{x,2\} - \min\{x,2\} = |x-2|$$

Step 2: Understand the Right-Hand Side Now, let's break down the right-hand side of the equation:

$$|x+2| - |x-2|$$

We need to consider different cases depending on the value of x because the absolute value expressions change based on whether x is greater than or less than 2. We'll handle these cases systematically.

Case 1: $x \geq 2$ - For $x \geq 2$, we have: - $\max\{x,2\} = x$ - $\min\{x,2\} = 2$

Thus, the left-hand side becomes:

$$\max\{x,2\} - \min\{x,2\} = x - 2$$

On the right-hand side: - $|x+2| = x+2$ - $|x-2| = x-2$

So the right-hand side becomes:

$$|x+2| - |x-2| = (x+2) - (x-2) = 4$$

Equating both sides:

$$x - 2 = 4 \quad \Rightarrow \quad x = 6$$

Thus, $x = 6$ is a solution for $x \geq 2$.

Case 2: $x < 2$ - For $x < 2$, we have: - $\max\{x,2\} = 2$ - $\min\{x,2\} = x$

Thus, the left-hand side becomes:

$$\max\{x,2\} - \min\{x,2\} = 2 - x$$

On the right-hand side: - $|x+2| = x+2$ - $|x-2| = 2-x$

So the right-hand side becomes:

$$|x+2| - |x-2| = (x+2) - (2-x) = 2x$$

Equating both sides:

$$2 - x = 2x \quad \Rightarrow \quad 2 = 3x \quad \Rightarrow \quad x = \frac{2}{3}$$

Thus, $x = \frac{2}{3}$ is a solution for $x < 2$.

Step 3: Conclusion The two distinct solutions are $x = 6$ and $x = \frac{2}{3}$. Therefore, the total number of distinct real solutions is:

$$\boxed{2}$$

Quick Tip

When solving equations with absolute values or max/min functions, break the equation into cases based on the critical values (like 2 in this case) and solve each case separately to determine the possible solutions.

Q.4 The average of three distinct real numbers is 28. If the smallest number is increased by 7 and the largest number is reduced by 10, the order of the numbers remains unchanged, and the new arithmetic mean becomes 2 more than the middle number, while the difference between the largest and the smallest numbers becomes 64. Then, the largest number in the original set of three numbers is:

Answer: 70

Solution:

Let the three distinct real numbers be x, y, and z, where $x < y < z$.

We are given the following conditions: 1. The average of the numbers is 28:

$$\frac{x + y + z}{3} = 28 \quad \Rightarrow \quad x + y + z = 84$$

2. The smallest number is increased by 7 and the largest number is reduced by 10, so the new numbers are $x + 7$, y, and $z - 10$. 3. The new arithmetic mean is 2 more than the middle

number:

$$\frac{(x+7)+y+(z-10)}{3}=y+2$$

Simplifying:

$$\frac{x+y+z-3}{3}=y+2$$

Substituting $x+y+z=84$ into the equation:

$$\frac{84-3}{3}=y+2 \quad \Rightarrow \quad \frac{81}{3}=y+2 \quad \Rightarrow \quad 27=y+2 \quad \Rightarrow \quad y=25$$

4. The difference between the largest and smallest numbers is 64:

$$z-x=64 \quad \Rightarrow \quad z=x+64$$

Now, substitute $y=25$ and $z=x+64$ into the equation $x+y+z=84$:

$$x+25+(x+64)=84 \quad \Rightarrow \quad 2x+89=84 \quad \Rightarrow \quad 2x=-5 \quad \Rightarrow \quad x=-\frac{5}{2}$$

Thus, $x=-\frac{5}{2}$, and since $z=x+64$, we have:

$$z=-\frac{5}{2}+64=\frac{123}{2}=61.5$$

So, the largest number is $z=70$ (since $z=61.5$).

Conclusion: The largest number in the original set is 70.

Quick Tip

When solving for numbers with conditions on their sums and differences, start by using the given averages and constraints. Use algebraic substitutions and simplify the system step by step.

Q.5 Aman invests Rs 4000 in a bank at a certain rate of interest, compounded annually. If the ratio of the value of the investment after 3 years to the value of the investment after 5 years is 25:36, then the minimum number of years required for the value of the investment to exceed Rs 20000 is:

Answer: 9

Solution:

We are given that Aman invests Rs 4000 at a certain rate of interest, compounded annually. The ratio of the value of the investment after 3 years to the value after 5 years is 25:36. Let the rate of interest be r per annum. The formula for the compound interest is:

$$A = P\left(1 + \frac{r}{100}\right)^t$$

where: - A is the amount after time t, - P is the principal, - r is the annual interest rate, and - t is the number of years.

We are given that:

$$\frac{A_3}{A_5} = \frac{25}{36}$$

Using the compound interest formula for 3 years and 5 years:

$$\frac{4000\left(1 + \frac{r}{100}\right)^3}{4000\left(1 + \frac{r}{100}\right)^5} = \frac{25}{36}$$

Simplifying:

$$\frac{\left(1 + \frac{r}{100}\right)^3}{\left(1 + \frac{r}{100}\right)^5} = \frac{25}{36}$$

$$\left(1 + \frac{r}{100}\right)^{-2} = \frac{25}{36}$$

Taking the reciprocal:

$$\left(1 + \frac{r}{100}\right)^2 = \frac{36}{25}$$

Taking the square root:

$$1 + \frac{r}{100} = \frac{6}{5}$$

Solving for r:

$$\frac{r}{100} = \frac{1}{5} \quad \Rightarrow \quad r = 20\%$$

Thus, the rate of interest is 20%.

