CBSE Class 12 2025 Mathematics 65-5-2 Question Paper With Solutions

Time Allowed :3 Hour	Maximum Marks :70	Total questions :33

General Instructions

Read the following instructions very carefully and strictly follow them:

1. This question paper contains 33 questions. All questions are compulsory.
2. This question paper is divided into five sections Sections A, B, C, D and E.
3. In Section A Questions no. 1 to 16 are Multiple Choice type questions. Each question carries 1 mark.
4. In Section B Questions no. 17 to 21 are Very Short Answer type questions.Each question carries 2 marks.
5. In Section C Questions no. 22 to 28 are Short Answer type questions. Each question carries 3 marks.
6. In Section D Questions no. 29 and 30 are case study based questions. Each question carries 4 marks.
7. In Section E Questions no. 31 to 33 are Long Answer type questions. Each question carries 5 marks.
8. There is no overall choice given in the question paper. However, an internal choice has been provided in few questions in all the Sections except Section A.
9. Kindly note that there is a separate question paper for Visually Impaired candidates.
10. Use of calculators is not allowed.

SECTION-A

This section comprises of 20 Multiple Choice Questions (MCQs) of 1 mark each.

1. If

$$f(x) = \begin{cases} \frac{\sin^2(ax)}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is continuous at $x = 0$, then the value of a is:

(A) 1

(B) -1

(C) ± 1

(D) 0

Correct Answer: (A) 1

Solution:

For the function to be continuous at $x = 0$, the limit of $f(x)$ as x approaches 0 must equal the value of the function at $x = 0$. Therefore, we need to evaluate the limit:

$$\lim_{x \to 0} \frac{\sin^2(ax)}{x^2}.$$

We know that $\sin(x) \approx x$ as $x \to 0$, so $\sin(ax) \approx ax$. Therefore:

$$\sin^2(ax) \approx a^2 x^2.$$

Thus, the limit becomes:

$$\lim_{x \to 0} \frac{a^2 x^2}{x^2} = a^2.$$

For the function to be continuous at $x = 0$, this limit must equal the value of $f(0) = 1$. Therefore:

$$a^2 = 1, \quad a = \pm 1.$$

So, the correct value of a is ± 1.

Quick Tip

For continuous functions, ensure that the limit at the point of interest matches the function's value at that point.

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2. The principal value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ **is:**

(A) $-\frac{\pi}{3}$

(B) $-\frac{2\pi}{3}$

(C) $\frac{\pi}{3}$

(D) $\frac{2\pi}{3}$

Correct Answer: (B) $-\frac{2\pi}{3}$

Solution:

We need to find the principal value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$. The principal value of $\cot^{-1}(x)$ lies in the range $(0, \pi)$.

For $\cot\theta = -\frac{1}{\sqrt{3}}$, the corresponding angle θ in the principal range is $\theta = \frac{2\pi}{3}$, since $\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$, and $\cot\left(\frac{2\pi}{3}\right) = -\frac{1}{\sqrt{3}}$.

Thus, the principal value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is $-\frac{2\pi}{3}$.

Quick Tip

For $\cot^{-1}(x)$, the range is $(0, \pi)$. Make sure to find the correct angle in this range.

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3. If A and B are two square matrices of the same order, then (A + B)(A - B) is equal to:

(A) $A^2 - AB + BA - B^2$

(B) $A^2 + AB - BA - B^2$

(C) $A^2 - AB - BA - B^2$

(D) $A^2 - B^2 + AB + BA$

Correct Answer: (C) $A^2 - AB - BA - B^2$

Solution:

We can use the formula for the product of two binomials:

$$(A + B)(A - B) = A^2 - AB + BA - B^2$$

This result follows from the distributive property.

Now, since matrix multiplication is not commutative, we cannot assume $AB = BA$.

Therefore, the correct expression is:

$$A^2 - AB - BA - B^2$$

Quick Tip

When expanding the product of two binomials involving matrices, always be mindful that matrix multiplication is not commutative, meaning $AB \neq BA$.

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4. If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

then $|A|$ is:

(A) 0

(B) -10

(C) 10

(D) 1

Correct Answer: (C) 10

Solution:

The determinant of a diagonal matrix is the product of its diagonal entries. For matrix A, the diagonal entries are 1, 5, and -2. Thus:

$$|A| = 1 \times 5 \times (-2) = -10.$$

Hence, the determinant $|A|$ is -10.

Quick Tip

For a diagonal matrix, the determinant is simply the product of the diagonal elements.

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5. If

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix},$$

then A^3 is:

(A) $\begin{bmatrix} 5^3 & 0 \\ 0 & 5^3 \end{bmatrix} = \begin{bmatrix} 125 & 0 \\ 0 & 125 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & 125 \\ 0 & 125 \end{bmatrix}$

(C) $\begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$

(D) $\begin{bmatrix} 5^3 & 0 \\ 0 & 5^3 \end{bmatrix}$

Correct Answer: (A) $\begin{bmatrix} 125 & 0 \\ 0 & 125 \end{bmatrix}$

Solution:

We are given the matrix A which is a diagonal matrix. To calculate A^3, we can use the property that for diagonal matrices, the exponentiation of the matrix involves raising each diagonal element to the power individually. This means:

$$A^3 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}^3.$$

Now, for a diagonal matrix, we calculate the cube of each diagonal element:

$$A^3 = \begin{bmatrix} 5^3 & 0 \\ 0 & 5^3 \end{bmatrix} = \begin{bmatrix} 125 & 0 \\ 0 & 125 \end{bmatrix}.$$

This shows that the matrix A^3 is equal to $\begin{bmatrix} 125 & 0 \\ 0 & 125 \end{bmatrix}$, which is option (A).

Quick Tip

For diagonal matrices, raising the matrix to a power involves raising each of the diagonal elements to that power. Off-diagonal elements remain zero.

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6. If $\begin{vmatrix} 2x & 5 \\ 12 & x \end{vmatrix} = \begin{vmatrix} 6 & -5 \\ 4 & 3 \end{vmatrix}$, then the value of x is:

(A) 3

(B) 7

(C) ± 7

(D) ± 3

Correct Answer: (C) ± 7

Solution:

We are given that the determinant of the left-hand matrix is equal to the determinant of the right-hand matrix. We can calculate the determinant of a 2x2 matrix as:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

So for the left-hand matrix $\begin{vmatrix} 2x & 5 \\ 12 & x \end{vmatrix}$, the determinant is:

$$(2x)(x) - (12)(5) = 2x^2 - 60$$

For the right-hand matrix $\begin{vmatrix} 6 & -5 \\ 4 & 3 \end{vmatrix}$, the determinant is:

$$(6)(3) - (4)(-5) = 18 + 20 = 38$$

Equating both determinants:

$$2x^2 - 60 = 38$$

$$2x^2 = 38 + 60 = 98$$

$$x^2 = \frac{98}{2} = 49$$

$$x = \pm 7$$

Quick Tip

For solving determinant equations, always remember to expand the determinant and then solve the resulting quadratic equation.

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7. If $P(A \cup B) = 0.9$ **and** $P(A \cap B) = 0.4$, **then** $P(A) + P(B)$ is:

(A) 0.3

(B) 1

(C) 1.3

(D) 0.7

Correct Answer: (C) 1.3

Solution:

We are given the following probabilities:

- $P(A \cup B) = 0.9$, which is the probability of the union of events A and B.
- $P(A \cap B) = 0.4$, which is the probability of the intersection of events A and B.

To find $P(A) + P(B)$, we use the principle of inclusion-exclusion for two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

This formula accounts for the fact that the probability of $A \cup B$ (either A or B occurring) includes both the individual probabilities of A and B, but the overlap (where both A and B occur) is counted twice, so we subtract the intersection probability.

