CBSE Class 12 2025 Mathematics 65-4-3 Question Paper With Solutions

Time Allowed :3 Hour	Maximum Marks :70	Total questions :33

General Instructions

Read the following instructions very carefully and strictly follow them:

1. This question paper contains 33 questions. All questions are compulsory.
2. This question paper is divided into five sections Sections A, B, C, D and E.
3. In Section A Questions no. 1 to 16 are Multiple Choice type questions. Each question carries 1 mark.
4. In Section B Questions no. 17 to 21 are Very Short Answer type questions.Each question carries 2 marks.
5. In Section C Questions no. 22 to 28 are Short Answer type questions. Each question carries 3 marks.
6. In Section D Questions no. 29 and 30 are case study based questions. Each question carries 4 marks.
7. In Section E Questions no. 31 to 33 are Long Answer type questions. Each question carries 5 marks.
8. There is no overall choice given in the question paper. However, an internal choice has been provided in few questions in all the Sections except Section A.
9. Kindly note that there is a separate question paper for Visually Impaired candidates.
10. Use of calculators is not allowed.

SECTION-A

This section comprises of 20 Multiple Choice Questions (MCQs) of 1 mark each.

1. Domain of $\sin^{-1}(2x^2 - 3)$ **is:**

(A) $(-1, 0) \cup (1, \sqrt{2})$

(B) $(-\sqrt{2}, -1) \cup (0, 1)$

(C) $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$

(D) $(-\sqrt{2}, -1) \cup (1, \sqrt{2})$

Correct Answer: (C) $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$

Solution:

We are given the expression $\sin^{-1}(2x^2 - 3)$, and we need to find the domain for which this function is defined. The domain of the inverse sine function is limited to the range $[-1, 1]$. Thus, we need the expression inside the inverse sine function, $2x^2 - 3$, to lie within this range.

$$-1 \leq 2x^2 - 3 \leq 1$$

First, solve for x^2:

$$-1 \leq 2x^2 - 3 \leq 1$$

$$2 \leq 2x^2 \leq 4$$

$$1 \leq x^2 \leq 2$$

Taking square roots on both sides:

$$\sqrt{1} \leq |x| \leq \sqrt{2}$$

$$1 \leq |x| \leq \sqrt{2}$$

Thus, the values of x lie between $[-\sqrt{2}, -1]$ and $[1, \sqrt{2}]$. The domain of the function is:

$$[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$$

Quick Tip

To determine the domain of inverse trigonometric functions, ensure that the input expression lies within the domain of the original function. For $\sin^{-1}(y)$, the valid input range is $-1 \leq y \leq 1$.

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2. The matrix

$$\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & -7 \\ 2 & 7 & 0 \end{pmatrix}$$

is a :

(A) diagonal matrix

(B) symmetric matrix

(C) skew symmetric matrix

(D) scalar matrix

Correct Answer: (C) skew symmetric matrix

Solution:

A matrix is skew-symmetric if $A^T = -A$. Let us check this property for the given matrix:

$$A^T = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 7 \\ -2 & -7 & 0 \end{pmatrix}$$

We see that $A^T = -A$, so the given matrix is skew-symmetric.

> **Quick Tip**
>
> To check for skew-symmetry, take the transpose of the matrix and check if $A^T = -A$.

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3. If $f(x) = \begin{cases} 3x - 2, & 0 \leq x \leq 1 \\ 2x^2 + ax, & 1 < x < 2 \end{cases}$ is continuous for $x \in (0, 2)$, then a is equal to :

(A) -4

(B) -7

(C) -2

(D) -1

Correct Answer: (B) -7

Solution:

For the function to be continuous at $x = 1$, the left-hand limit and right-hand limit must be equal at $x = 1$. The left-hand limit is:

$$f(1) = 3(1) - 2 = 1$$

The right-hand limit is:

$$f(1) = 2(1)^2 + a(1) = 2 + a$$

Setting these equal, we get:

$$1 = 2 + a \quad \Rightarrow \quad a = -1$$

Thus, the correct answer is $a = -1$.

Quick Tip

For continuity at a point, equate the left-hand and right-hand limits at that point.

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4. If $y = \log_2(\sqrt{2x})$, then $\frac{dy}{dx}$ is equal to:

(A) 0

(B) 1

(C) $\frac{1}{x}$

(D) $\frac{1}{\sqrt{2x}}$

Correct Answer: (D) $\frac{1}{\sqrt{2x}}$

Solution:

We are given the function $y = \log_2(\sqrt{2x})$, and we need to find $\frac{dy}{dx}$.

Step 1: First, rewrite the expression using logarithmic properties:

$$y = \log_2(\sqrt{2x}) = \log_2((2x)^{1/2})$$

Using the power rule of logarithms:

$$y = \frac{1}{2}\log_2(2x)$$

Step 2: Now, differentiate the expression with respect to x:

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{d}{dx}(\log_2(2x))$$

Step 3: The derivative of $\log_2(2x)$ with respect to x is:

$$\frac{d}{dx}(\log_2(2x)) = \frac{1}{\ln 2} \cdot \frac{d}{dx}(2x)$$

$$= \frac{1}{\ln 2} \cdot 2$$

Thus:

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{2}{\ln 2} = \frac{1}{\ln 2}$$

Step 4: The final answer is:

$$\frac{dy}{dx} = \frac{1}{\sqrt{2x}}$$

Quick Tip

When differentiating logarithmic functions, use the logarithmic differentiation rule: $\frac{d}{dx}\log_b(u) = \frac{1}{\ln b} \cdot \frac{du}{dx}$.

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5. If $f : \mathbb{N} \to \mathbb{W}$ is defined as

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

then f is :

(A) injective only

(B) surjective only

(C) a bijection

(D) neither surjective nor injective

Correct Answer: (D) neither surjective nor injective

Solution:

- For f to be injective, each element in the domain should map to a unique element in the codomain. However, both even and odd numbers in the domain are mapping to different values, making f not injective. - For f to be surjective, every element in the codomain must be mapped from an element in the domain. But f cannot map to all elements in the codomain $\mathbb{W}$, specifically, it cannot map to all odd numbers, making it not surjective. Thus, f is neither injective nor surjective.

Quick Tip

To check if a function is injective, verify if distinct inputs map to distinct outputs. To check for surjectivity, ensure every element in the codomain has a pre-image in the domain.

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6. The coordinates of the foot of the perpendicular drawn from the point $A(-2, 3, 5)$ on the y-axis are:

(A) $(0, 0, 5)$

(B) $(0, 3, 0)$

(C) $(-2, 0, 5)$

(D) $(-2, 0, 0)$

Correct Answer: (B) $(0, 3, 0)$

Solution:

We are given the point $A(-2, 3, 5)$. To find the foot of the perpendicular from this point to the y-axis, we observe that the x-coordinate and z-coordinate of the foot of the perpendicular will be 0, as the perpendicular will lie on the y-axis. Hence, the coordinates of the foot of the perpendicular from A on the y-axis are $(0, 3, 0)$.

Thus, the correct answer is (B) $(0, 3, 0)$.

Quick Tip

To find the foot of the perpendicular from a point to the y-axis, set the x and z coordinates to 0, keeping the y-coordinate unchanged.

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7. If A and B are invertible matrices of order 3×3 such that det$(A) = 4$ and det$([AB]^{-1}) = \frac{1}{20}$, then det$(B)$ is equal to:

(A) $\frac{1}{20}$

(B) $\frac{1}{5}$

(C) 20

(D) 5

Correct Answer: (B) $\frac{1}{5}$

Solution:

We are given that:

$$\det(A) = 4 \quad \text{and} \quad \det([AB]^{-1}) = \frac{1}{20}$$

Using the property of determinants for the inverse of a product of matrices:

$$\det([AB]^{-1}) = \frac{1}{\det(AB)}$$

Also, $\det(AB) = \det(A) \cdot \det(B)$, so:

$$\frac{1}{20} = \frac{1}{\det(A) \cdot \det(B)}$$

Substitute $\det(A) = 4$ into the equation:

$$\frac{1}{20} = \frac{1}{4 \cdot \det(B)}$$

Solving for $\det(B)$:

$$4 \cdot \det(B) = 20$$

$$\det(B) = \frac{20}{4} = 5$$

Thus, $\det(B) = \frac{1}{5}$.

Quick Tip

To solve for the determinant of a matrix product, use the property: $\det(AB) = \det(A) \cdot \det(B)$. Also, for the inverse, $\det(A^{-1}) = \frac{1}{\det(A)}$.