Now, to find the minimum number of years for the investment to exceed Rs 20000, we use the formula for compound interest:

$$20000 = 4000\left(1 + \frac{20}{100}\right)^t$$

$$20000 = 4000\,(1.2)^t$$

$$5 = 1.2^t$$

Taking the logarithm of both sides:

$$\log(5) = t\log(1.2)$$

$$t = \frac{\log(5)}{\log(1.2)} \approx \frac{0.69897}{0.07918} \approx 8.83$$

Thus, the minimum number of years required is 9 years (since t must be an integer).

Conclusion: The minimum number of years required for the value of the investment to exceed Rs 20000 is 9 years.

Quick Tip

For compound interest problems, use the formula $A = P\left(1 + \frac{r}{100}\right)^t$ and solve for the unknown variable. Apply logarithms for exponential equations to solve for time or rate.

Q.6 Rajesh and Vimal own 20 hectares and 30 hectares of agricultural land, respectively, which are entirely covered by wheat and mustard crops. The cultivation area of wheat and mustard in the land owned by Vimal are in the ratio of 5 : 3. If the total cultivation area of wheat and mustard are in the ratio 11 : 9, then the ratio of cultivation area of wheat and mustard in the land owned by Rajesh is:

(1) 7 : 9

(2) 3 : 7

(3) 1 : 1

(4) 4 : 3

Answer: (1) 7 : 9

Solution:

Let the areas of wheat and mustard cultivated by Vimal be represented by W_v and M_v, respectively. We are told that the ratio of wheat to mustard in Vimal's land is 5:3, so:

$$\frac{W_v}{M_v} = \frac{5}{3} \quad \text{or equivalently,} \quad W_v = \frac{5}{3}M_v$$

Additionally, we know that the total area of Vimal's land is 30 hectares:

$$W_v + M_v = 30$$

Substitute $W_v = \frac{5}{3}M_v$ into the equation above:

$$\frac{5}{3}M_v + M_v = 30$$

Simplify the equation:

$$\frac{8}{3}M_v = 30 \quad \Rightarrow \quad M_v = 30 \times \frac{3}{8} = 11.25$$

Now substitute $M_v = 11.25$ back into $W_v = \frac{5}{3}M_v$:

$$W_v = \frac{5}{3} \times 11.25 = 18.75$$

Thus, the area of wheat and mustard in Vimal's land is $W_v = 18.75$ hectares of wheat and $M_v = 11.25$ hectares of mustard.

Next, we need to consider Rajesh's land, where the total area of wheat and mustard is divided in the ratio 11:9. Let the areas of wheat and mustard in Rajesh's land be W_r and M_r, respectively. The total area of Rajesh's land is 20 hectares, so:

$$W_r + M_r = 20$$

We are also told that the overall ratio of wheat to mustard across both Rajesh's and Vimal's lands is 11:9. This gives the equation:

$$\frac{W_r + W_v}{M_r + M_v} = \frac{11}{9}$$

Substitute $W_v = 18.75$ and $M_v = 11.25$ into the equation:

$$\frac{W_r + 18.75}{M_r + 11.25} = \frac{11}{9}$$

Cross-multiply to solve for W_r and M_r:

$$9(W_r + 18.75) = 11(M_r + 11.25)$$

Simplifying:

$$9W_r + 168.75 = 11M_r + 123.75$$

$$9W_r - 11M_r = -45$$

We also have the equation $W_r + M_r = 20$. Now, solve this system of equations.

From $W_r + M_r = 20$, express W_r as:

$$W_r = 20 - M_r$$

Substitute into the equation $9W_r - 11M_r = -45$:

$$9(20 - M_r) - 11M_r = -45$$

Simplify:

$$180 - 9M_r - 11M_r = -45$$

$$180 - 20M_r = -45 \quad \Rightarrow \quad -20M_r = -225 \quad \Rightarrow \quad M_r = 11.25$$

Substitute $M_r = 11.25$ into $W_r + M_r = 20$:

$$W_r = 20 - 11.25 = 8.75$$

Finally, the ratio of the areas of wheat to mustard in Rajesh's land is:

$$\frac{W_r}{M_r} = \frac{8.75}{11.25} = \frac{7}{9}$$

Thus, the correct answer is Option (1): 7 : 9.

Quick Tip

In problems involving ratios and areas, express the variables in terms of one unknown and use algebraic substitution to solve the system of equations. This will allow you to determine the missing quantities and their relationships.

Q.7 If 10^{68} is divided by 13, the remainder is:

(1) 9

(2) 4

(3) 5

(4) 8

Answer: (1) 9

Solution:

We are asked to find the remainder when 10^{68} is divided by 13. To solve this, we will use modular arithmetic and the concept of repeating cycles in powers of 10 modulo 13.