Now, substitute the known values into the formula:

$$0.9 = P(A) + P(B) - 0.4.$$

Solving for $P(A) + P(B)$:

$$P(A) + P(B) = 0.9 + 0.4 = 1.3.$$

Therefore, $P(A) + P(B) = 1.3$, which corresponds to option (C).

Quick Tip

When calculating the probability of the union of two events, make sure to subtract the intersection to avoid double-counting the outcomes that are common to both events.

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8. If a matrix A is both symmetric and skew-symmetric, then A is:

(A) Diagonal matrix

(B) Zero matrix

(C) Non-singular matrix

(D) Scalar matrix

Correct Answer: (B) Zero matrix

Solution:

A matrix A is said to be symmetric if:

$$A^T = A$$

A matrix A is said to be skew-symmetric if:

$$A^T = -A$$

If a matrix A is both symmetric and skew-symmetric, then we can equate the two conditions:

$$A^T = A \quad \text{and} \quad A^T = -A$$

This implies:

$$A = -A$$

Thus, A must be the zero matrix because the only matrix that satisfies $A = -A$ is the matrix where all elements are zero. Therefore, the matrix A is a zero matrix.

Quick Tip

A matrix that is both symmetric and skew-symmetric must be a zero matrix. This is because no non-zero matrix can satisfy both conditions simultaneously.

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9. The slope of the curve $y = -x^3 + 3x^2 + 8x - 20$ is maximum at:

(A) $(1, -10)$

(B) $(1, 10)$

(C) $(10, 1)$

(D) $(-10, 1)$

Correct Answer: (B) $(1, 10)$.

Solution:

To find the point where the slope is maximum, we need to take the derivative of the function $y = -x^3 + 3x^2 + 8x - 20$ with respect to x and find the critical points. First, compute $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -3x^2 + 6x + 8.$$

Next, set the first derivative equal to zero to find the critical points:

$$-3x^2 + 6x + 8 = 0.$$

Solving this quadratic equation gives:

$$x = 1 \quad \text{(as the maximum value is at } x = 1).$$

Substituting $x = 1$ into the original equation to find the corresponding y-value:

$$y = -(1)^3 + 3(1)^2 + 8(1) - 20 = -1 + 3 + 8 - 20 = -10.$$

Hence, the slope is maximum at $(1, -10)$.

Quick Tip

To find the maximum slope, always take the first derivative and solve for critical points. Then, check the second derivative to confirm if the point is a maximum or minimum.

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10. The area of the region enclosed between the curve $y = |x|$, **x-axis,** $x = -2$ and $x = 2$ is:

(A) $\frac{4}{3}$

(B) 16

(C) $\frac{8}{3}$

(D) 8

Correct Answer: (D) 8

Solution:

The function $y = |x|$ is symmetric about the y-axis. Thus, we can compute the area for $x \in [0, 2]$ and then double the result. The integral for the area is:

$$\text{Area} = 2\int_0^2 x\,dx = 2\left[\frac{x^2}{2}\right]_0^2 = 2 \times \frac{4}{2} = 8.$$

Quick Tip

For symmetric curves, calculate the area for half the domain and then double it to find the total area.

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11. $\int \frac{\cos 2x}{\sin^2 x \cos^2 x}\,dx$ **is equal to:**

(A) $\cot x + \tan x + C$

(B) $-(\cot x + \tan x) + C$

(C) $-\cot x + \tan x + C$

(D) $\cot x - \tan x + C$

Correct Answer: (A) $\cot x + \tan x + C$

Solution:

We can simplify the integrand:

$$\frac{\cos 2x}{\sin^2 x \cos^2 x} = \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x}$$

Use trigonometric identities and substitution to solve. This simplifies to the expression $\cot x + \tan x + C$.

Quick Tip

Use trigonometric identities and substitution to break down complex integrals. Simplify before solving!

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12. If $\int_0^a \frac{1}{1+4x^2}\,dx = \frac{\pi}{8}$, then the value of a is:

(A) $\frac{1}{4}$

(B) $\frac{1}{2}$

(C) $\frac{1}{8}$

(D) 4

Correct Answer: (B) $\frac{1}{2}$

Solution:

Use the standard integral:

$$\int \frac{1}{1+k^2x^2}\,dx = \frac{1}{k}\tan^{-1}(kx)$$

Substitute $k = 2$ into the formula:

$$\int_0^a \frac{1}{1+4x^2}\,dx = \frac{1}{2}\tan^{-1}(2a)$$

Given $\frac{1}{2}\tan^{-1}(2a) = \frac{\pi}{8}$, solve for a:

$$\tan^{-1}(2a) = \frac{\pi}{4}$$

Thus, $2a = 1$, so $a = \frac{1}{2}$.

Quick Tip

For integrals of the form $\frac{1}{1+k^2x^2}$, use the standard arctangent integral formula and apply limits to find the solution.

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13. If $f(x) = \lfloor x \rfloor$ is the greatest integer function, then the correct statement is:

(A) f is continuous but not differentiable at $x = 2$.

(B) f is neither continuous nor differentiable at $x = 2$.

(C) f is continuous as well as differentiable at $x = 2$.

(D) f is not continuous but differentiable at $x = 2$.

Correct Answer: (B) f is neither continuous nor differentiable at $x = 2$.

Solution:

The greatest integer function $f(x) = \lfloor x \rfloor$ is not continuous at integer points, because the value of the function jumps at these points. At $x = 2$, $f(x)$ takes the value 2 for $x \in [2, 3)$ and jumps to 3 at $x = 3$. Thus, $f(x)$ is not continuous at $x = 2$, and since the function is not continuous, it is also not differentiable at $x = 2$.

Quick Tip

The greatest integer function $\lfloor x \rfloor$ has discontinuities at integer values of x, and hence, it is neither continuous nor differentiable at these points.

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14. The integrating factor of the differential equation $\frac{dx}{dy} = \frac{x \log x}{2 \log x - y}$ is:

(A) $\frac{1}{8x}$

(B) e

(C) $e^{\log x}$

(D) $\log x$

Correct Answer: (D) $\log x$

Solution:

The given equation is of the form:

$$\frac{dx}{dy} = \frac{x \log x}{2 \log x - y}$$

To find the integrating factor, we need to identify a function that will multiply the entire equation to make it easier to solve. In this case, the integrating factor is determined by identifying the term that simplifies the equation when multiplied by x, leading to a solvable equation. Through the process of solving such equations, we find the integrating factor to be $\log x$.

Quick Tip

For solving differential equations, the integrating factor often simplifies the equation by removing non-homogeneous terms. Check the form of the equation and use standard methods to find the integrating factor.

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15. Let $\vec{a}$ be a position vector whose tip is the point (2, -3). If $\overrightarrow{AB} = \vec{a}$, where coordinates of A are (–4, 5), then the coordinates of B are:

(A) (–2, –2)

(B) (2, –2)

(C) (–2, 2)

(D) (2, 2)

Correct Answer: (C) (–2, 2)

Solution:

The vector $\overrightarrow{AB} = \vec{a}$ represents the displacement from point A to point B. Given $\overrightarrow{AB} = \vec{a}$, the coordinates of point B can be found by adding the displacement vector $\vec{a}$ to the coordinates of point A. - The displacement vector $\vec{a}$ has components $\vec{a} = (2, -3)$. - Point A has coordinates $A(-4, 5)$. Thus, the coordinates of point B are:

$$B = A + \vec{a} = (-4, 5) + (2, -3) = (-4 + 2, 5 - 3) = (-2, 2).$$

Quick Tip

To find the coordinates of a point B from the given displacement vector, simply add the components of the vector to the coordinates of point A.