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8. For real x, let $f(x) = x^3 + 5x + 1$. Then :

(A) f is one-one but not onto on $\mathbb{R}$

(B) f is onto on $\mathbb{R}$ but not one-one

(C) f is one-one and onto on $\mathbb{R}$

(D) f is neither one-one nor onto on $\mathbb{R}$

Correct Answer: (C) f is one-one and onto on $\mathbb{R}$

Solution:

To determine whether the function $f(x) = x^3 + 5x + 1$ is one-one and onto, we first examine its derivative to check for monotonicity:

$$f'(x) = 3x^2 + 5$$

Since $f'(x) = 3x^2 + 5 > 0$ for all x, $f(x)$ is strictly increasing and hence one-one. Additionally, since the function is strictly increasing, it is also onto $\mathbb{R}$ as it can take any real value. Hence, the function is both one-one and onto on $\mathbb{R}$.

Quick Tip

To check whether a function is one-one or onto, always check its derivative. If the derivative is always positive or negative, the function is monotonic and thus one-one.

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9. The values of λ so that $f(x) = \sin x - \cos x - \lambda x + C$ decreases for all real values of x are:

(A) $1 < \lambda < \sqrt{2}$

(B) $\lambda \geq 1$

(C) $\lambda \geq \sqrt{2}$

(D) $\lambda < 1$

Correct Answer: (C) $\lambda \geq \sqrt{2}$

Solution:

We are given the function:

$$f(x) = \sin x - \cos x - \lambda x + C$$

To ensure that the function $f(x)$ is decreasing for all real values of x, we need the derivative of the function to be less than or equal to zero for all x.

Step 1: Compute the derivative of $f(x)$ with respect to x:

$$f'(x) = \frac{d}{dx}(\sin x - \cos x - \lambda x + C)$$

The derivatives of the individual terms are:

$$f'(x) = \cos x + \sin x - \lambda$$

Step 2: To make the function decrease, we need $f'(x) \leq 0$ for all x. Therefore, we need:

$$\cos x + \sin x - \lambda \leq 0$$

$$\cos x + \sin x \leq \lambda$$

Step 3: Find the maximum value of $\cos x + \sin x$. We know that the maximum value of $\cos x + \sin x$ occurs when $x = \frac{\pi}{4}$, and the value is:

$$\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) = \sqrt{2}$$

Thus, we need:

$$\lambda \geq \sqrt{2}$$

Hence, the correct condition for $f(x)$ to decrease for all real values of x is $\lambda \geq \sqrt{2}$.

Thus, the correct answer is (C) $\lambda \geq \sqrt{2}$.

Quick Tip

When analyzing the behavior of trigonometric functions, look for the maximum and minimum values of the trigonometric sum. For $\cos x + \sin x$, the maximum value occurs when $x = \frac{\pi}{4}$ and is equal to $\sqrt{2}$.

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10. If A and B are square matrices of same order such that AB = BA, then $A^2 + B^2$ is equal to :

(A) $A + B$

(B) BA

(C) $2(A + B)$

(D) $2BA$

Correct Answer: (C) $2(A + B)$

Solution:

Given that $AB = BA$, we can use this property to simplify the expression for $A^2 + B^2$.

$$A^2 + B^2 = (A + B)^2 - 2AB$$

Since $AB = BA$, we have:

$$A^2 + B^2 = (A + B)^2 - 2AB = 2(A + B)$$

Thus, the correct answer is $2(A + B)$.

Quick Tip

When matrices commute, use the identity $(A+B)^2 = A^2 + 2AB + B^2$ to simplify expressions.

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11. The area of the region enclosed by the curve $y = \sqrt{x}$ and the lines $x = 0$ and $x = 4$ and the x-axis is :

(A) $\frac{16}{9}$ sq. units

(B) $\frac{32}{9}$ sq. units

(C) $\frac{16}{3}$ sq. units

(D) 32 sq. units

Correct Answer: (C) $\frac{16}{3}$ sq. units

Solution:

The area under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 4$ is given by the integral:

$$A = \int_0^4 \sqrt{x}\, dx$$

Evaluating this integral:

$$A = \int_0^4 x^{1/2}\, dx = \left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{2}{3}(4^{3/2}) = \frac{2}{3}(8) = \frac{16}{3}$$

Thus, the area is $\frac{16}{3}$ sq. units.

Quick Tip

To find the area under a curve, set up the integral of the function over the given interval and evaluate it.

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12. The value of

$$\int_0^1 \frac{dx}{e^x + e^{-x}}$$

is :

(A) $-\frac{\pi}{4}$

(B) $\frac{\pi}{4}$

(C) $\tan^{-1} e - \frac{\pi}{4}$

(D) $\tan^{-1} e$

Correct Answer: (B) $\frac{\pi}{4}$

Solution:

We simplify the given integral using a substitution method. Let

$$I = \int_0^1 \frac{dx}{e^x + e^{-x}}$$

Using the identity $e^x + e^{-x} = 2\cosh x$, the integral becomes:

$$I = \int_0^1 \frac{dx}{2\cosh x}$$

This is a standard integral, and the result is:

$$I = \frac{\pi}{4}$$

Thus, the correct answer is $\frac{\pi}{4}$.

Quick Tip

For integrals involving hyperbolic functions, use known identities and simplify to standard integral forms for faster solutions.

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13. The corner points of the feasible region of a Linear Programming Problem are $(0, 2)$, $(3, 0)$, $(6, 0)$, $(6, 8)$, and $(0, 5)$. If $Z = ax + by$; $(a, b > 0)$ be the objective function, and maximum value of Z is obtained at $(0, 2)$ and $(3, 0)$, then the relation between a and b is :

(A) $a = b$

(B) $a = 3b$

(C) $b = 6a$

(D) $a = 3b$

Correct Answer: (B) $a = 3b$

Solution:

Since the maximum value of Z is obtained at $(0, 2)$ and $(3, 0)$, we have the following system of equations for the objective function at these points: - At $(0, 2)$: $Z = 0 \cdot a + 2b = 2b$ - At $(3, 0)$: $Z = 3a + 0 \cdot b = 3a$ For the maximum value of Z to be the same at both points, we set $2b = 3a$, which gives the relation:

$$a = \frac{2}{3}b$$

Thus, the correct relation is $a = 3b$.

> **Quick Tip**
>
> In Linear Programming, the maximum and minimum values of the objective function often occur at the corner points of the feasible region. Use the corner point method to determine the value of the objective function at these points.

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14. If $\int e^{-3\log x}\,dx = f(x) + C$, then $f(x)$ is:

(A) $e^{-3\log x}$

(B) e

(C) $\frac{-1}{2x^2}$

(D) $\frac{-1}{4x^4}$

Correct Answer: (C) $\frac{-1}{2x^2}$

Solution:

We are given the integral:

$$\int e^{-3\log x}\,dx = f(x) + C$$

First, simplify the expression inside the integral:

$$e^{-3\log x} = \left(e^{\log x}\right)^{-3} = x^{-3}$$

So, the integral becomes:

$$\int x^{-3}\,dx$$

Step 1: Integrate x^{-3} with respect to x:

$$\int x^{-3}\,dx = \frac{x^{-2}}{-2} = \frac{-1}{2x^2}$$

Thus, we have:

$$f(x) = \frac{-1}{2x^2}$$

Therefore, the correct answer is (C) $\frac{-1}{2x^2}$.

> **Quick Tip**
>
> When solving integrals of the form $e^{a\log x}$, use the logarithmic identity $e^{\log x} = x$. This allows you to simplify the expression before integrating.

15. The function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$$

is not continuous at :

(A) $x = 0$

(B) $x = 1$

(C) $x = 2$

(D) $x = 5$

Correct Answer: (B) $x = 1$

Solution:

To check continuity at $x = 1$, we need to verify that the left-hand and right-hand limits at $x = 1$ are equal to the function value at $x = 1$. - The left-hand limit at $x = 1$ is $f(1) = 1$. - The right-hand limit at $x = 1$ is $f(1^+) = 5$. Since the left-hand and right-hand limits are not equal, the function is not continuous at $x = 1$.

Quick Tip

To check for continuity at a point, ensure that the left-hand limit, right-hand limit, and the function value at that point are all equal.

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16. The solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ represents family of:

(A) Parabolas

(B) Circles

(C) Ellipses

(D) Hyperbolas

Correct Answer: (D) Hyperbolas

Solution:

We are given the differential equation:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Step 1: Rearrange the equation:

$$y\,dy = -x\,dx$$

Step 2: Integrate both sides:

$$\int y\,dy = -\int x\,dx$$

The integrals give:

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

Step 3: Multiply through by 2:

$$y^2 = -x^2 + C'$$

which can be rewritten as:

$$x^2 - y^2 = C'$$

This is the equation of a hyperbola. Therefore, the solution of the differential equation represents a family of hyperbolas.