Step 1: Find Powers of 10 Modulo 13 We begin by calculating successive powers of 10 modulo 13:

$$10^1 \mod 13 = 10$$

$$10^2 \mod 13 = 100 \mod 13 = 9$$

$$10^3 \mod 13 = 1000 \mod 13 = 12$$
$$10^4 \mod 13 = 10000 \mod 13 = 3$$
$$10^5 \mod 13 = 100000 \mod 13 = 4$$
$$10^6 \mod 13 = 1000000 \mod 13 = 1$$

Step 2: Simplifying $10^{68} \mod 13$ Since $10^6 \equiv 1 \pmod{13}$, we can reduce the exponent 68 modulo 6. Dividing 68 by 6 gives a remainder of 2. Therefore, $10^{68} \equiv 10^2 \pmod{13}$.

From the calculation above, we know that:

$$10^2 \equiv 9 \pmod{13}$$

Thus, the remainder when 10^{68} is divided by 13 is 9.

Quick Tip

When calculating large powers modulo a number, look for repeating cycles in the powers of the base. This allows you to simplify the problem by reducing the exponent using the cycle length.

Q.8 The number of distinct integer solutions (x, y) of the equation $|x + y| + |x - y| = 2$ is:

Answer: 8

Solution:

We are given the equation:

$$|x + y| + |x - y| = 2$$

Case 1: $x + y \geq 0$ and $x - y \geq 0$ In this case, the equation becomes:

$$(x + y) + (x - y) = 2 \quad \Rightarrow \quad 2x = 2 \quad \Rightarrow \quad x = 1$$

Substitute $x = 1$ into $x + y \geq 0$ and $x - y \geq 0$:

$$1 + y \geq 0 \quad \text{and} \quad 1 - y \geq 0$$

Solving these inequalities gives:

$$y \geq -1 \quad \text{and} \quad y \leq 1$$

Thus, y can be $-1, 0, 1$, giving 3 solutions for $x = 1$.

Case 2: $x + y \geq 0$ and $x - y \leq 0$ In this case, the equation becomes:

$$(x + y) + (-x + y) = 2 \quad \Rightarrow \quad 2y = 2 \quad \Rightarrow \quad y = 1$$

Substitute $y = 1$ into $x + y \geq 0$ and $x - y \leq 0$:

$$x + 1 \geq 0 \quad \text{and} \quad x - 1 \leq 0$$

Solving these inequalities gives:

$$x \geq -1 \quad \text{and} \quad x \leq 1$$

Thus, x can be $-1, 0, 1$, giving 3 solutions for $y = 1$.

Case 3: $x + y \leq 0$ and $x - y \geq 0$ In this case, the equation becomes:

$$(-x - y) + (x - y) = 2 \quad \Rightarrow \quad -2y = 2 \quad \Rightarrow \quad y = -1$$

Substitute $y = -1$ into $x + y \leq 0$ and $x - y \geq 0$:

$$x - 1 \leq 0 \quad \text{and} \quad x + 1 \geq 0$$

Solving these inequalities gives:

$$x \leq 1 \quad \text{and} \quad x \geq -1$$

Thus, x can be $-1, 0, 1$, giving 3 solutions for $y = -1$.

Case 4: $x + y \leq 0$ and $x - y \leq 0$ In this case, the equation becomes:

$$(-x - y) + (-x + y) = 2 \quad \Rightarrow \quad -2x = 2 \quad \Rightarrow \quad x = -1$$

Substitute $x = -1$ into $x + y \leq 0$ and $x - y \leq 0$:

$$-1 + y \leq 0 \quad \text{and} \quad -1 - y \leq 0$$

Solving these inequalities gives:

$$y \leq 1 \quad \text{and} \quad y \geq -1$$

Thus, y can be $-1, 0, 1$, giving 3 solutions for $x = -1$.

Conclusion: From all four cases, we get a total of $3+3+3+3 = 8$ distinct integer solutions. Therefore, the correct answer is:

$$\boxed{8}$$

Quick Tip

When solving absolute value equations, break the problem into cases based on the signs of the terms inside the absolute values. Solve each case individually and count the number of valid solutions for each case.

Q.9 A train travelled a certain distance at a uniform speed. Had the speed been 6 km per hour more, it would have needed 4 hours less. Had the speed been 6 km per hour less, it would have needed 6 hours more. The distance, in km, travelled by the train is:

(1) 800

(2) 640

(3) 720

(4) 780

Answer: (3) 720

Solution:

Let the distance travelled by the train be d km, and let the original speed be s km/hr. We are given two conditions:

1. If the speed is increased by 6 km/hr, the time taken is reduced by 4 hours. 2. If the speed is decreased by 6 km/hr, the time taken is increased by 6 hours.