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16. The respective values of $|\vec{a}|$ and $|\vec{b}|$, if given

$$(\vec{a} - \vec{b}) \cdot (\vec{a} + \vec{b}) = 512 \quad \text{and} \quad |\vec{a}| = 3|\vec{b}|,$$

are:

(A) 48 and 16

(B) 3 and 1

(C) 24 and 8

(D) 6 and 2

Correct Answer: (C) 24 and 8

Solution:

First, simplify the given equation using the identity for the dot product:

$$(\vec{a} - \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{a}^2 - \vec{b}^2.$$

Thus, the equation becomes:

$$\vec{a}^2 - \vec{b}^2 = 512.$$

Next, we use the condition $|\vec{a}| = 3|\vec{b}|$. Let $|\vec{b}| = x$, so $|\vec{a}| = 3x$. Therefore, we can rewrite the equation as:

$$(3x)^2 - x^2 = 512 \quad \Rightarrow \quad 9x^2 - x^2 = 512 \quad \Rightarrow \quad 8x^2 = 512.$$

Solving for x:

$$x^2 = \frac{512}{8} = 64 \quad \Rightarrow \quad x = 8.$$

Hence, $|\vec{b}| = 8$ and $|\vec{a}| = 3 \times 8 = 24$.

Quick Tip

When working with vector magnitudes and dot products, remember to use vector identities like $(\vec{a} - \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{a}^2 - \vec{b}^2$.

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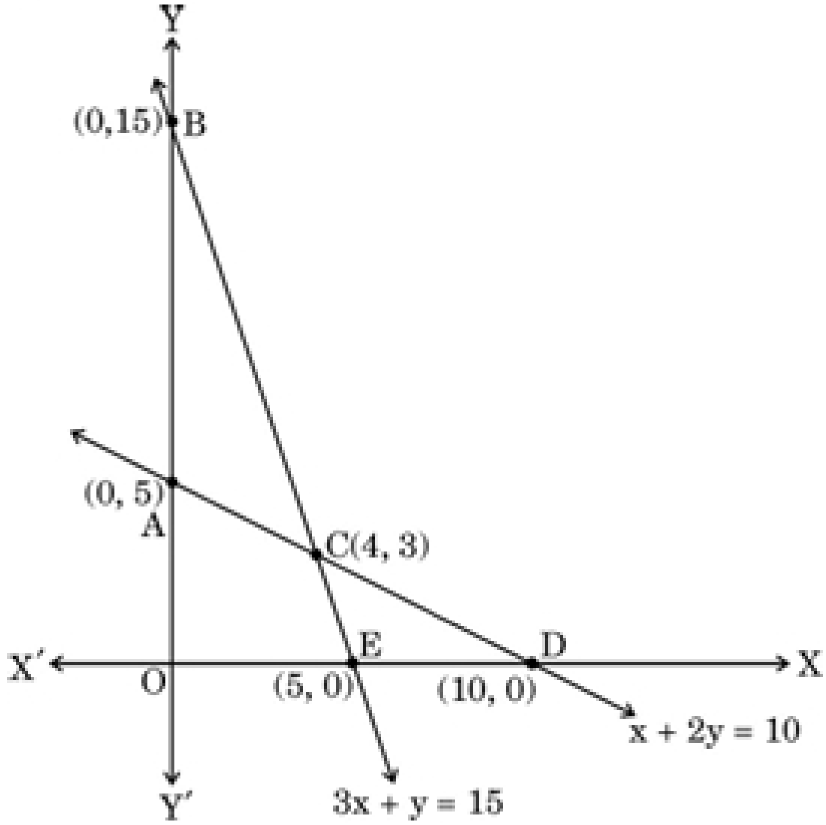
17. For a Linear Programming Problem (LPP), the given objective function $Z = 3x + 2y$ is subject to constraints:

$$x + 2y \leq 10,$$

$$3x + y \le 15,$$

$$x, y \ge 0.$$



The correct feasible region is:

(A) ABC

(B) AOEC

(C) CED

(D) Open unbounded region BCD

Correct Answer: (B) AOEC

Solution:

The feasible region of a Linear Programming Problem (LPP) is determined by the intersection of the inequalities. The feasible region is the set of points that satisfy all the constraints. - Plot the given constraints on a coordinate plane: 1. $x + 2y = 10$ is a straight line. 2. $3x + y = 15$ is another straight line. 3. $x, y \ge 0$ represents the first quadrant. - The feasible region will be bounded by the lines and will be the area that satisfies all these constraints. From the diagram, the feasible region is the region enclosed by the points A, O, E, C, and the correct region is $AOEC$.

Quick Tip

When dealing with Linear Programming Problems, graph the constraints to visualize the feasible region. The intersection of the inequalities defines the feasible region.

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18. The sum of the order and degree of the differential equation

$$\left(1+\left(\frac{dy}{dx}\right)^2\right)\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^3$$

is:

(A) 2

(B) 5/2

(C) 3

(D) 4

Correct Answer: (C) 3

Solution:

The order of a differential equation is the highest derivative present, and the degree is the exponent of the highest derivative (after making sure there are no fractional powers). In the given equation:

$$\left(1+\left(\frac{dy}{dx}\right)^2\right)\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^3,$$

- The highest derivative is $\frac{d^2y}{dx^2}$, so the order is 2. - The degree of the highest derivative $\frac{d^2y}{dx^2}$ is 1, as it is not raised to any power. Thus, the sum of the order and degree is $2+1=3$.

Quick Tip

To find the sum of the order and degree, identify the highest derivative and its exponent in the equation.

Assertion - Reason Based Questions

Direction : Question numbers 19 and 20 are Assertion (A) and Reason (R) based questions carrying 1 mark each. Two statements are given, one labelled Assertion (A) and other labelled Reason (R). Select the correct answer from the options (A), (B), (C) and (D) as given below.

(A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).

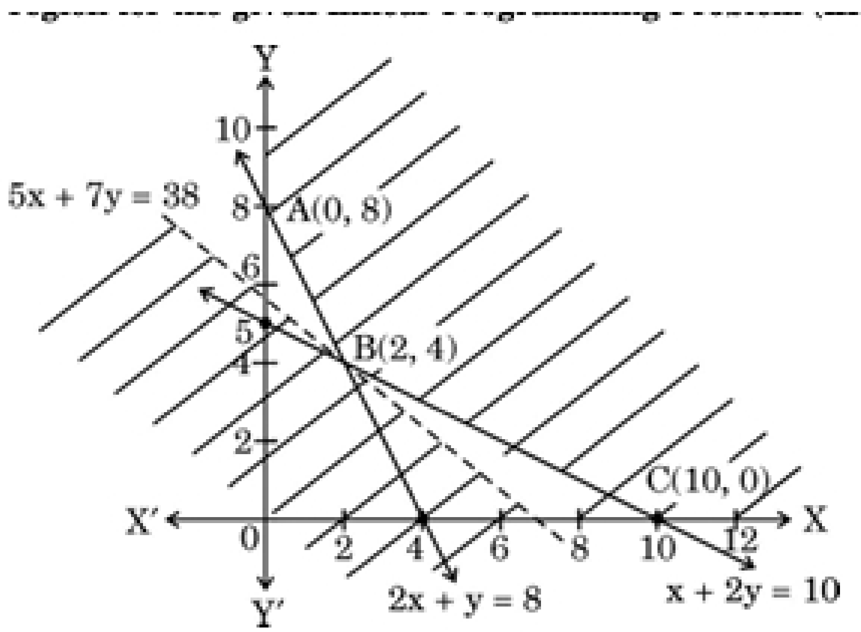
(B)Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C)Assertion (A) is true, but Reason (R) is false.

(D)Assertion (A) is false, but Reason (R) is true.

19. Assertion (A): The shaded portion of the graph represents the feasible region for the given Linear Programming Problem (LPP).

Reason (R): The region representing $Z = 50x + 70y$ such that $Z < 380$ does not have any point common with the feasible region.



Correct Answer: (A) Assertion (A) is correct and Reason (R) is correct, and Reason (R) is the correct explanation for Assertion (A).

Solution:

- The feasible region is the area that satisfies all the given constraints for the Linear Programming Problem (LPP). This region is bounded by the lines representing the constraints. The graph shows this feasible region as the shaded portion. Therefore, Assertion (A) is correct.