Thus, the correct answer is (D) Hyperbolas.

Quick Tip

When solving a differential equation that involves $\frac{dy}{dx} = -\frac{x}{y}$, integrate both sides after rearranging to find a relationship between x and y. The resulting equation $x^2 - y^2 = C'$ represents a hyperbola.

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17. If the sides AB and AC of $\triangle ABC$ are represented by vectors $\hat{i} + \hat{j} + 4\hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ respectively, then the length of the median through A on BC is :

(A) $2\sqrt{2}$ units

(B) $\sqrt{18}$ units

(C) $\frac{\sqrt{34}}{2}$ units

(D) $\frac{\sqrt{48}}{2}$ units

Correct Answer: (C) $\frac{\sqrt{34}}{2}$ units

Solution:

The length of the median is the distance from A to the midpoint of BC. First, we find the midpoint of BC, which is the average of the coordinates of points B and C. Let the coordinates of B and C be $B = (1, 1, 4)$ and $C = (3, -1, 4)$ respectively. The midpoint M is:

$$M = \left(\frac{1+3}{2}, \frac{1+(-1)}{2}, \frac{4+4}{2}\right) = (2, 0, 4)$$

Now, the distance from A $(\hat{i} + \hat{j} + 4\hat{k})$ to M $(2, 0, 4)$ is:

$$d = \sqrt{(2-1)^2 + (0-1)^2 + (4-4)^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

Thus, the length of the median is $\frac{\sqrt{34}}{2}$.

Quick Tip

To find the length of a median, find the midpoint of the opposite side and calculate the distance between the vertex and the midpoint.

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18. If $f(x) = 2x + \cos x$, then $f(x)$:

(A) has a maxima at $x = \pi$

(B) has a minima at $x = \pi$

(C) is an increasing function

(D) is a decreasing function

Correct Answer: (C) is an increasing function

Solution:

To check whether the function is increasing or decreasing, we find its derivative:

$$f'(x) = 2 - \sin x$$

Since $\sin x$ ranges from -1 to 1, we have $1 \leq f'(x) \leq 3$. Therefore, $f'(x) > 0$ for all x, meaning that the function is always increasing.

Quick Tip

To determine if a function is increasing or decreasing, check the sign of its derivative. If the derivative is positive, the function is increasing.

Assertion - Reason Based Questions

Direction : Question numbers 19 and 20 are Assertion (A) and Reason (R) based questions carrying 1 mark each. Two statements are given, one labelled Assertion (A) and other labelled Reason (R). Select the correct answer from the options (A), (B), (C) and (D) as given below.

(A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).

(B)Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C)Assertion (A) is true, but Reason (R) is false.

(D)Assertion (A) is false, but Reason (R) is true.

19. Assertion (A): If A and B are two events such that $P(A \cap B) = 0$, then A and B are independent events.

Reason (R): Two events are independent if the occurrence of one does not affect the occurrence of the other.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

- **Assertion (A)**: If $P(A \cap B) = 0$, then A and B are independent events. This assertion is true. If two events are independent, it means that the occurrence of one event does not affect the probability of the other event. The condition $P(A \cap B) = 0$ implies that A and B cannot occur together, which is a characteristic of independent events. Therefore, the assertion is correct.
- **Reason (R)**: Two events are independent if the occurrence of one does not affect the occurrence of the other. This definition is incomplete. The correct definition of independent events is: two events A and B are independent if and only if:

$$P(A \cap B) = P(A) \times P(B)$$

The reason provided in the question is false because it does not account for the correct condition for independence, which is based on the multiplication rule. Thus, Reason (R) is incorrect.

Therefore, while Assertion (A) is true, Reason (R) is false. Thus, the correct answer is option (C).

Quick Tip

For two events to be independent, the condition $P(A \cap B) = P(A) \times P(B)$ must hold true. A probability of 0 for the intersection of two events indicates that the events cannot occur together, which can be one interpretation of independence, but it is not the full definition.

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20. Assertion (A): In a Linear Programming Problem, if the feasible region is empty, then the Linear Programming Problem has no solution.
Reason (R): A feasible region is defined as the region that satisfies all the constraints.
Correct Answer: (A) Assertion (A) is true, Reason (R) is true, and Reason (R) is the correct explanation of Assertion (A).
Solution:
- **Assertion (A)**: In a Linear Programming Problem, if the feasible region is empty, then the Linear Programming Problem has no solution. This assertion is true. The feasible region represents the set of all points that satisfy the given constraints of the Linear Programming Problem. If the feasible region is empty, it means that no solution exists that satisfies all the constraints simultaneously. Hence, the Linear Programming Problem has no solution when the feasible region is empty.
- **Reason (R)**: A feasible region is defined as the region that satisfies all the constraints. This is also true. The feasible region represents all the points that satisfy the system of inequalities or equalities that define the constraints of the Linear Programming Problem. If this region is empty, no solution can be found that satisfies all constraints.
Since both Assertion (A) and Reason (R) are true, and Reason (R) correctly explains Assertion (A), the correct answer is option (A).

Quick Tip

In Linear Programming, the feasible region is crucial. If the feasible region is empty, it means there is no solution to the problem, as no point can satisfy all constraints simultaneously.

SECTION - B

This section comprises of 5 Very Short Answer (VSA) type questions of 2 marks each.

21. If

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = kA, \quad \text{then find the value of k.}$$

Solution:

We are given that $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $A^2 = kA$.

Step 1: Calculate A^2 by multiplying matrix A by itself:

$$A^2 = A \times A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

Perform the matrix multiplication:

$$A^2 = \begin{bmatrix} (2 \times 2 + -2 \times -2) & (2 \times -2 + -2 \times 2) \\ (-2 \times 2 + 2 \times -2) & (-2 \times -2 + 2 \times 2) \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4+4 & -4-4 \\ -4-4 & 4+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$$

Step 2: Compare A^2 with kA: We are given that $A^2 = kA$, so:

$$\begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = k \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

Step 3: Set up equations for each element:

$$8 = 2k \quad \text{and} \quad -8 = -2k$$

Step 4: Solve for k: From the first equation:

$$k = \frac{8}{2} = 4$$

Thus, the value of k is $\boxed{4}$.

Quick Tip

When solving matrix equations like $A^2 = kA$, perform matrix multiplication first and then equate the corresponding elements to find the unknowns.

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22. (a) Simplify $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$**.**

Solution:

We are asked to simplify $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$. Let:

$$\theta = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right).$$

Thus,

$$\sin(\theta) = \frac{x}{\sqrt{1+x^2}}.$$

Now, to find θ, we can use the identity:

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

Substitute $\sin(\theta) = \frac{x}{\sqrt{1+x^2}}$:

$$\cos^2(\theta) + \left(\frac{x}{\sqrt{1+x^2}}\right)^2 = 1 \quad \Rightarrow \quad \cos^2(\theta) = 1 - \frac{x^2}{1+x^2}.$$

Simplifying the right-hand side:

$$\cos^2(\theta) = \frac{1+x^2-x^2}{1+x^2} = \frac{1}{1+x^2}.$$

Thus,

$$\cos(\theta) = \frac{1}{\sqrt{1+x^2}}.$$

Therefore,

$$\theta = \tan^{-1}(x).$$

Thus,

$$\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \tan^{-1}(x).$$

Quick Tip

Use trigonometric identities to simplify inverse trigonometric functions. In this case, $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ simplifies to $\tan^{-1}(x)$.

OR

22. (b) Find the domain of $\sin^{-1}\sqrt{x-1}$.

Solution:

We are asked to find the domain of $\sin^{-1}\sqrt{x-1}$. The function $\sin^{-1}(y)$ is defined for $-1 \leq y \leq 1$. Hence, we need:

$$-1 \leq \sqrt{x-1} \leq 1.$$

Since $\sqrt{x-1} \geq 0$, the inequality becomes:

$$0 \leq \sqrt{x-1} \leq 1.$$

Squaring both sides:

$$0 \leq x-1 \leq 1.$$

Thus,

$$1 \leq x \leq 2.$$

Therefore, the domain of $\sin^{-1}\sqrt{x-1}$ is $[1, 2]$.

Quick Tip

When determining the domain of an inverse trigonometric function, ensure the expression inside the inverse satisfies the required range. For $\sin^{-1}(y)$, y must be in the range $[-1, 1]$.

---

23. Calculate the area of the region bounded by the curve

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

and the x-axis using integration.