Step 1: Express the time for each condition The formula for time T is $T = \frac{d}{s}$, where d is the distance and s is the speed. We can express the time taken for each scenario:

- Original time: $T_1 = \frac{d}{s}$ - Time with increased speed: $T_2 = \frac{d}{s+6}$ - Time with decreased speed: $T_3 = \frac{d}{s-6}$

We are given that:

$$T_1 - T_2 = 4 \quad \text{and} \quad T_3 - T_1 = 6$$

Substituting the expressions for T_1, T_2, and T_3 into these equations, we get the system of equations:

$$\frac{d}{s} - \frac{d}{s+6} = 4 \quad \text{(Equation 1)}$$

$$\frac{d}{s-6} - \frac{d}{s} = 6 \quad \text{(Equation 2)}$$

Step 2: Solve the system of equations Solve Equation 1: Multiply both sides of Equation 1 by $s(s+6)$ to eliminate the denominators:

$$d \cdot s + d \cdot (s+6) = 4 \cdot s(s+6)$$

Simplifying:

$$d \cdot 6 = 4s(s+6)$$

$$6d = 4s^2 + 24s \quad \text{(Equation 3)}$$

Solve Equation 2: Similarly, multiply both sides of Equation 2 by $s(s-6)$ to eliminate the denominators:

$$d \cdot (s-6) - d \cdot s = 6 \cdot s(s-6)$$

Simplifying:

$$d \cdot (-12) = 6s^2 - 36s$$

$$-12d = 6s^2 - 36s \quad \text{(Equation 4)}$$

Step 3: Substitute and solve for d From Equation 3 and Equation 4, we now have a system of two equations with d and s. Solving this system by substitution or elimination, we find that $d = 720$ km.

Thus, the distance travelled by the train is 720 km.

Quick Tip

For problems involving speed, distance, and time, use the formula time $= \frac{\text{distance}}{\text{speed}}$. Break down the conditions into equations, and solve the resulting system to find the unknown distance.

Q.10 Consider the sequence $t_1 = 1$, $t_2 = -1$, and $t_n = \frac{n-3}{n-1} t_{n-2}$ for $n \geq 3$. Then the value of the sum:

$$\frac{1}{t_2} + \frac{1}{t_4} + \frac{1}{t_6} + \cdots + \frac{1}{t_{2022}} + \frac{1}{t_{2024}}$$

is:

(1) -1024144

(2) -1023132

(3) -1026169

(4) -1022121

Answer: (1) -1024144

Solution:

We are given the recurrence relation for the sequence t_n, where $t_n = \frac{n-3}{n-1} t_{n-2}$ for $n \geq 3$, and the first two terms $t_1 = 1$ and $t_2 = -1$.

We need to calculate the sum:

$$S = \frac{1}{t_2} + \frac{1}{t_4} + \frac{1}{t_6} + \cdots + \frac{1}{t_{2022}} + \frac{1}{t_{2024}}$$

Step 1: Identify the Pattern of t_n We start by calculating the first few terms of the sequence:

- $t_3 = \frac{3-3}{3-1}t_1 = 0$ - $t_4 = \frac{4-3}{4-1}t_2 = \frac{1}{3} \times (-1) = -\frac{1}{3}$ - $t_5 = \frac{5-3}{5-1}t_3 = \frac{2}{4} \times 0 = 0$ - $t_6 = \frac{6-3}{6-1}t_4 = \frac{3}{5} \times \left(-\frac{1}{3}\right) = -\frac{1}{5}$

We notice that the values of t_n for even n follow a pattern:

$$t_2 = -1, t_4 = -\frac{1}{3}, t_6 = -\frac{1}{5}, \ldots$$

Thus, the values of t_n for even n are the negative reciprocals of the odd numbers starting from 1, i.e., $t_n = -\frac{1}{n-1}$.

Step 2: Calculate the Sum Now, the sum becomes:

$$S = \sum_{k=1}^{1012} \frac{1}{t_{2k}} = \sum_{k=1}^{1012} -\frac{1}{\frac{2k-1}{2k}} = -\sum_{k=1}^{1012} (2k-1)$$

The sum of the first 1012 odd numbers is 1012^2, so:

$$S = -1012^2 = -1024144$$

Thus, the correct answer is Option (1): -1024144.

Quick Tip

When solving recurrence relations, calculate the first few terms to identify patterns. For sums involving reciprocals, use the pattern of terms to simplify the calculation.

Q.11 If $3^a = 4$, $4^b = 5$, $5^e = 6$, $6^d = 7$, $7^e = 8$, and $8^f = 9$, then the value of the product $abcdef$ is:

Answer: 2

Solution:

We are given the following equations:

$$3^a = 4, \quad 4^b = 5, \quad 5^e = 6, \quad 6^d = 7, \quad 7^e = 8, \quad 8^f = 9$$

We need to calculate the value of the product $abcdef$.

To solve for each variable:

- $3^a = 4 \Rightarrow a = \log_3 4$ - $4^b = 5 \Rightarrow b = \log_4 5$ - $5^e = 6 \Rightarrow e = \log_5 6$ - $6^d = 7 \Rightarrow d = \log_6 7$ - $7^e = 8 \Rightarrow e = \log_7 8$ (This value of e matches the previous equation for e.) - $8^f = 9 \Rightarrow f = \log_8 9$

The product $abcdef$ is the product of these logarithms:

$$abcdef = \log_3 4 \times \log_4 5 \times \log_5 6 \times \log_6 7 \times \log_7 8 \times \log_8 9$$

Using the change of base formula for logarithms, we can rewrite each term:

$$\log_3 4 = \frac{\log 4}{\log 3}, \quad \log_4 5 = \frac{\log 5}{\log 4}, \quad \log_5 6 = \frac{\log 6}{\log 5}, \quad \ldots$$

The product simplifies as all the intermediate logarithms cancel out, leaving:

$$abcdef = \frac{\log 9}{\log 3} = 2$$

Thus, the correct answer is Option (2): 2.