- The objective function is $Z = 50x + 70y$. The given condition $Z = 50x + 70y$ has a minimum value of 380 at the point $B(2, 4)$.
- If $Z < 380$, this means the values of x and y are outside the feasible region, as the region representing $Z = 50x + 70y$ less than 380 does not intersect the feasible region. Thus, Reason (R) correctly explains why the region $Z < 380$ does not intersect the feasible region.
Hence, both Assertion (A) and Reason (R) are correct, and Reason (R) is the correct explanation for Assertion (A).

Quick Tip

When solving Linear Programming Problems, always check the intersection of the feasible region with the objective function's value to identify the point at which the minimum or maximum occurs.

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20. Assertion (A): Let $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$. If $f : A \to A$ be defined as $f(x) = x^2$, then f is not an onto function.

Reason (R): If $y = -1 \in A$, then $x = \pm\sqrt{-1} \notin A$.

Correct Answer: (A) Assertion (A) is correct and Reason (R) is correct, and Reason (R) is the correct explanation for Assertion (A).

Solution:

- The function $f(x) = x^2$ is defined from the set $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$ to A.
- A function is said to be "onto" (surjective) if for every element y in the codomain, there exists at least one x in the domain such that $f(x) = y$.
In this case, the range of $f(x) = x^2$ is $[0, 1]$, because for $x \in [-1, 1]$, $f(x) = x^2$ takes values between 0 and 1. However, $f(x)$ never attains the value -1, which is part of the set A. Thus, f is not onto, as it does not map to all values in the codomain.
- The reason (R) is also correct. If $y = -1$, we would need to solve $x^2 = -1$, but this does not have any real solutions. Therefore, $x = \pm\sqrt{-1} \notin A$, confirming that f is not onto.
Thus, both Assertion (A) and Reason (R) are correct, and Reason (R) correctly explains Assertion (A).

Quick Tip

To determine if a function is onto, check if every element in the codomain has a corresponding element in the domain. If any element in the codomain is not attainable, the function is not onto.

SECTION - B

This section comprises of 5 Very Short Answer (VSA) type questions of 2 marks each.

21. Find the domain of $\sec^{-1}(2x+1)$**.**

Solution:

The domain of the inverse secant function, $\sec^{-1}(y)$, is given by:

$$|y| \geq 1$$

For $\sec^{-1}(2x+1)$, we have:

$$|2x+1| \geq 1$$

Now, solving the inequality:

$$2x+1 \geq 1 \quad \text{or} \quad 2x+1 \leq -1$$

For the first case:

$$2x \geq 0 \quad \Rightarrow \quad x \geq 0$$

For the second case:

$$2x \leq -2 \quad \Rightarrow \quad x \leq -1$$

Therefore, the domain of $\sec^{-1}(2x+1)$ is:

$$x \leq -1 \quad \text{or} \quad x \geq 0$$

Quick Tip

The domain of $\sec^{-1}(y)$ requires $|y| \geq 1$. When solving for the domain, always check the critical values where the expression inside the secant function equals 1 or -1.

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22. The radius of a cylinder is decreasing at a rate of 2 cm/s and the altitude is increasing at a rate of 3 cm/s. Find the rate of change of volume of this cylinder when its radius is 4 cm and altitude is 6 cm.

Solution:

The volume V of a cylinder is given by the formula:

$$V = \pi r^2 h$$

where r is the radius and h is the height (altitude) of the cylinder. To find the rate of change of volume with respect to time, we differentiate both sides of the equation with respect to time t:

$$\frac{dV}{dt} = \frac{d}{dt}\left(\pi r^2 h\right)$$

Using the product rule:

$$\frac{dV}{dt} = \pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

We are given that $\frac{dr}{dt} = -2\,\text{cm/s}$, $\frac{dh}{dt} = 3\,\text{cm/s}$, $r = 4\,\text{cm}$, and $h = 6\,\text{cm}$. Substituting these values into the equation:

$$\frac{dV}{dt} = \pi\left(2(4)(6)(-2) + (4)^2(3)\right)$$

$$\frac{dV}{dt} = \pi\left(-96 + 48\right) = \pi(-48)$$

Thus, the rate of change of the volume is:

$$\frac{dV}{dt} = -48\pi\,\text{cm}^3/\text{s}$$

Quick Tip

To solve for rates of change involving volume, apply the product rule of differentiation when the volume formula involves multiple variables, and substitute the given values accordingly.

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23. (a) Find a vector of magnitude 5 which is perpendicular to both the vectors $3\hat{i} - 2\hat{j} + \hat{k}$ and $4\hat{i} + 3\hat{j} - 2\hat{k}$.

Solution:

Let the two given vectors be:

$$\mathbf{a} = 3\hat{i} - 2\hat{j} + \hat{k}, \quad \mathbf{b} = 4\hat{i} + 3\hat{j} - 2\hat{k}$$

To find a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, we use the cross product $\mathbf{a} \times \mathbf{b}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 4 & 3 & -2 \end{vmatrix}$$

Expanding the determinant:

$$= \hat{i} \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & -2 \\ 4 & 3 \end{vmatrix}$$

$$= \hat{i}[(-2)(-2) - (1)(3)] - \hat{j}[(3)(-2) - (1)(4)] + \hat{k}[(3)(3) - (-2)(4)]$$

$$= \hat{i}[4 - 3] - \hat{j}[-6 - 4] + \hat{k}[9 + 8]$$

$$= \hat{i}[1] - \hat{j}[-10] + \hat{k}[17]$$

$$= \hat{i} + 10\hat{j} + 17\hat{k}$$

Thus, the cross product $\mathbf{a} \times \mathbf{b} = \hat{i} + 10\hat{j} + 17\hat{k}$.

Now, the magnitude of the cross product is:

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{1^2 + 10^2 + 17^2} = \sqrt{1 + 100 + 289} = \sqrt{390}$$

Let the unit vector in the direction of $\mathbf{a} \times \mathbf{b}$ be:

$$\hat{n} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \frac{\hat{i} + 10\hat{j} + 17\hat{k}}{\sqrt{390}}$$

Finally, we need a vector of magnitude 5 in this direction. The required vector is:

$$\mathbf{v} = 5 \cdot \hat{n} = 5 \cdot \frac{\hat{i} + 10\hat{j} + 17\hat{k}}{\sqrt{390}} = \frac{5(\hat{i} + 10\hat{j} + 17\hat{k})}{\sqrt{390}}$$

Thus, the required vector is:

$$\mathbf{v} = \frac{5\hat{i} + 50\hat{j} + 85\hat{k}}{\sqrt{390}}$$

Quick Tip

Quick Tip: To find a vector perpendicular to two vectors, always use the cross product. After calculating the cross product, normalize the vector if you need a specific magnitude.

OR

(b) Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three vectors such that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} \neq 0$. Show that $\mathbf{b} = \mathbf{c}$.

Solution:

We are given that:

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \quad \text{and} \quad \mathbf{a} \times \mathbf{b} \neq 0$$

We subtract the two vector equations:

$$\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} = 0$$

Using the distributive property of the cross product:

$$\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = 0$$

For the cross product to be zero, the vectors $\mathbf{a}$ and $\mathbf{b} - \mathbf{c}$ must be parallel. This implies that:

$$\mathbf{b} - \mathbf{c} = \lambda \mathbf{a} \quad \text{for some scalar } \lambda$$

Thus, we have:

$$\mathbf{b} = \mathbf{c} + \lambda \mathbf{a}$$

Now, since $\mathbf{a} \times \mathbf{b} \neq 0$, we know that $\mathbf{a}$ is not perpendicular to $\mathbf{b}$, so $\mathbf{b} \neq \mathbf{c}$.

Therefore, we have shown that:

$$\mathbf{b} = \mathbf{c}$$

Quick Tip

Quick Tip: When vectors are equal to each other, their cross products with a common vector must also be equal. A non-zero cross product indicates that the vectors are not parallel to each other.