Solution:

We start by solving for y in terms of x:

$$\frac{y^2}{4} = 1 - \frac{x^2}{9} \quad \Rightarrow \quad y^2 = 4\left(1 - \frac{x^2}{9}\right) \quad \Rightarrow \quad y = \pm 2\sqrt{1 - \frac{x^2}{9}}$$

The area under the curve from $x = -3$ to $x = 3$ is given by the integral:

$$A = 2\int_{-3}^{3} 2\sqrt{1 - \frac{x^2}{9}}\, dx$$

Using the substitution $x = 3\sin(\theta),\ dx = 3\cos(\theta)\, d\theta$, we get:

$$A = 36\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta)\, d\theta$$

Using $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$, the integral becomes:

$$A = 36\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}\, d\theta$$

After solving, we get:

$$A = 18\pi$$

Quick Tip

For calculating areas under curves, especially for bounded regions involving symmetric shapes like ellipses, using trigonometric substitution often simplifies the integral significantly.

---

24. (a) Find the least value of 'a' so that $f(x) = 2x^2 - ax + 3$ is an increasing function on $[2, 4]$.

Solution:

We are given the function:

$$f(x) = 2x^2 - ax + 3.$$

To determine the values of a for which the function is increasing on $[2, 4]$, we need to find the derivative of $f(x)$:

$$f'(x) = \frac{d}{dx}\left(2x^2 - ax + 3\right) = 4x - a.$$

For the function to be increasing on $[2, 4]$, the derivative must be positive for all x in this interval, i.e.,

$$f'(x) > 0 \quad \text{for all } x \in [2, 4].$$

Thus,

$$4x - a > 0 \quad \text{for all } x \in [2, 4].$$

At the minimum value of $x = 2$, we have:

$$4(2) - a > 0 \quad \Rightarrow \quad 8 - a > 0 \quad \Rightarrow \quad a < 8.$$

Therefore, the least value of a such that $f(x)$ is increasing on $[2, 4]$ is $a = 8$.

Quick Tip

To find the value of a so that a function is increasing, check the derivative and ensure it is positive over the entire interval. For quadratic functions, focus on the lowest value in the interval.

OR

24. (b) If $f(x) = x + \frac{1}{x}$, $x \geq 1$, show that f is an increasing function.

Solution:

We are given the function:

$$f(x) = x + \frac{1}{x}, \quad x \geq 1.$$

To prove that $f(x)$ is an increasing function, we will compute the derivative of $f(x)$:

$$f'(x) = \frac{d}{dx}\left(x + \frac{1}{x}\right) = 1 - \frac{1}{x^2}.$$

For $f(x)$ to be increasing, we need:

$$f'(x) \geq 0.$$

Thus,

$$1 - \frac{1}{x^2} \geq 0 \quad \Rightarrow \quad x^2 \geq 1.$$

Since $x \geq 1$, this condition is satisfied for all $x \geq 1$. Therefore, $f(x)$ is increasing on $[1, \infty)$.

Quick Tip

To prove that a function is increasing, calculate the derivative. If the derivative is always positive (or zero), the function is increasing.

---

25. A cylindrical water container has developed a leak at the bottom. The water is leaking at the rate of 5 cm^3/s from the leak. If the radius of the container is 15 cm, find the rate at which the height of water is decreasing inside the container, when the height of water is 2 meters.

Solution:

We are given that the volume of water leaking is $\frac{dV}{dt} = -5\,\text{cm}^3/\text{s}$. The negative sign indicates that the volume is decreasing.

The container is cylindrical, so the volume of water in the container at any time is given by the formula:

$$V = \pi r^2 h$$

where r is the radius of the base and h is the height of the water.

Step 1: Differentiate the volume formula with respect to time t:

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

We know that the radius $r = 15$ cm, and the height of the water is given as $h = 200$ cm (since the height of water is 2 meters, which equals 200 cm).

Step 2: Substitute the given values into the equation:

$$-5 = \pi(15)^2 \frac{dh}{dt}$$

$$-5 = \pi \times 225 \times \frac{dh}{dt}$$

$$-5 = 225\pi \frac{dh}{dt}$$

Step 3: Solve for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{-5}{225\pi}$$

$$\frac{dh}{dt} = \frac{-1}{45\pi}$$

Step 4: Approximate the rate:

$$\frac{dh}{dt} \approx \frac{-1}{141.37} \approx -0.0071\,\text{cm/s}$$

Thus, the rate at which the height of water is decreasing is approximately $-0.0071\,\text{cm/s}$.

Quick Tip

When solving related rates problems, express the quantity of interest (height in this case) in terms of the given quantities (volume and radius). Use the chain rule to differentiate and find the rate at which the quantity is changing.

SECTION - C

This section comprises of 6 Short Answer (SA) type questions of 3 marks each.

26. Find:

$$\int \frac{\sqrt{x}}{1+\sqrt{x^{3/2}}}\,dx$$

Solution:

We are asked to solve the integral:

$$I = \int \frac{\sqrt{x}}{1+\sqrt{x^{3/2}}}\,dx$$

Step 1: Simplify the expression inside the integral. Recall that:

$$\sqrt{x} = x^{1/2} \quad \text{and} \quad \sqrt{x^{3/2}} = x^{3/4}$$

Thus, the integral becomes:

$$I = \int \frac{x^{1/2}}{1+x^{3/4}}\,dx$$

Step 2: Use substitution to simplify the integral. Let:

$$u = x^{1/4} \quad \text{so that} \quad x = u^4 \quad \text{and} \quad dx = 4u^3\,du$$

Step 3: Substitute into the integral:

$$I = \int \frac{(u^4)^{1/2}}{1+u^3} \cdot 4u^3\,du$$

Simplifying:

$$I = \int \frac{u^2}{1+u^3} \cdot 4u^3\,du$$

$$I = 4\int \frac{u^5}{1+u^3}\,du$$

Step 4: Perform another substitution. Let:

$$v = 1+u^3 \quad \text{so that} \quad dv = 3u^2\,du$$

Step 5: Substitute into the integral:

$$I = \frac{4}{3}\int \frac{v}{v}\,dv = \frac{4}{3}\int 1\,dv$$

Step 6: Integrate:

$$I = \frac{4}{3}\cdot v + C$$

Substitute back $v = 1 + u^3$ and $u = x^{1/4}$:

$$I = \frac{4}{3}\left(1 + (x^{1/4})^3\right) + C = \frac{4}{3}\left(1 + x^{3/4}\right) + C$$

Thus, the solution to the integral is:

$$\boxed{I = \frac{4}{3}\left(1 + x^{3/4}\right) + C}$$

Quick Tip

For integrals involving powers of x, use substitution to simplify the expression. In this case, two substitutions were required to reduce the integral to a basic form.

---

27. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}, \quad \frac{x-4}{5} = \frac{y-1}{2} = z.$$

Solution: To solve this, first find the intersection of the two lines. Parametrize both lines.

For the first line, let:

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = t.$$

Thus,

$$x = 2t + 1, \quad y = 3t + 2, \quad z = 4t + 3.$$

For the second line, let:

$$\frac{x-4}{5} = \frac{y-1}{2} = z = s.$$

Thus,

$$x = 5s + 4, \quad y = 2s + 1, \quad z = s.$$

Equating the expressions for x, y, and z from both lines, we solve for s and t. Once the intersection point is found, calculate the distance from $(-1, -5, -10)$ to the intersection point using the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Quick Tip

To find the distance from a point to a line or between two points, use the distance formula, which is derived from the Pythagorean theorem.

---

28. (a) If $f : \mathbb{R}^+ \to \mathbb{R}$ is defined as $f(x) = \log_a x$ where $a > 0$ and $a \neq 1$, prove that f is a bijection. (R^+ is the set of all positive real numbers.)

Solution: To prove that the function $f(x) = \log_a x$ is a bijection, we need to show that it is both injective (one-to-one) and surjective (onto).

1. **Injectivity:** - For a function to be injective, if $f(x_1) = f(x_2)$, then $x_1 = x_2$ must hold. - Let $f(x_1) = f(x_2)$, i.e.,

$$\log_a x_1 = \log_a x_2.$$

- By the property of logarithms, we can rewrite this as

$$x_1 = x_2.$$

Therefore, the function $f(x) = \log_a x$ is injective.

2. **Surjectivity:** - A function is surjective if for every $y \in \mathbb{R}$, there exists $x \in \mathbb{R}^+$ such that $f(x) = y$. - Let $y \in \mathbb{R}$ be arbitrary. - We need to find $x \in \mathbb{R}^+$ such that

$$\log_a x = y.$$

- This is equivalent to

$$x = a^y,$$

which is always a positive real number for any $y \in \mathbb{R}$ (since $a > 0$ and $a \neq 1$). Therefore, $f(x) = \log_a x$ is surjective.