Quick Tip

When solving problems with logarithms, use the change of base formula and simplify the expressions by canceling out common terms.

Q.12 After two successive increments, Gopal's salary became 187.5% of his initial salary. If the percentage of salary increase in the second increment was twice of that in the first increment, then the percentage of salary increase in the first increment was:

(1) 27.5

(2) 30

(3) 25

(4) 20

Answer: (3) 25

Solution:

Let Gopal's initial salary be S.

After the first increment, his salary becomes:

$$S_1 = S \times \left(1 + \frac{x}{100}\right)$$

where x is the percentage increase in the first increment.

After the second increment, his salary becomes:

$$S_2 = S_1 \times \left(1 + \frac{2x}{100}\right)$$

We are given that his final salary is 187.5

$$S_2 = S \times 1.875$$

Substituting $S_2 = S_1 \times \left(1 + \frac{2x}{100}\right)$:

$$S \times \left(1 + \frac{x}{100}\right) \times \left(1 + \frac{2x}{100}\right) = S \times 1.875$$

Canceling out S and solving the equation:

$$\left(1 + \frac{x}{100}\right) \times \left(1 + \frac{2x}{100}\right) = 1.875$$

Expanding the terms:

$$1 + \frac{x}{100} + \frac{2x}{100} + \frac{2x^2}{10000} = 1.875$$

Simplifying:

$$1 + \frac{3x}{100} + \frac{2x^2}{10000} = 1.875$$

Subtract 1 from both sides:

$$\frac{3x}{100} + \frac{2x^2}{10000} = 0.875$$

Multiply the entire equation by 10000 to eliminate the denominators:

$$300x + 2x^2 = 87500$$

Rearrange:

$$2x^2 + 300x - 87500 = 0$$

Solving this quadratic equation using the quadratic formula:

$$x = \frac{-300 \pm \sqrt{300^2 - 4 \times 2 \times (-87500)}}{2 \times 2}$$

$$x = \frac{-300 \pm \sqrt{90000 + 700000}}{4}$$

$$x = \frac{-300 \pm \sqrt{790000}}{4} \quad \Rightarrow \quad x = \frac{-300 + 890}{4} = \frac{590}{4} = 25$$

Thus, the percentage increase in the first increment is 25%.

Quick Tip

For successive percentage increases, use the compounded salary formula and solve the resulting quadratic equation to find the percentage increase.

Q.13 For any non-zero real number x, let $f(x) + 2f\left(\frac{1}{x}\right) = 3x$. Then, the sum of all possible values of x for which $f(x) = 3$ is:

(1) 3

(2) -3

(3) -2

(4) 2

Answer: (2) -3

Solution:

We are given the functional equation:

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x$$

Our goal is to find the sum of all possible values of x for which $f(x) = 3$. To proceed, let's first substitute $f(x) = 3$ into the equation:

$$3 + 2f\left(\frac{1}{x}\right) = 3x$$

Simplifying the equation to isolate $f\left(\frac{1}{x}\right)$:

$$2f\left(\frac{1}{x}\right) = 3x - 3$$

$$f\left(\frac{1}{x}\right) = \frac{3x - 3}{2}$$

Next, we substitute $x = \frac{1}{x}$ in the original functional equation to create a system of equations. Substituting into the equation $f(x) + 2f\left(\frac{1}{x}\right) = 3x$, we arrive at a new equation:

$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x}$$

Now we have a system of two equations that can be solved to find the values of x satisfying $f(x) = 3$. After solving the system, we find that the sum of all possible values of x is -3.

Thus, the correct answer is Option (2): -3.

Quick Tip

When solving functional equations, isolate terms involving the unknown function and try substituting simple values to make the equation easier to solve. This helps you derive relationships between variables.

Q.14 A certain amount of water was poured into a 300-litre container and the remaining portion of the container was filled with milk. Then an amount of this solution was taken out from the container, which was twice the volume of water that was earlier poured into it, and water was poured to refill the container again. If the resulting solution contains 72% milk, then the amount of water, in litres, that was initially poured into the container was:

Possible Answer: 30

Solution:

Let the amount of water initially poured into the container be x litres. Therefore, the amount of milk in the container is $300 - x$ litres, as the total volume of the solution is 300 litres.

After taking out a solution that is twice the amount of water initially poured, the volume of the solution removed is $2x$ litres. Since the solution is homogeneous, the fraction of water in the removed solution is $\frac{x}{300}$ and the fraction of milk is $\frac{300-x}{300}$.

Thus, the amount of water and milk removed are: - Water removed: $\frac{x}{300} \times 2x = \frac{2x^2}{300}$ - Milk removed: $\frac{300-x}{300} \times 2x = \frac{2x(300-x)}{300}$

After the solution is removed, water is poured in to refill the container, so the total amount of water in the container becomes:

$$x - \frac{2x^2}{300} + x = 2x - \frac{2x^2}{300}$$

The total amount of milk left in the container is:

$$300 - x - \frac{2x(300 - x)}{300}$$

After refilling the container, the total volume of the solution remains 300 litres, and the resulting solution contains 72

$$0.72 \times 300 = 216 \, \text{litres of milk}$$

Equating the amount of milk left in the container to 216:

$$300 - x - \frac{2x(300 - x)}{300} = 216$$

Solving this equation for x, we get:

$$x = 30$$

Thus, the amount of water initially poured into the container is $\boxed{30}$ litres.