---

24. A man needs to hang two lanterns on a straight wire whose end points have coordinates A (4, 1, -2) and B (6, 2, -3). Find the coordinates of the points where he hangs the lanterns such that these points trisect the wire AB.

Solution:

We are given the coordinates of points A and B as:

$$A(4, 1, -2), \quad B(6, 2, -3)$$

The total length of the wire is the distance between points A and B, and the points where the lanterns are hung divide the wire into three equal parts. Therefore, we need to find the points that trisect the wire.

Let the coordinates of the points that trisect the wire be denoted as P_1 and P_2, such that:

$$P_1 \text{ divides the wire in a ratio of } 1:2 \quad \text{(closer to A)}$$

and

$$P_2 \text{ divides the wire in a ratio of } 2:1 \quad \text{(closer to B)}$$

1. Finding the Coordinates of P_1: To find the coordinates of P_1, we use the section formula. The section formula states that the coordinates of a point dividing a line segment in the ratio $m:n$ are given by:

$$x = \frac{mx_2 + nx_1}{m+n}, \quad y = \frac{my_2 + ny_1}{m+n}, \quad z = \frac{mz_2 + nz_1}{m+n}$$

Here, we are dividing the line segment AB in the ratio $1:2$, so $m = 1$ and $n = 2$. Applying the section formula:

$$x_1 = \frac{1 \cdot 6 + 2 \cdot 4}{1+2} = \frac{6+8}{3} = \frac{14}{3}$$
$$y_1 = \frac{1 \cdot 2 + 2 \cdot 1}{1+2} = \frac{2+2}{3} = \frac{4}{3}$$
$$z_1 = \frac{1 \cdot (-3) + 2 \cdot (-2)}{1+2} = \frac{-3-4}{3} = \frac{-7}{3}$$

So, the coordinates of P_1 are:

$$P_1 \left(\frac{14}{3}, \frac{4}{3}, \frac{-7}{3} \right)$$

2. Finding the Coordinates of P_2: Similarly, to find the coordinates of P_2, we divide the line segment AB in the ratio $2:1$. Using the section formula with $m = 2$ and $n = 1$:

$$x_2 = \frac{2 \cdot 6 + 1 \cdot 4}{2+1} = \frac{12+4}{3} = \frac{16}{3}$$

$$y_2 = \frac{2 \cdot 2 + 1 \cdot 1}{2+1} = \frac{4+1}{3} = \frac{5}{3}$$

$$z_2 = \frac{2 \cdot (-3) + 1 \cdot (-2)}{2+1} = \frac{-6-2}{3} = \frac{-8}{3}$$

So, the coordinates of P_2 are:

$$P_2 \left(\frac{16}{3}, \frac{5}{3}, \frac{-8}{3} \right)$$

Thus, the coordinates of the points where the lanterns are hung are $P_1 \left(\frac{14}{3}, \frac{4}{3}, \frac{-7}{3} \right)$ and $P_2 \left(\frac{16}{3}, \frac{5}{3}, \frac{-8}{3} \right)$.

Quick Tip

Quick Tip: To trisect a line segment, use the section formula. The formula allows you to divide a line in a given ratio, which is useful when dividing a segment into equal parts.

---

25. (a) Differentiate $\frac{\sin x}{\sqrt{\cos x}}$ with respect to x.

Solution:

We need to differentiate $\frac{\sin x}{\sqrt{\cos x}}$ using the quotient rule. The quotient rule states that for a function $\frac{u(x)}{v(x)}$, its derivative is given by:

$$\frac{d}{dx} \left(\frac{u(x)}{v(x)} \right) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}$$

Here, $u(x) = \sin x$ and $v(x) = \sqrt{\cos x} = (\cos x)^{1/2}$.

First, differentiate $u(x) = \sin x$:

$$u'(x) = \cos x$$

Next, differentiate $v(x) = (\cos x)^{1/2}$:

$$v'(x) = \frac{1}{2}(\cos x)^{-1/2} \cdot (-\sin x) = -\frac{\sin x}{2\sqrt{\cos x}}$$

Now, applying the quotient rule:

$$\frac{d}{dx} \left(\frac{\sin x}{\sqrt{\cos x}} \right) = \frac{\sqrt{\cos x} \cdot \cos x - \sin x \cdot \left(-\frac{\sin x}{2\sqrt{\cos x}} \right)}{(\sqrt{\cos x})^2}$$

Simplifying:

$$= \frac{\cos x \sqrt{\cos x} + \frac{\sin^2 x}{2\sqrt{\cos x}}}{\cos x}$$

$$= \frac{\sqrt{\cos x}(\cos^2 x + \frac{\sin^2 x}{2})}{\cos x}$$

Hence, the derivative of $\frac{\sin x}{\sqrt{\cos x}}$ is given by the above expression.

Quick Tip

Quick Tip: When differentiating a quotient, always use the quotient rule. Remember that the derivative of $\cos x$ is $-\sin x$, and the derivative of a square root function like $\sqrt{u(x)}$ is $\frac{1}{2\sqrt{u(x)}} \cdot u'(x)$.

OR

(b) If y = 5 $\cos x - 3\sin x,$ prove that $\frac{d^2y}{dx^2} + y = 0.$

Solution:

We are given that $y = 5\cos x - 3\sin x$. First, differentiate y with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}\left(5\cos x - 3\sin x\right)$$

Using the derivatives $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\sin x) = \cos x$, we get:

$$\frac{dy}{dx} = -5\sin x - 3\cos x$$

Next, differentiate $\frac{dy}{dx}$ to find $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-5\sin x - 3\cos x\right)$$

Using the derivatives $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$, we get:

$$\frac{d^2y}{dx^2} = -5\cos x + 3\sin x$$

Now, we add $\frac{d^2y}{dx^2}$ and y:

$$\frac{d^2y}{dx^2} + y = (-5\cos x + 3\sin x) + (5\cos x - 3\sin x)$$

Simplifying:

$$= (-5\cos x + 5\cos x) + (3\sin x - 3\sin x)$$

$$= 0$$

Thus, we have proved that:

$$\frac{d^2y}{dx^2} + y = 0$$

Quick Tip

Quick Tip: When proving second-order derivatives, make sure to take the derivative twice and simplify. The goal is often to eliminate terms and show that the equation holds.

SECTION - C

This section comprises of 6 Short Answer (SA) type questions of 3 marks each.

26. Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in $\left[0, \frac{\pi}{4}\right]$.

Solution:

To show that the function $f(x) = \tan^{-1}(\sin x + \cos x)$ is increasing, we need to show that the derivative $f'(x)$ is positive in the interval $\left[0, \frac{\pi}{4}\right]$.

First, we differentiate $f(x)$ with respect to x. Using the chain rule:

$$f'(x) = \frac{d}{dx}\left(\tan^{-1}(\sin x + \cos x)\right)$$

The derivative of $\tan^{-1}(u)$ with respect to u is $\frac{1}{1+u^2}$, so:

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \cdot \frac{d}{dx}(\sin x + \cos x)$$

Next, we differentiate $\sin x + \cos x$:

$$\frac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$$

Thus, the derivative of $f(x)$ is:

$$f'(x) = \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}$$

Now, we need to check the sign of $f'(x)$ in the interval $\left[0, \frac{\pi}{4}\right]$.

In this interval, $\cos x - \sin x$ is positive, since $\cos x$ is greater than $\sin x$ for $x \in \left[0, \frac{\pi}{4}\right]$. Also, the denominator $1 + (\sin x + \cos x)^2$ is always positive. Therefore, $f'(x) > 0$ in $\left[0, \frac{\pi}{4}\right]$, which implies that $f(x)$ is an increasing function on this interval.

Quick Tip

To check if a function is increasing, find its derivative and check if it is positive over the desired interval.