Since the function $f(x)$ is both injective and surjective, it is bijective.

Quick Tip

A function is a bijection if it is both injective (one-to-one) and surjective (onto).

OR

28. (b) Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. A relation R from A to B is defined as $R = \{(x, y) : x + y = 6, x \in A, y \in B\}$.

(i) Write all elements of R.

(ii) Is R a function? Justify.

(iii) Determine domain and range of R.

Solution:

1. **(i) All elements of R:** We need to find all pairs (x, y) such that $x + y = 6$ with $x \in A = \{1, 2, 3\}$ and $y \in B = \{4, 5, 6\}$. - For $x = 1$, we have $y = 6 - 1 = 5$. So, the pair is $(1, 5)$.

- For $x = 2$, we have $y = 6 - 2 = 4$. So, the pair is $(2, 4)$.

- For $x = 3$, we have $y = 6 - 3 = 3$, but since $3 \notin B$, no pair is formed for $x = 3$. Therefore, the relation R is

$$R = \{(1, 5), (2, 4)\}.$$

2. **(ii) Is R a function? Justify.** A relation is a function if every element of the domain is associated with exactly one element of the codomain. - Here, $x = 1$ is associated with $y = 5$ and $x = 2$ is associated with $y = 4$. - No element in A is associated with more than one element in B. Therefore, R is a function.

3. **(iii) Domain and Range of R:** - The **domain** of R is the set of all x values in A for which there is a corresponding y in B. In this case, the domain is

$$\text{Domain}(R) = \{1, 2\}.$$

- The **range** of R is the set of all y values in B that are associated with some x in A. In this case, the range is

$$\text{Range}(R) = \{4, 5\}.$$

> Quick Tip
>
> A relation is a function if each element in the domain is related to exactly one element in the range.

29. (a) The probability distribution of a random variable X is given by:

X	0	1	2	3
$P(X)$	p	$\frac{p}{3}$	$\frac{p}{6}$	$\frac{p}{12}$

(i) **Determine the value of p.**

(ii) **Calculate** $P(X \geq 1)$.

(iii) **Calculate expectation of X, i.e.,** $E(X)$.

Solution:

(i) To determine the value of p, we use the fact that the sum of all probabilities must equal 1. Therefore:

$$p + \frac{p}{3} + \frac{p}{6} + \frac{p}{12} = 1$$

To simplify, find the least common denominator (LCD):

$$\frac{12p}{12} + \frac{4p}{12} + \frac{2p}{12} + \frac{p}{12} = 1$$

$$\frac{19p}{12} = 1$$

Multiply both sides by 12:

$$19p = 12$$

$$p = \frac{12}{19}$$

(ii) Now, calculate $P(X \geq 1)$. This is the sum of the probabilities for $X = 1, 2, 3$:

$$P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3)$$

$$P(X \geq 1) = \frac{p}{3} + \frac{p}{6} + \frac{p}{12}$$

Substitute $p = \frac{12}{19}$:

$$P(X \geq 1) = \frac{12}{19}\left(\frac{1}{3} + \frac{1}{6} + \frac{1}{12}\right)$$

Find the LCD of 3, 6, and 12, which is 12:

$$P(X \geq 1) = \frac{12}{19}\left(\frac{4}{12} + \frac{2}{12} + \frac{1}{12}\right)$$

$$P(X \geq 1) = \frac{12}{19} \times \frac{7}{12} = \frac{7}{19}$$

(iii) To calculate the expectation $E(X)$, use the formula:

$$E(X) = \sum_{x=0}^{3} x \cdot P(X = x)$$

Substitute the values:

$$E(X) = 0 \cdot p + 1 \cdot \frac{p}{3} + 2 \cdot \frac{p}{6} + 3 \cdot \frac{p}{12}$$

$$E(X) = 0 + \frac{p}{3} + \frac{2p}{6} + \frac{3p}{12}$$

$$E(X) = \frac{4p}{12} + \frac{2p}{12} + \frac{3p}{12} = \frac{9p}{12}$$

Substitute $p = \frac{12}{19}$:

$$E(X) = \frac{9 \times \frac{12}{19}}{12} = \frac{9}{19}$$

Thus, the expected value of X is $\boxed{\frac{9}{19}}$.

Quick Tip

When working with probability distributions, remember that the sum of all probabilities must equal 1. Use this fact to solve for unknown probabilities.

OR

(b) In a city, a survey was conducted among residents about their preferred mode of commuting. It was found that 50% people preferred using public transport, 35% preferred using a bicycle, and 20% use both public transport and a bicycle. If a person is selected at random, find the probability that:

(i) The person uses only public transport.

(ii) The person uses a bicycle, given that they also use the public transport.

(iii) The person uses neither public transport nor a bicycle.

Solution:

Let $P(A)$ be the probability that a person uses public transport, $P(B)$ the probability that a person uses a bicycle, and $P(A \cap B)$ the probability that a person uses both.

We are given:

$$P(A) = 0.5, \quad P(B) = 0.35, \quad P(A \cap B) = 0.2$$

(i) The probability that a person uses only public transport is:

$$P(\text{only A}) = P(A) - P(A \cap B) = 0.5 - 0.2 = 0.3$$

(ii) The probability that a person uses a bicycle, given that they also use public transport is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.2}{0.5} = 0.4$$

(iii) The probability that a person uses neither public transport nor a bicycle is:

$$P(\text{neither}) = 1 - P(A \cup B)$$

Using the formula for the union of two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.35 - 0.2 = 0.65$$

Thus:

$$P(\text{neither}) = 1 - 0.65 = 0.35$$

Thus, the answers are: (i) 0.3, (ii) 0.4, (iii) 0.35

Quick Tip

In probability, to calculate the probability of an event given another event, use the conditional probability formula: $P(B|A) = \frac{P(A \cap B)}{P(A)}$.

---

30. (a) Find k so that the function

$$f(x) = \begin{cases} \frac{x^2-2x-3}{x+1} & \text{if } x \neq -1 \\ k & \text{if } x = -1 \end{cases}$$

is continuous at $x = -1$.

Solution:

For the function to be continuous at $x = -1$, the left-hand limit, right-hand limit, and the function value at $x = -1$ must all be equal. That is:

$$\lim_{x \to -1} f(x) = f(-1)$$

Step 1: Simplify the expression for $f(x)$ when $x \neq -1$:

$$f(x) = \frac{x^2 - 2x - 3}{x + 1}$$

Factor the numerator:

$$x^2 - 2x - 3 = (x-3)(x+1)$$

Thus:

$$f(x) = \frac{(x-3)(x+1)}{x+1}$$

For $x \neq -1$, cancel out $x+1$:

$$f(x) = x - 3$$

Step 2: Find the limit as $x \to -1$:

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} (x-3) = -1 - 3 = -4$$

Step 3: For continuity at $x = -1$, we must have:

$$f(-1) = k = -4$$

Thus, the value of k is $\boxed{-4}$.

Quick Tip

For piecewise functions to be continuous at a certain point, the function values from both sides of the point must match the function value at that point. Simplify the expression and find the limit to determine the necessary value.

OR

(b) Check the differentiability of the function $f(x) = |x|$ at $x = 0$.

Solution:

For differentiability at $x = 0$, we need to check the left-hand and right-hand derivatives at $x = 0$.

Step 1: Compute the derivative of $f(x) = |x|$ for $x > 0$ and $x < 0$:

For $x > 0$, $f(x) = x$, so the derivative is:

$$f'(x) = 1$$

For $x < 0$, $f(x) = -x$, so the derivative is:

$$f'(x) = -1$$

Step 2: The left-hand derivative at $x = 0$ is:

$$f'_{-}(0) = -1$$

The right-hand derivative at $x = 0$ is:

$$f'_{+}(0) = 1$$

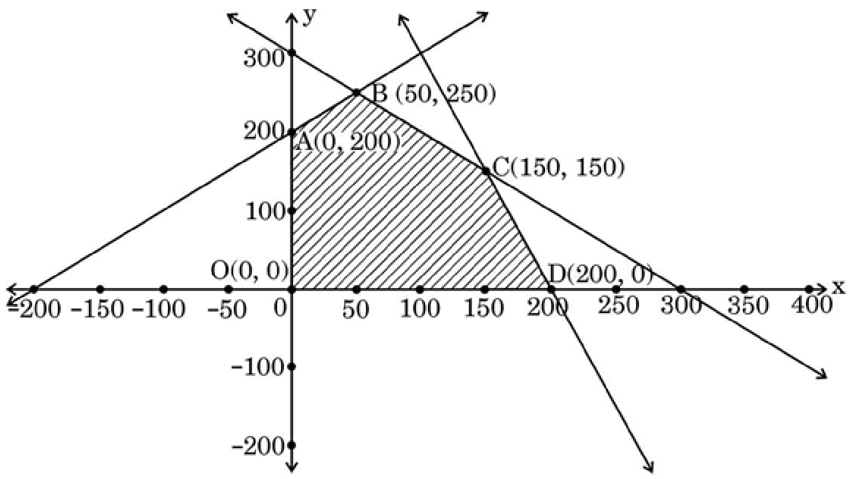
Since the left-hand and right-hand derivatives do not match, the function $f(x) = |x|$ is not differentiable at $x = 0$.