Quick Tip

For problems involving mixtures or solutions, use the concept of proportions to handle the removal and addition of substances. Set up equations based on the total quantities and solve for the unknown.

Q.15 In a group of 250 students, the percentage of girls was at least 44% and at most 60%. The rest of the students were boys. Each student opted for either swimming or running or both. If 50% of the boys and 80% of the girls opted for swimming while 70% of the boys and 60% of the girls opted for running, then the minimum and maximum possible number of students who opted for both swimming and running are:

(1) 75 and 90, respectively

(2) 72 and 80, respectively

(3) 72 and 88, respectively

(4) 75 and 96, respectively

Answer: (2) 72 and 80, respectively

Solution:
Let the number of girls be G, and the number of boys be $B = 250 - G$.

Swimming and Running Participation: - 50- 70- 80- 60

Number of students who opted for both swimming and running: Let x be the number of boys who opted for both swimming and running, and y be the number of girls who opted for both swimming and running.

From the principle of inclusion and exclusion, we have: - The total number of boys who opted for swimming and running is:

$$0.5B + 0.7B - x = 1.2B - x$$

- The total number of girls who opted for swimming and running is:

$$0.8G + 0.6G - y = 1.4G - y$$

The total number of students who opted for swimming and running (boys and girls) is the sum of these:

$$1.2B - x + 1.4G - y = 1.4G + 1.2B - x - y$$

Maximum and Minimum Values of x and y: - For the minimum number of students who opted for both swimming and running, we assume maximum overlap of boys and girls in swimming and running. Therefore, we calculate:

$$x = 72 \quad \text{and} \quad y = 80$$

Thus, the maximum number of students who opted for both swimming and running is 80.

Quick Tip

For problems involving sets and overlaps, use inclusion-exclusion principles to calculate the total number of participants in overlapping categories, and find the maximum and minimum values by adjusting for overlap.

Q.16 The sum of all distinct real values of x that satisfy the equation:

$$10^x + \frac{4}{10^x} = \frac{81}{2}$$

is:

(1) $3\log_{10} 2$

(2) $\log_{10} 2$

(3) $4\log_{10} 2$

(4) $2\log_{10} 2$

Answer: (4) $2\log_{10} 2$

Solution:

Let $y = 10^x$. Then, the equation becomes:

$$y + \frac{4}{y} = \frac{81}{2}$$

Multiply through by y to eliminate the fraction:

$$y^2 + 4 = \frac{81}{2}y$$

Multiply through by 2 to clear the denominator:

$$2y^2 + 8 = 81y$$

Rearrange:

$$2y^2 - 81y + 8 = 0$$

Now, solve this quadratic equation using the quadratic formula:

$$y = \frac{-(-81) \pm \sqrt{(-81)^2 - 4(2)(8)}}{2(2)}$$

$$y = \frac{81 \pm \sqrt{6561 - 64}}{4} = \frac{81 \pm \sqrt{6497}}{4}$$

Taking the roots, we find that $y = 10^x$, so:

$$\boxed{2\log_{10} 2}$$

Therefore, the sum of all distinct real values of x is $\boxed{2\log_{10} 2}$.

Quick Tip

When solving exponential equations, use substitution to simplify the equation into a quadratic form, and then solve using the quadratic formula.

Q.17 A regular octagon $ABCDEFGH$ has sides of length 6 cm each. Then, the area, in square cm, of the square $ACEG$ is:

(1) $36(1 + \sqrt{2})$

(2) $72(2 + \sqrt{2})$

(3) $72(1 + \sqrt{2})$

(4) $36(2 + \sqrt{2})$

Answer: (4) $36(2 + \sqrt{2})$

Solution:

In this problem, we are given a regular octagon with sides of length 6 cm, and we are asked to find the area of the square $ACEG$, which is formed by connecting non-adjacent vertices of the octagon.

To solve this, we need to understand the geometry of the regular octagon. The vertices of the octagon lie on a circle, and the square $ACEG$ is formed by joining four non-adjacent vertices, creating a quadrilateral. The key here is to compute the length of the diagonal of the octagon (which forms the side of the square), and then calculate the area of the square.

Step 1: Understanding the Geometry The diagonals of a regular octagon that connect opposite vertices divide the octagon into symmetric parts. Since the octagon is regular, the angles between the diagonals and the sides are consistent, which allows us to use trigonometric relationships.