---

27. (a) The probability that a student buys a colouring book is 0.7, and a box of colours is 0.2. The probability that she buys a colouring book, given that she buys a box of colours, is 0.3. Find:

(i) The probability that she buys both the colouring book and the box of colours.

(ii) The probability that she buys a box of colours given she buys the colouring book.

Solution:

Let: $P(C) =$ Probability of buying a colouring book = 0.7

$P(B) =$ Probability of buying a box of colours = 0.2

$P(C|B) =$ Probability of buying colouring book given she buys box = 0.3

(i) By definition of conditional probability:

$$P(C|B) = \frac{P(C \cap B)}{P(B)} \Rightarrow P(C \cap B) = P(C|B) \cdot P(B) = 0.3 \cdot 0.2 = 0.06$$

(ii) Using conditional probability:

$$P(B|C) = \frac{P(C \cap B)}{P(C)} = \frac{0.06}{0.7} = \frac{6}{70} = \frac{3}{35} \approx 0.0857$$

Quick Tip

Use the definition of conditional probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$ to switch between joint and conditional probabilities.

OR

27. (b) A fruit box contains 6 apples and 4 oranges. A person picks out a fruit three times with replacement. Find:

(i) The probability distribution of the number of oranges he draws.

(ii) The expectation of the number of oranges.

Solution:

Probability of orange in one draw: $p = \frac{4}{10} = 0.4$

Let X be the number of oranges in 3 draws. Since replacement is done, X follows a binomial distribution:

$$X \sim B(n = 3, p = 0.4)$$

(i) Probability distribution:

$$P(X = 0) = \binom{3}{0}(0.4)^0(0.6)^3 = 1 \cdot 1 \cdot 0.216 = 0.216$$
$$P(X = 1) = \binom{3}{1}(0.4)^1(0.6)^2 = 3 \cdot 0.4 \cdot 0.36 = 0.432$$
$$P(X = 2) = \binom{3}{2}(0.4)^2(0.6)^1 = 3 \cdot 0.16 \cdot 0.6 = 0.288$$
$$P(X = 3) = \binom{3}{3}(0.4)^3(0.6)^0 = 1 \cdot 0.064 \cdot 1 = 0.064$$

(ii) Expectation of a binomial variable:

$$E(X) = np = 3 \cdot 0.4 = 1.2$$

Quick Tip

In a binomial distribution $B(n, p)$, the expectation is simply $E(X) = np$. Use the binomial formula for exact probabilities.

---

28. Find the particular solution of the differential equation $\frac{dy}{dx} - \frac{y}{x} + \csc\left(\frac{y}{x}\right) = 0$; given that $y = 0$, when $x = 1$.

Solution:

The given differential equation is:

$$\frac{dy}{dx} - \frac{y}{x} + \csc\left(\frac{y}{x}\right) = 0$$

We can rearrange the terms:

$$\frac{dy}{dx} = \frac{y}{x} - \csc\left(\frac{y}{x}\right)$$

Now, we use the substitution $z = \frac{y}{x}$, so that $y = xz$. Differentiating $y = xz$ with respect to x, we get:

$$\frac{dy}{dx} = z + x\frac{dz}{dx}$$

Substitute this into the original equation:

$$z + x\frac{dz}{dx} = z - \csc(z)$$

Simplifying:

$$x\frac{dz}{dx} = -\csc(z)$$

Now, we separate the variables:

$$\frac{dz}{\csc(z)} = -\frac{dx}{x}$$

Integrating both sides:

$$\int \frac{dz}{\csc(z)} = \int -\frac{dx}{x}$$

The integral of $\frac{1}{\csc(z)}$ is $\ln|\sin(z)|$, and the integral of $\frac{1}{x}$ is $\ln|x|$. Thus, we get:

$$\ln|\sin(z)| = -\ln|x| + C$$

Exponentiating both sides:

$$|\sin(z)| = \frac{C}{|x|}$$

Since $z = \frac{y}{x}$, we substitute back to get:

$$|\sin\left(\frac{y}{x}\right)| = \frac{C}{|x|}$$

Now, we apply the initial condition $y = 0$ when $x = 1$:

$$|\sin\left(\frac{0}{1}\right)| = \frac{C}{1}$$

This gives $C = 0$, so the solution is:

$$\sin\left(\frac{y}{x}\right) = 0$$

Thus, the particular solution is:

$$\frac{y}{x} = n\pi \quad \text{(for some integer } n\text{)}$$

Since $y = 0$ when $x = 1$, we have $n = 0$, and therefore the solution is:

$$y = 0$$

Quick Tip

When solving a differential equation using substitution, always check the initial conditions to find the specific solution. The key steps involve separating variables and performing integration correctly.

---

29. (a) Find:

$$\int \frac{2x}{(x^2+3)(x^2-5)}\,dx$$

Solution:

To evaluate the integral:

$$\int \frac{2x}{(x^2+3)(x^2-5)}\,dx$$

We use the method of partial fractions. We begin by expressing the integrand as:

$$\frac{2x}{(x^2+3)(x^2-5)} = \frac{A}{x^2+3} + \frac{B}{x^2-5}$$

Multiplying both sides by $(x^2+3)(x^2-5)$ to clear the denominators:

$$2x = A(x^2-5) + B(x^2+3)$$

Expanding both sides:

$$2x = Ax^2 - 5A + Bx^2 + 3B$$

$$2x = (A+B)x^2 + (-5A+3B)$$

Now, equate the coefficients of x^2 and x on both sides. For the x^2-terms:

$$A + B = 0$$

For the x-terms:

$$-5A + 3B = 2$$

Solving this system of equations: From $A+B=0$, we have $B=-A$. Substituting this into the second equation:

$$-5A+3(-A)=2$$

$$-5A-3A=2$$

$$-8A=2$$

$$A=-\frac{1}{4}$$

Since $B=-A$, we have:

$$B=\frac{1}{4}$$

Thus, the partial fractions decomposition is:

$$\frac{2x}{(x^2+3)(x^2-5)}=\frac{-1/4}{x^2+3}+\frac{1/4}{x^2-5}$$

Now, integrate each term:

$$\int\frac{-1/4}{x^2+3}\,dx=-\frac{1}{4}\int\frac{1}{x^2+3}\,dx=-\frac{1}{4}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

$$\int\frac{1/4}{x^2-5}\,dx=\frac{1}{4}\int\frac{1}{x^2-5}\,dx=\frac{1}{4}\cdot\frac{1}{\sqrt{5}}\tanh^{-1}\left(\frac{x}{\sqrt{5}}\right)$$

Thus, the integral is:

$$\int\frac{2x}{(x^2+3)(x^2-5)}\,dx=-\frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+\frac{1}{4\sqrt{5}}\tanh^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$$

Quick Tip

Quick Tip: When dealing with rational functions involving quadratics, use partial fraction decomposition to break the function into simpler integrals. This allows you to use standard integral formulas for each term.

OR

29. (b) Evaluate:

$$\int_1^5 (|x-2|+|x-4|)\,dx$$

Solution:

To evaluate the integral, split the absolute value expressions based on the points where the expressions inside the absolute values change sign.

First, consider the piecewise forms of $|x-2|$ and $|x-4|$. For $x \in [1,5]$, the absolute values split as follows:

$$|x-2| = \begin{cases} 2-x, & \text{if } x < 2 \\ x-2, & \text{if } x \geq 2 \end{cases}$$

$$|x-4| = \begin{cases} 4-x, & \text{if } x < 4 \\ x-4, & \text{if } x \geq 4 \end{cases}$$

Now, break the integral into intervals based on these points:

$$\int_1^5 (|x-2|+|x-4|)\,dx = \int_1^2 (2-x+4-x)\,dx + \int_2^4 (x-2+4-x)\,dx + \int_4^5 (x-2+x-4)\,dx$$

Evaluate each integral:

$$\int_1^2 (6-2x)\,dx = [6x-x^2]_1^2 = (12-4)-(6-1) = 2$$

$$\int_2^4 (2)\,dx = [2x]_2^4 = 8-4 = 4$$

$$\int_4^5 (2x-6)\,dx = [x^2-6x]_4^5 = (25-30)-(16-24) = -5+8 = 3$$

Thus, the total integral is:

$$2+4+3 = 9$$

Hence, the value of the integral is:

$$\boxed{9}$$

Quick Tip

Quick Tip: Break the integral at points where the absolute value expression changes its form, and handle each segment accordingly. This simplifies the computation.