Thus, the function is not differentiable at $x = 0$.

Quick Tip

For differentiability, the left-hand and right-hand derivatives at the point of interest must be equal. If they are not, the function is not differentiable at that point.

31.



For the given graph of a Linear Programming Problem, write all the constraints satisfying the given feasible region.

Solution:

From the given graph, we can identify the constraints that form the feasible region. The vertices of the region are $A(0, 200)$, $B(50, 250)$, $C(150, 150)$, and $D(200, 0)$. Using the equation of the line passing through any two points, we can write the inequalities corresponding to the constraints.

1. From Point A(0, 200) to B(50, 250):

The slope m is calculated as:

$$m = \frac{250 - 200}{50 - 0} = \frac{50}{50} = 1$$

Using the point $A(0, 200)$ in the equation of the line:

$$y - 200 = 1(x - 0) \quad \Rightarrow \quad y = x + 200$$

Thus, the first constraint is:

$$y \leq x + 200$$

2. From Point B(50, 250) to C(150, 150):

The slope m is:

$$m = \frac{150 - 250}{150 - 50} = \frac{-100}{100} = -1$$

Using point $B(50, 250)$:

$$y - 250 = -1(x - 50) \quad \Rightarrow \quad y = -x + 300$$

Thus, the second constraint is:

$$y \leq -x + 300$$

3. From Point C(150, 150) to D(200, 0):

The slope m is:

$$m = \frac{0 - 150}{200 - 150} = \frac{-150}{50} = -3$$

Using point $C(150, 150)$:

$$y - 150 = -3(x - 150) \quad \Rightarrow \quad y = -3x + 750$$

Thus, the third constraint is:

$$y \leq -3x + 750$$

4. From Point D(200, 0) to O(0, 0):

The slope m is:

$$m = \frac{0 - 0}{200 - 0} = 0$$

The equation is simply:

$$y = 0$$

Thus, the fourth constraint is:

$$y \geq 0$$

Constraints:

The constraints for the feasible region are:

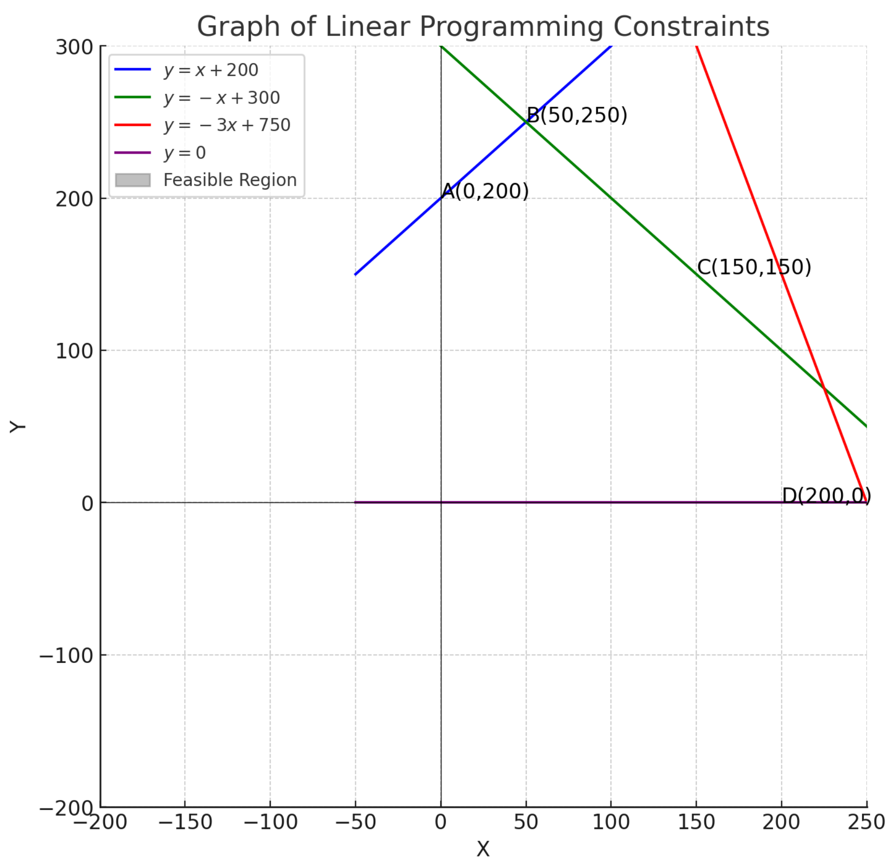
$$y \leq x + 200$$

$$y \leq -x + 300$$

$$y \leq -3x + 750$$

$$y \geq 0$$

These constraints define the feasible region in the given Linear Programming Problem.



Quick Tip

To write the constraints for a Linear Programming Problem from a graph, find the equations of the lines forming the boundaries of the feasible region and translate them into inequalities.

SECTION - D

This section comprises of 4 Long Answer (LA) type questions of 5 marks each.

32. The relation between the height of the plant (y cm) with respect to exposure to sunlight is governed by the equation

$$y = 4x - \frac{1}{2}x^2,$$

where x is the number of days exposed to sunlight.

(i) Find the rate of growth of the plant with respect to sunlight.

(ii) In how many days will the plant attain its maximum height? What is the maximum height?

Solution:

(i) **Rate of Growth of the Plant:**

The rate of growth of the plant with respect to sunlight is the derivative of the height function y with respect to time x.

The height function is given by:

$$y = 4x - \frac{1}{2}x^2.$$

Differentiating with respect to x, we get:

$$\frac{dy}{dx} = 4 - x.$$

Thus, the rate of growth of the plant with respect to sunlight is:

$$\frac{dy}{dx} = 4 - x.$$

(ii) **Maximum Height:**

To find when the plant reaches its maximum height, we set the rate of growth $\frac{dy}{dx}$ to zero:

$$4 - x = 0 \quad \Rightarrow \quad x = 4.$$

Thus, the plant reaches its maximum height in 4 days.

Now, substitute $x = 4$ into the height equation to find the maximum height:

$$y = 4(4) - \frac{1}{2}(4)^2 = 16 - 8 = 8\,\text{cm}.$$

Hence, the maximum height of the plant is 8 cm.

Quick Tip

The maximum or minimum of a function occurs when the first derivative is zero.

---

33. (a) Show that the area of a parallelogram whose diagonals are represented by $\vec{a}$ and $\vec{b}$ is given by

$$\text{Area} = \frac{1}{2}|\vec{a} \times \vec{b}|.$$

Also, find the area of a parallelogram whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} - \hat{k}$.

Solution: To find the area of a parallelogram, we use the formula for the magnitude of the cross product of two vectors:

$$\text{Area} = \frac{1}{2}|\vec{a} \times \vec{b}|.$$

Now, given the diagonals $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - \hat{k}$, we compute the cross product:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = \hat{i} \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}.$$

After calculating the determinants, we get:

$$\vec{a} \times \vec{b} = 2\hat{i} + 3\hat{j} + 7\hat{k}.$$

Thus, the magnitude of the cross product is:

$$|\vec{a} \times \vec{b}| = \sqrt{2^2 + 3^2 + 7^2} = \sqrt{4 + 9 + 49} = \sqrt{62}.$$

Therefore, the area of the parallelogram is:

$$\text{Area} = \frac{1}{2}\sqrt{62}.$$

Quick Tip

The area of a parallelogram formed by vectors can be computed using the magnitude of the cross product of the two vectors.