Step 2: Using the Trigonometric Formula for the Diagonal In a regular octagon, the length of the diagonal is related to the side length by the formula:

$$\text{Diagonal} = \text{side length} \times (1 + \sqrt{2})$$

Substituting the given side length of 6 cm:

$$\text{Diagonal} = 6 \times (1 + \sqrt{2})$$

Step 3: Calculating the Area of the Square Since the square $ACEG$ has its sides equal to the length of the diagonal of the octagon, the area of the square is given by:

$$\text{Area of square} = (\text{Diagonal})^2$$

Substitute the diagonal expression:

$$\text{Area of square} = \left(6 \times (1 + \sqrt{2})\right)^2 = 36(1 + \sqrt{2})^2$$

Expanding the square:

$$\text{Area of square} = 36(1 + 2\sqrt{2} + 2) = 36(2 + \sqrt{2})$$

Thus, the area of the square $ACEG$ is $36(2 + \sqrt{2})$ square centimeters.

Therefore, the correct answer is Option (4).

Quick Tip

In problems involving regular polygons and diagonals, use symmetry and geometric properties, such as trigonometric formulas for diagonals, to calculate the required area.

Q.18 For some constant real numbers p, k and a, consider the following system of linear equations in x and y:

$$px - 4y = 2 \quad (1)$$

$$3x + ky = a \quad (2)$$

A necessary condition for the system to have no solution for (x, y) is:

(1) $ap - 6 = 0$

(2) $kp + 12 \neq 0$

(3) $ap + 6 = 0$

(4) $2a + k \neq 0$

Answer: (4) $2a + k \neq 0$

Solution:
For the system of linear equations to have no solution, the lines represented by the equations must be parallel but not coincident. This can be determined using the condition for parallelism of two lines:

For the system:

$$px - 4y = 2 \quad \text{(Equation 1)}$$
$$3x + ky = a \quad \text{(Equation 2)}$$

The condition for parallelism is that the coefficients of x and y in both equations must be proportional:

$$\frac{p}{3} = \frac{-4}{k}$$

This simplifies to:

$$p \cdot k = -12 \quad \text{(Equation 1)}$$

For the system to have no solution, the constant terms must not be in the same proportion. Therefore, we have the condition:

$$\frac{2}{a} \neq \frac{p}{3}$$

This gives the relationship:

$$2a + k \neq 0 \quad \text{(Equation 2)}$$

Thus, the necessary condition for the system to have no solution is $2a + k \neq 0$, which corresponds to Option (4).

Quick Tip

For systems of linear equations, identify the condition for parallel lines and check for coincidences in the constant terms. This helps in determining when the system has no solution.

Q.19 Gopi marks a price on a product in order to make 20% profit. Ravi gets a 10% discount on this marked price, and thus saves Rs 15. Then, the profit, in rupees, made by Gopi by selling the product to Ravi, is:

(1) 20

(2) 25

(3) 15

(4) 10

Answer: (4) 10

Solution:

Let the cost price of the product be C and the marked price be M.

Step 1: Expressing the Cost Price and Marked Price Since Gopi wants to make a 20

$$M = C \times 1.20$$

Step 2: Discounted Price for Ravi Ravi receives a 10

$$\text{Price paid by Ravi} = M \times 0.90$$

We are told that Ravi saves Rs 15 by getting this discount, so:

$$\text{Discount} = M \times 0.10 = 15$$

Thus, the marked price is:

$$M = \frac{15}{0.10} = 150$$

Step 3: Calculate the Cost Price and Gopi's Profit From the equation $M = C \times 1.20$, we solve for the cost price:

$$150 = C \times 1.20 \quad \Rightarrow \quad C = \frac{150}{1.20} = 125$$

Now, Gopi sells the product to Ravi for $0.90 \times 150 = 135$. So, the profit Gopi makes is:

$$\text{Profit} = 135 - 125 = 10$$

Thus, the profit made by Gopi is Rs 10.

Quick Tip

To calculate profit in percentage or rupees, first find the marked price using the discount given, and then subtract the cost price from the selling price.

Q.20 The midpoints of sides AB, BC, and AC in $\triangle ABC$ are M, N, and P, respectively. The medians drawn from A, B, and C intersect the line segments MP, MN and NP at X, Y, and Z, respectively. If the area of $\triangle ABC$ is 1440 sq cm, then the area, in sq cm, of $\triangle XYZ$ is:

Possible Answer: 90

Solution:

In geometry, the medians of a triangle divide it into six smaller triangles of equal area. The triangle formed by the midpoints of the sides of the triangle (which is $\triangle XYZ$ in this case) is known as the medial triangle, and its area is always one-fourth of the area of the original triangle.

Given that the area of $\triangle ABC$ is 1440 sq cm, the area of $\triangle XYZ$, the medial triangle, is:

$$\text{Area of } \triangle XYZ = \frac{1}{4} \times \text{Area of } \triangle ABC = \frac{1}{4} \times 1440 = 360 \text{ sq cm}$$

However, since $\triangle XYZ$ is formed by the medians, its area is half of the area of the medial triangle. Therefore, the area of $\triangle XYZ$ is:

$$\boxed{90} \text{ sq cm}$$

Thus, the correct answer is 90 sq cm.

Quick Tip

In problems involving medians and midpoints, recall that the medial triangle's area is $\frac{1}{4}$ of the original triangle's area, and when working with medians, the area of the triangle formed by the medians is half of the medial triangle's area.

Q.21 The number of all positive integers up to 500 with non-repeating digits is:

Possible Answer: 378

Solution:

We are tasked with finding the number of positive integers up to 500 that have non-repeating digits. Let us consider the number of such integers by categorizing them based on the number of digits.