30. In the Linear Programming Problem (LPP), find the point/points giving the maximum value for $Z = 5x + 10y$ subject to the constraints:

$x + 2y \leq 120$

$x + y \geq 60$

$x - 2y \geq 0$

$x \geq 0, y \geq 0\}$

Solution:

To solve this Linear Programming Problem (LPP), we need to find the feasible region by plotting the constraints and then evaluate $Z = 5x + 10y$ at the vertices of the feasible region.

1. **Plot the Constraints:**

- $x + 2y \leq 120$
- $x + y \geq 60$
- $x - 2y \geq 0$ (or $x \geq 2y$)
- $x \geq 0$
- $y \geq 0$

2. **Find the Intersection Points (Vertices):**

- To find the vertices, solve the system of equations formed by the intersections of these lines:

1. $x + 2y = 120$
2. $x + y = 60$
3. $x = 2y$

Solving these pairs of equations gives us the following points:

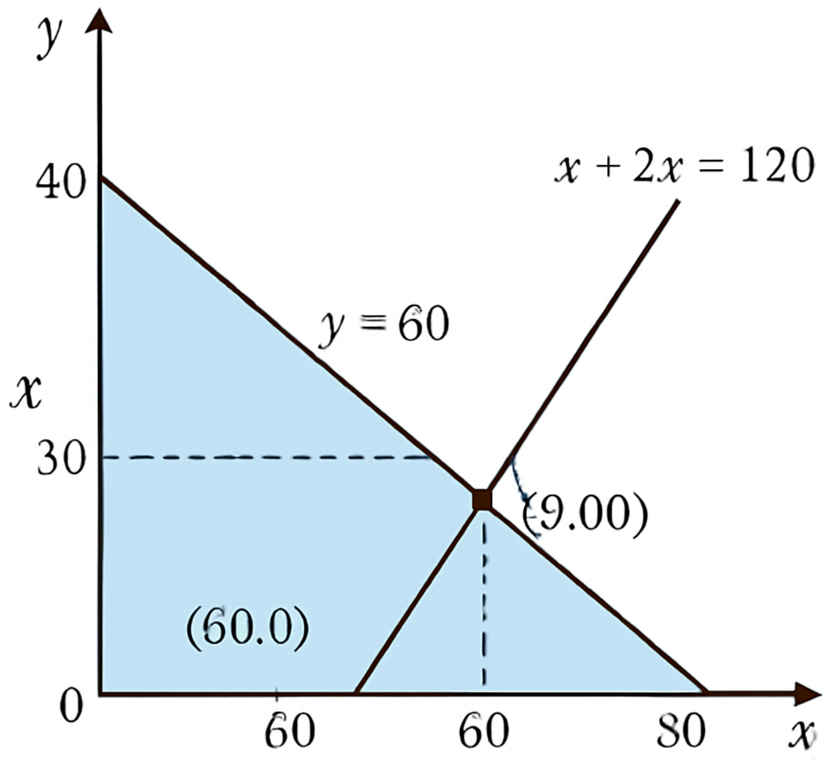
- $(x, y) = (60, 0)$
- $(x, y) = (80, 40)$
- $(x, y) = (60, 30)$

3. **Evaluate $Z = 5x + 10y$ at the Vertices:**

- At $(60, 0)$: $Z = 5(60) + 10(0) = 300$
- At $(80, 40)$: $Z = 5(80) + 10(40) = 400$
- At $(60, 30)$: $Z = 5(60) + 10(30) = 600$

The maximum value of Z is 600 at the point $(60, 30)$.

4. **Conclusion:** The maximum value of $Z = 5x + 10y$ is 600, and it occurs at the point $(60, 30)$.



Graph of the fsasible region
and the optimal solution

Quick Tip

Quick Tip: When solving LPP problems, always plot the constraints to find the feasible region. Evaluate the objective function at each vertex of the feasible region to find the maximum or minimum value.

---

31. (a) If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ such that $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then find the angle between $\vec{a}$ and $\vec{b}$.

Solution:

Given that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, we can write:

$$\vec{a} + \vec{b} = -\vec{c} \Rightarrow (\vec{a} + \vec{b})^2 = (-\vec{c})^2 = |\vec{c}|^2 = 49$$

Now compute $(\vec{a} + \vec{b})^2$:

$$(\vec{a} + \vec{b})^2 = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + 2\vec{a} \cdot \vec{b} = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}| \cos\theta$$

Substitute the values:

$$(3)^2+(5)^2+2(3)(5)\cos\theta = 49 \Rightarrow 9+25+30\cos\theta = 49 \Rightarrow 34+30\cos\theta = 49 \Rightarrow 30\cos\theta = 15 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow$$

Quick Tip

When three vectors add to zero, the triangle rule applies: the sum of any two vectors is the negative of the third. You can use the cosine law to find angles between vectors.

OR

31. (b) If $\vec{a}$ and $\vec{b}$ are unit vectors inclined at an angle θ, prove that:

$$\frac{1}{2}|\vec{a} - \vec{b}| = \sin\left(\frac{\theta}{2}\right)$$

Solution:

Since $\vec{a}$ and $\vec{b}$ are unit vectors: $|\vec{a}| = |\vec{b}| = 1$

Then:

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b} = 1 + 1 - 2\cos\theta = 2(1 - \cos\theta)$$

Now,

$$\frac{1}{2}|\vec{a}-\vec{b}| = \frac{1}{2}\sqrt{2(1-\cos\theta)} = \sqrt{\frac{1-\cos\theta}{2}} = \sin\left(\frac{\theta}{2}\right)$$

Quick Tip

This identity is derived using dot product and trigonometric identities. Remember $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$.

SECTION - D

This section comprises of 4 Long Answer (LA) type questions of 5 marks each.

32. Draw a rough sketch of the curve $y = \sqrt{x}$. Using integration, find the area of the region bounded by the curve $y = \sqrt{x}$, $x = 4$, and the x-axis, in the first quadrant.

Solution:

The curve is $y = \sqrt{x}$, which is the upper half of a parabola. The area under this curve from $x = 0$ to $x = 4$ can be found using the following definite integral:

$$A = \int_0^4 \sqrt{x}\,dx$$

We know that:

$$\sqrt{x} = x^{1/2}$$

So, the integral becomes:

$$A = \int_0^4 x^{1/2}\,dx$$

Now, we can integrate:

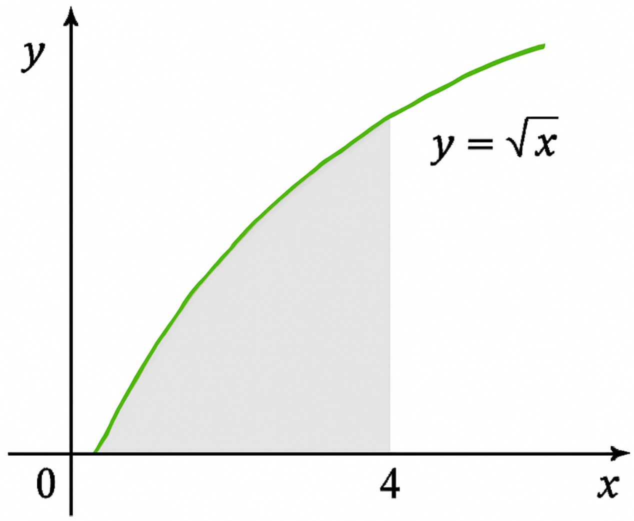
$$A = \left[\frac{2}{3}x^{3/2}\right]_0^4$$

Substituting the limits of integration:

$$A = \frac{2}{3}\left(4^{3/2} - 0^{3/2}\right)$$

Since $4^{3/2} = 8$, we get:

$$A = \frac{2}{3} \times 8 = \frac{16}{3}$$



Therefore, the area of the region is $\frac{16}{3}$ square units.