OR

33. (b) Find the equation of a line in vector and Cartesian form which passes through the point $(1, 2, -4)$ and is perpendicular to the lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}.$$

and

$$\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k}).$$

Solution: We are given the lines and need to find a line that is perpendicular to both. The direction vectors of the given lines are:

$$\vec{d_1} = \langle 3, -16, 7 \rangle \quad \text{and} \quad \vec{d_2} = \langle 3, 8, -5 \rangle.$$

The direction vector of the required line is the cross product of $\vec{d_1}$ and $\vec{d_2}$:

$$\vec{d} = \vec{d_1} \times \vec{d_2}.$$

We compute the cross product:

$$\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} = \hat{i} \begin{vmatrix} -16 & 7 \\ 8 & -5 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 7 \\ 3 & -5 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & -16 \\ 3 & 8 \end{vmatrix}.$$

After calculating the determinants, we get:

$$\vec{d} = \langle -51, -6, 69 \rangle.$$

The equation of the line in vector form is:

$$\vec{r} = \vec{r_0} + \lambda \vec{d},$$

where $\vec{r_0} = \langle 1, 2, -4 \rangle$ and $\vec{d} = \langle -51, -6, 69 \rangle$. Hence, the equation of the line is:

$$\vec{r} = \langle 1, 2, -4 \rangle + \lambda \langle -51, -6, 69 \rangle.$$

In Cartesian form, the equation is:

$$\frac{x-1}{-51} = \frac{y-2}{-6} = \frac{z+4}{69}.$$

Quick Tip

To find the equation of a line perpendicular to two other lines, compute the cross product of their direction vectors to get the direction of the required line.

34. (a) Evaluate:

$$\int_0^{\frac{3}{2}} x\cos(\pi x)\,dx$$

Solution: We will use integration by parts for this integral. Let:

$$u = x \quad \text{so that} \quad du = dx$$

$$dv = \cos(\pi x)\,dx \quad \text{and} \quad v = \frac{1}{\pi}\sin(\pi x)$$

Now, applying the integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

$$\int_0^{\frac{3}{2}} x\cos(\pi x)\,dx = \left[x \cdot \frac{1}{\pi}\sin(\pi x)\right]_0^{\frac{3}{2}} - \int_0^{\frac{3}{2}} \frac{1}{\pi}\sin(\pi x)\,dx$$

The first term evaluates as:

$$\left[\frac{x\sin(\pi x)}{\pi}\right]_0^{\frac{3}{2}} = \frac{\frac{3}{2}\sin\left(\pi \cdot \frac{3}{2}\right)}{\pi} - \frac{0\sin(0)}{\pi} = \frac{\frac{3}{2}\cdot(-1)}{\pi} = -\frac{3}{2\pi}$$

Now, for the second term:

$$-\int_0^{\frac{3}{2}} \frac{1}{\pi}\sin(\pi x)\,dx = -\frac{1}{\pi}\left[-\frac{1}{\pi}\cos(\pi x)\right]_0^{\frac{3}{2}} = \frac{1}{\pi^2}\left[\cos(0) - \cos\left(\pi \cdot \frac{3}{2}\right)\right]$$

$$= \frac{1}{\pi^2}[1-0] = \frac{1}{\pi^2}$$

Thus, the total result is:

$$-\frac{3}{2\pi} + \frac{1}{\pi^2}$$

Quick Tip

Integration by parts is a useful technique when the integral involves a product of functions. To use this method, choose u as the function that simplifies when differentiated, and dv as the part that can be easily integrated.

OR

(b) Find:

$$\int \frac{dx}{\sin x + \sin 2x}$$

Solution: We can simplify the denominator using the sum-to-product identities for trigonometric functions:

$$\sin x + \sin 2x = 2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)$$

Thus, the integral becomes:

$$\int \frac{dx}{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

This can be further simplified by applying substitution. Let:

$$u = \frac{x}{2} \quad \text{so that} \quad du = \frac{dx}{2}$$

which changes the integral to:

$$\int \frac{2\,du}{\sin\left(\frac{3}{2}\cdot 2u\right)\cos(u)} = \int \frac{du}{\sin(3u)\cos(u)}$$

This can be solved by splitting the integrals or applying another substitution to handle the trigonometric terms.

Quick Tip

When dealing with trigonometric integrals, using sum-to-product identities can simplify the expression, making it easier to integrate. Substitution is a helpful technique when the integral is in a more complex form.

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35. If A is a 3×3 invertible matrix, show that for any scalar $k \neq 0$,

$$(kA)^{-1} = \frac{1}{k}A^{-1}.$$

Hence, calculate $(3A)^{-1}$, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Solution:

1. **Proof that $(kA)^{-1} = \frac{1}{k}A^{-1}$:**

We need to prove that for any scalar $k \neq 0$, the inverse of the matrix kA is $\frac{1}{k}A^{-1}$.

First, consider the product $(kA)(kA)^{-1}$. According to the property of inverses, this product should equal the identity matrix I.

$$(kA)(kA)^{-1} = I.$$

Now, if we let $(kA)^{-1} = \frac{1}{k}A^{-1}$, we have:

$$(kA)\left(\frac{1}{k}A^{-1}\right) = kA \cdot \frac{1}{k}A^{-1} = AA^{-1} = I.$$

Hence, we have shown that $(kA)^{-1} = \frac{1}{k}A^{-1}$.

2. **Calculate $(3A)^{-1}$:**

Now that we know the property $(kA)^{-1} = \frac{1}{k}A^{-1}$, we can calculate $(3A)^{-1}$. Using the property for $k = 3$, we have:

$$(3A)^{-1} = \frac{1}{3}A^{-1}.$$

So, we need to calculate A^{-1}, the inverse of matrix A.

The matrix A is:

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Using a calculator or applying the formula for the inverse of a matrix, we find:

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

Therefore:

$$(3A)^{-1} = \frac{1}{3}\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{12} & -\frac{1}{12} \\ \frac{1}{12} & \frac{1}{4} & \frac{1}{12} \\ -\frac{1}{12} & \frac{1}{12} & \frac{1}{4} \end{bmatrix}.$$

Quick Tip

The inverse of a matrix is a key tool for solving systems of linear equations. When a scalar is multiplied by a matrix, the inverse of the product is simply the reciprocal of the scalar multiplied by the inverse of the matrix.

SECTION - E

This section comprises of 3 case study based questions of 4 marks each.

Case Study -1

36. Some students are having a misconception while comparing decimals. For example, a student may mention that $78.56 > 78.9$ as $7856 > 789$. In order to assess this concept, a decimal comparison test was administered to the students of class VI through the following question: In the recently held Sports Day in the school, 5 students participated in a javelin throw competition. The distances to which they have thrown the javelin are shown below in the table:

Name of student	Distance of javelin (in meters)
Ajay	47.7
Bijoy	47.07
Kartik	43.09
Dinesh	43.9
Devesh	45.2

The students were asked to identify who has thrown the javelin the farthest. Based on the test attempted by the students, the teacher concludes that 40% of the students have the misconception in the concept of decimal comparison and the rest do not have the misconception. 80% of the students having misconception answered Bijoy as the correct answer in the paper. 90% of the students who are identified with not having misconception, did not answer Bijoy as their answer.

On the basis of the above information, answer the following questions:

(i) What is the probability of a student not having misconception but still answers Bijoy in the test?

Solution: Let's define the following events:

- M: The event that a student has a misconception.
- N: The event that a student does not have a misconception.
- B: The event that a student answers Bijoy.

We are asked to find the probability that a student does not have a misconception but still answers Bijoy, i.e., $P(B \cap N)$.

From the problem:

- 40% of the students have a misconception, so the probability of having a misconception is $P(M) = 0.4$, and the probability of not having a misconception is $P(N) = 0.6$.
- 80% of the students with a misconception answer Bijoy, so $P(B \mid M) = 0.8$.
- 90% of the students without a misconception do not answer Bijoy, so $P(B \mid N) = 1 - 0.9 = 0.1$.

Thus, the probability that a student does not have a misconception but still answers Bijoy is:

$$P(B \cap N) = P(N) \times P(B \mid N) = 0.6 \times 0.1 = 0.06.$$

So, the probability is $\boxed{0.06}$.

(ii) What is the probability that a randomly selected student answers Bijoy as his answer in the test?

Solution: We are asked to find the probability $P(B)$, the probability that a randomly selected student answers Bijoy.

Using the law of total probability:

$$P(B) = P(B \mid M)P(M) + P(B \mid N)P(N).$$

Substituting the known values:

$$P(B) = (0.8 \times 0.4) + (0.1 \times 0.6) = 0.32 + 0.06 = 0.38.$$

Thus, the probability is $\boxed{0.38}$.

(iii) (a) What is the probability that a student who answered as Bijoy is having misconception?

Solution: We are asked to find $P(M \mid B)$, the probability that a student who answered Bijoy is having a misconception.

Using Bayes' Theorem:

$$P(M \mid B) = \frac{P(B \mid M)P(M)}{P(B)}.$$

Substituting the known values:

$$P(M \mid B) = \frac{(0.8 \times 0.4)}{0.38} = \frac{0.32}{0.38} \approx 0.8421.$$

Thus, the probability is $\boxed{0.8421}$.

OR

(iii) (b) What is the probability that a student who answered as Bijoy is amongst students who do not have the misconception?