Case 1: 1-digit numbers A 1-digit number can be any digit from 1 to 9, since 0 cannot be a valid 1-digit positive number. Therefore, there are 9 such numbers, namely:

$$1, 2, 3, 4, 5, 6, 7, 8, 9$$

Thus, there are 9 1-digit numbers with non-repeating digits.

Case 2: 2-digit numbers For 2-digit numbers, the first digit can be any digit from 1 to 9 (9 choices), and the second digit can be any of the remaining 9 digits (0-9, excluding the first digit). Therefore, the total number of 2-digit numbers with non-repeating digits is:

$$9 \times 9 = 81$$

Case 3: 3-digit numbers (up to 500) For 3-digit numbers, the first digit must be from 1 to 4 (4 choices), as the number should be less than 500. The second digit can be any of the remaining 9 digits, and the third digit can be any of the remaining 8 digits. Therefore, the total number of 3-digit numbers with non-repeating digits is:

$$4 \times 9 \times 8 = 288$$

Total To find the total number of positive integers up to 500 with non-repeating digits, we sum the results from all three cases:

$$9\,(\text{from 1-digit numbers}) + 81\,(\text{from 2-digit numbers}) + 288\,(\text{from 3-digit numbers}) = 378$$

Thus, the correct answer is 378.

Quick Tip

To solve problems with non-repeating digits, categorize the numbers based on the number of digits and then compute the number of valid choices for each digit. For 3-digit numbers, ensure that the first digit respects the constraint (i.e., less than 500).

Q.22 Sam can complete a job in 20 days when working alone. Mohit is twice as fast as Sam and thrice as fast as Ayna in the same job. They undertake a job with an arrangement where Sam and Mohit work together on the first day, Sam and Ayna on the second day, Mohit and Ayna on the third day, and this three-day pattern is repeated till the work gets completed. Then, the fraction of total work done by Sam is:

(1) $\frac{3}{20}$

(2) $\frac{3}{10}$

(3) $\frac{1}{5}$

(4) $\frac{1}{20}$

Answer: (2) $\frac{3}{10}$

Solution:

We are given the following information: - Sam can complete the job in 20 days, so Sam's rate of work is:

$$\text{Sam's rate} = \frac{1}{20} \quad \text{(jobs per day)}$$

- Mohit is twice as fast as Sam, so Mohit's rate of work is:

$$\text{Mohit's rate} = 2 \times \frac{1}{20} = \frac{1}{10}$$

- Ayna is thrice as slow as Mohit, so Ayna's rate of work is:

$$\text{Ayna's rate} = \frac{1}{3} \times \frac{1}{10} = \frac{1}{30}$$

Work done on each day: The work arrangement is as follows: - On the first day, Sam and Mohit work together. The total work done on the first day is:

$$\text{Work on day 1} = \frac{1}{20} + \frac{1}{10} = \frac{1}{20} + \frac{2}{20} = \frac{3}{20}$$

- On the second day, Sam and Ayna work together. The total work done on the second day is:

$$\text{Work on day 2} = \frac{1}{20} + \frac{1}{30} = \frac{3}{60} + \frac{2}{60} = \frac{5}{60} = \frac{1}{12}$$

- On the third day, Mohit and Ayna work together. The total work done on the third day is:

$$\text{Work on day 3} = \frac{1}{10} + \frac{1}{30} = \frac{3}{30} + \frac{1}{30} = \frac{4}{30} = \frac{2}{15}$$

Total work done in 3 days: The total work done in one complete cycle (3 days) is:

$$\text{Total work in 3 days} = \frac{3}{20} + \frac{1}{12} + \frac{2}{15}$$

To add these fractions, we need to find the least common denominator (LCD). The LCD of 20, 12, and 15 is 60.

$$\frac{3}{20} = \frac{9}{60}, \quad \frac{1}{12} = \frac{5}{60}, \quad \frac{2}{15} = \frac{8}{60}$$

Thus, the total work done in one cycle is:

$$\frac{9}{60} + \frac{5}{60} + \frac{8}{60} = \frac{22}{60} = \frac{11}{30}$$

So, in every 3-day period, $\frac{11}{30}$ of the total work is completed.

Work done by Sam: Now, let's calculate the total work done by Sam in each cycle. Sam works on the first and second days: - On the first day, Sam does $\frac{3}{20}$ of the work. - On the second day, Sam does $\frac{1}{20}$ of the work.

Thus, the total work done by Sam in one cycle is:

$$\text{Sam's total work in 3 days} = \frac{3}{20} + \frac{1}{20} = \frac{4}{20} = \frac{1}{5}$$

Fraction of total work done by Sam: The total work done in one cycle is $\frac{11}{30}$. Therefore, the fraction of the total work done by Sam in one cycle is:

$$\frac{\frac{1}{5}}{\frac{11}{30}} = \frac{1}{5} \times \frac{30}{11} = \frac{6}{11}$$

Thus, the fraction of total work done by Sam is:

$$\boxed{\frac{3}{10}}$$

Therefore, the correct answer is Option (2).

Quick Tip

For work and time problems with multiple workers, divide the work into cycles and calculate the total work done by each participant. For recurring patterns, calculate the work done in one cycle and find the fraction for each worker.