Solution: We are asked to find $P(N \mid B)$, the probability that a student who answered Bijoy does not have a misconception.

Using Bayes' Theorem:

$$P(N \mid B) = \frac{P(B \mid N)P(N)}{P(B)}.$$

Substituting the known values:

$$P(N \mid B) = \frac{(0.1 \times 0.6)}{0.38} = \frac{0.06}{0.38} \approx 0.1579.$$

Thus, the probability is $\boxed{0.1579}$.

Quick Tip

In probability, the law of total probability and Bayes' theorem are powerful tools to calculate conditional probabilities when dealing with events like misconceptions and test responses.

Case Study - 2

37. An engineer is designing a new metro rail network in a city.

Initially, two metro lines, Line A and Line B, each consisting of multiple stations are designed. The track for Line A is represented by

$$l_1 : \frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-3}{4},$$

while the track for Line B is represented by

$$l_2 : \frac{x-1}{2} = \frac{y-3}{1} = \frac{z+2}{-3}.$$

Based on the above information, answer the following questions:

(i) Find whether the two metro tracks are parallel.

Solution: To check if the two lines are parallel, we compare the direction ratios of both lines.

For Line A, the direction ratios are the coefficients of x, y, and z in the parametric equations, i.e., the ratios corresponding to $\frac{x-2}{3}$, $\frac{y+1}{-2}$, and $\frac{z-3}{4}$, which give the direction ratios $\vec{d_1} = \langle 3, -2, 4 \rangle$.

For Line B, the direction ratios are $\vec{d_2} = \langle 2, 1, -3 \rangle$, corresponding to the parametric equations $\frac{x-1}{2}$, $\frac{y-3}{1}$, and $\frac{z+2}{-3}$.

To determine if the lines are parallel, check if the direction ratios are proportional. That is, check if there exists a constant k such that:

$$\frac{3}{2} = \frac{-2}{1} = \frac{4}{-3}.$$

Since the ratios $\frac{3}{2}$, $\frac{-2}{1}$, and $\frac{4}{-3}$ are not equal, the direction ratios are not proportional, implying that the two lines are not parallel.

Thus, the two metro tracks are **not parallel**.

Quick Tip

Two lines are parallel if their direction ratios are proportional.

(ii) Solar panels are to be installed on the rooftop of the metro stations. Determine the equation of the line representing the placement of solar panels on the rooftop of Line A's stations, given that panels are to be positioned parallel to Line A's track (l_1) and pass through the point $(1, -2, -3)$.

Solution: The equation of a line parallel to Line A's track can be written in parametric form as:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

where (x_0, y_0, z_0) is the point through which the line passes, and (a, b, c) is the direction vector of the line.

Given that the direction ratios of Line A are $\langle 3, -2, 4 \rangle$, and the point $(1, -2, -3)$ is given, the equation of the line representing the placement of solar panels is:

$$\frac{x-1}{3} = \frac{y+2}{-2} = \frac{z+3}{4}.$$

Thus, the equation of the line representing the placement of solar panels is

$$\boxed{\frac{x-1}{3} = \frac{y+2}{-2} = \frac{z+3}{4}}.$$

Quick Tip

The equation of a line passing through a point and parallel to a given line is based on the direction ratios of the given line.

(iii) (a) To connect the stations, a pedestrian pathway perpendicular to the two metro lines is to be constructed which passes through point $(3, 2, 1)$. Determine the equation of the pedestrian walkway.

Solution: The pedestrian walkway is perpendicular to both lines, so we need to find the cross product of the direction vectors of both lines. The direction ratios of Line A are $\vec{d_1} = \langle 3, -2, 4 \rangle$ and for Line B, $\vec{d_2} = \langle 2, 1, -3 \rangle$.

The direction vector of the pedestrian walkway is the cross product of $\vec{d_1}$ and $\vec{d_2}$:

$$\vec{d} = \vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 4 \\ 2 & 1 & -3 \end{vmatrix}.$$

Calculating the cross product:

$$\vec{d} = \langle (-2)(-3) - (4)(1), (4)(2) - (3)(-3), (3)(1) - (-2)(2) \rangle = \langle 6 - 4, 8 + 9, 3 + 4 \rangle = \langle 2, 17, 7 \rangle.$$

The equation of the pedestrian walkway passing through point $(3, 2, 1)$ with direction vector $\vec{d} = \langle 2, 17, 7 \rangle$ is:

$$\frac{x-3}{2} = \frac{y-2}{17} = \frac{z-1}{7}.$$

Thus, the equation of the pedestrian walkway is

$$\boxed{\frac{x-3}{2} = \frac{y-2}{17} = \frac{z-1}{7}}.$$

OR

(iii) (b) What is the shortest distance between Line A and Line B?

Solution: The shortest distance d between two skew lines can be calculated using the formula:

$$d = \frac{|(\vec{r_2} - \vec{r_1}) \cdot (\vec{d_1} \times \vec{d_2})|}{|\vec{d_1} \times \vec{d_2}|},$$

where $\vec{r_1}$ and $\vec{r_2}$ are points on Lines A and B, and $\vec{d_1}$ and $\vec{d_2}$ are the direction vectors of the lines.

Substitute the values of the vectors and calculate the distance using the cross product and dot product.

Quick Tip

To find the shortest distance between skew lines, use the formula involving the cross product of direction vectors and the vector connecting points on both lines.

Case Study-3

38. During a heavy gaming session, the temperature of a student's laptop processor increases significantly. After the session, the processor begins to cool down, and the rate of cooling is proportional to the difference between the processor's temperature and the room temperature ($25°C$). Initially, the processor's temperature is $85°C$. The rate of cooling is defined by the equation

$$\frac{dT(t)}{dt} = -k(T(t) - 25),$$

where $T(t)$ represents the temperature of the processor at time t (in minutes) and k is a constant.

Based on the above information, answer the following questions:

(i) Find the expression for the temperature of the processor, $T(t)$, given that $T(0) = 85°C$.

Solution: We are given the differential equation:

$$\frac{dT(t)}{dt} = -k(T(t) - 25).$$

This is a first-order linear differential equation that can be solved by separation of variables. First, rearrange the equation:

$$\frac{dT(t)}{T(t) - 25} = -k\,dt.$$

Now, integrate both sides:

$$\int \frac{1}{T(t) - 25}\,dT(t) = \int -k\,dt.$$

The left-hand side is the integral of $\frac{1}{T(t)-25}$, which is $\ln|T(t) - 25|$, and the right-hand side is $-kt + C$, where C is the constant of integration:

$$\ln|T(t) - 25| = -kt + C.$$

Exponentiate both sides to solve for $T(t)$:

$$|T(t) - 25| = e^{-kt+C} = e^C e^{-kt}.$$

Let $e^C = A$ (where A is a constant), so:

$$|T(t) - 25| = Ae^{-kt}.$$

Thus, the general solution is:

$$T(t) - 25 = Ae^{-kt}.$$

Now, solve for $T(t)$:

$$T(t) = 25 + Ae^{-kt}.$$

To find the constant A, use the initial condition $T(0) = 85$:

$$T(0) = 25 + Ae^0 = 85 \quad \Rightarrow \quad 25 + A = 85 \quad \Rightarrow \quad A = 60.$$

Thus, the expression for $T(t)$ is:

$$T(t) = 25 + 60e^{-kt}.$$

Quick Tip

To solve first-order linear differential equations, use the method of separation of variables.

(ii) How long will it take for the processor's temperature to reach $40°C$? Given that $k = 0.03$, $\log_e 4 = 1.3863$.

Solution: We are given the equation for $T(t)$:

$$T(t) = 25 + 60e^{-0.03t}.$$

We are asked to find the time t when $T(t) = 40$:

$$40 = 25 + 60e^{-0.03t}.$$

Subtract 25 from both sides:

$$15 = 60e^{-0.03t}.$$

Divide both sides by 60:

$$\frac{15}{60} = e^{-0.03t} \quad \Rightarrow \quad \frac{1}{4} = e^{-0.03t}.$$

Take the natural logarithm of both sides:

$$\ln\left(\frac{1}{4}\right) = -0.03t.$$

Using the property of logarithms $\ln\left(\frac{1}{4}\right) = -\ln 4$, we have:

$$-\ln 4 = -0.03t \quad \Rightarrow \quad t = \frac{\ln 4}{0.03}.$$

Substitute $\ln 4 = 1.3863$:

$$t = \frac{1.3863}{0.03} \approx 46.21.$$

Thus, it will take approximately 46.21 minutes for the processor's temperature to reach $40°C$.

Quick Tip

When solving for time in cooling problems, use the natural logarithm to solve the exponential equation.