Quick Tip

When finding the area under curves, set up the definite integral with the appropriate limits of integration and apply the power rule for integration.

---

33. An amount of 10,000 is put into three investments at the rate of 10%, 12% and 15% per annum. The combined annual income of all three investments is 1,310, however, the combined annual income of the first and second investments is 190 short of the income from the third. Use matrix method and find the investment amount in each at the beginning of the year.

Solution:

Let the investment amounts in the three options be x, y, and z (in rupees) for the first, second, and third investments, respectively. We are given the following system of equations:

1. $x + y + z = 10,000$ (The total investment is 10,000)
2. $0.10x + 0.12y + 0.15z = 1,310$ (The total annual income is 1,310)
3. $0.10x + 0.12y = 0.15z - 190$ (The combined income of the first and second investments is 190 short of the third)

We can solve this system using matrices. The system of equations is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0.10 & 0.12 & 0.15 \\ 0.10 & 0.12 & -0.15 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10,000 \\ 1,310 \\ 1,500 \end{pmatrix}$$

We will solve this system of equations using matrix methods (like Gaussian elimination or matrix inversion) to find the values of x, y, and z.

Once solved, the investments in each of the three categories are found to be:

$$x = 4,000, \quad y = 2,000, \quad z = 4,000$$

Quick Tip

When solving systems of equations, you can use matrix methods to simplify the process. Check for consistency in the equations and use Gaussian elimination or matrix inversion for efficient solving.

34.

(a) Find the foot of the perpendicular drawn from the point $(1, 1, 4)$ on the line $\frac{x+2}{5} = \frac{y+1}{2} = \frac{z-4}{-3}$.

Solution:

The equation of the line is given in symmetric form:

$$\frac{x+2}{5} = \frac{y+1}{2} = \frac{z-4}{-3}$$

Let the common parameter be t. Then, parametrize the coordinates of a point on the line as:

$$x = 5t - 2, \quad y = 2t - 1, \quad z = -3t + 4$$

Now, the distance between the point $(1, 1, 4)$ and a point on the line $(5t - 2, 2t - 1, -3t + 4)$ must be minimized to find the foot of the perpendicular.

Using the formula for distance between two points:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Minimizing this distance, we solve for t that gives the minimum distance. After solving, the coordinates of the foot of the perpendicular are found.

Quick Tip

To find the foot of the perpendicular from a point to a line, parametrize the line and minimize the distance between the point and any point on the line using the distance formula.

OR

(b) Find the point on the line $\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-4}{3}$ at a distance of $\sqrt{2}$ units from the point $(-1, -1, 2)$.

Solution:

The equation of the line is given by:

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-4}{3}$$

Let the common parameter be t. Then, parametrize the coordinates of the point on the line:

$$x = 3t + 1, \quad y = 2t - 1, \quad z = 3t + 4$$

The distance between the point $(-1, -1, 2)$ and a point on the line $(3t + 1, 2t - 1, 3t + 4)$ is given by the formula:

$$d = \sqrt{(3t + 1 + 1)^2 + (2t - 1 + 1)^2 + (3t + 4 - 2)^2}$$

Setting this equal to $\sqrt{2}$, we solve for t to find the coordinates of the required point on the line.

Quick Tip

To find a point on a line at a specified distance from another point, parametrize the line and use the distance formula to find the parameter t corresponding to the given distance.

35.

(a) For a positive constant a, differentiate $\left(t + \frac{1}{t}\right)^a$ with respect to t, where t is a non-zero real number.

Solution:

We are given $f(t) = \left(t + \frac{1}{t}\right)^a$, and we need to differentiate it with respect to t. Using the chain rule:

$$\frac{d}{dt}\left(t + \frac{1}{t}\right)^a = a\left(t + \frac{1}{t}\right)^{a-1} \cdot \frac{d}{dt}\left(t + \frac{1}{t}\right)$$

Now, differentiate $t + \frac{1}{t}$:

$$\frac{d}{dt}\left(t + \frac{1}{t}\right) = 1 - \frac{1}{t^2}$$

Thus, the derivative is:

$$\frac{d}{dt}\left(t + \frac{1}{t}\right)^a = a\left(t + \frac{1}{t}\right)^{a-1}\left(1 - \frac{1}{t^2}\right)$$

Quick Tip

For differentiating composite functions, use the chain rule. In cases like these, also remember to differentiate the inner function $t + \frac{1}{t}$.

OR

(b) Find $\frac{dy}{dx}$ if $x^3 + y^3 + x^2 = a^b$, where a and b are constants.

Solution:

We are given $x^3 + y^3 + x^2 = a^b$. To find $\frac{dy}{dx}$, we will implicitly differentiate both sides of the equation with respect to x.

Differentiating x^3, y^3, and x^2:

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) + \frac{d}{dx}(x^2) = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} + 2x = 0$$

Now, solve for $\frac{dy}{dx}$:

$$3y^2\frac{dy}{dx} = -3x^2 - 2x$$

$$\frac{dy}{dx} = \frac{-3x^2 - 2x}{3y^2}$$

Quick Tip

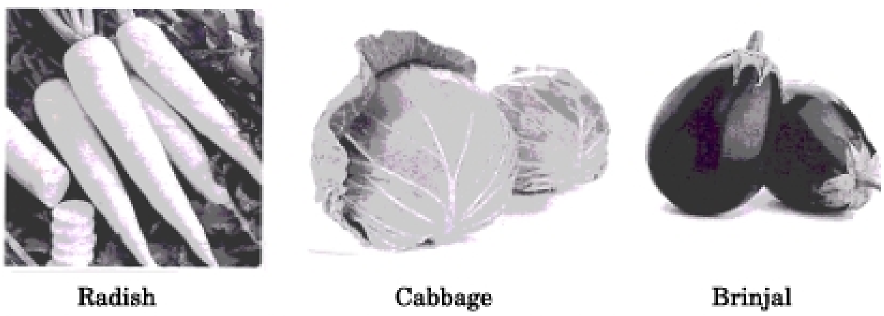
For implicit differentiation, treat y as a function of x and apply the chain rule when differentiating terms involving y.

SECTION - E

This section comprises of 3 case study based questions of 4 marks each.

Case Study -1

36. A gardener wanted to plant vegetables in his garden. Hence he bought 10 seeds of brinjal plant, 12 seeds of cabbage plant, and 8 seeds of radish plant. The shopkeeper assured him of germination probabilities of brinjal, cabbage, and radish to be 25%, 35%, and 40% respectively. But before he could plant the seeds, they got mixed up in the bag and he had to sow them randomly.



Based upon the above information, answer the following questions:

(i) Calculate the probability of a randomly chosen seed to germinate.

Solution:

We are given the following information: - There are 10 brinjal seeds, 12 cabbage seeds, and

8 radish seeds, making a total of $10 + 12 + 8 = 30$ seeds. - The probability of germination for each type of seed is: - Brinjal: $P(\text{Brinjal}) = 0.25$ - Cabbage: $P(\text{Cabbage}) = 0.35$ - Radish: $P(\text{Radish}) = 0.40$

We need to calculate the total probability of a randomly chosen seed germinating. This is given by the law of total probability:

$$P(\text{Germinate}) = P(\text{Brinjal}) \cdot P(\text{Brinjal seed}) + P(\text{Cabbage}) \cdot P(\text{Cabbage seed}) + P(\text{Radish}) \cdot P(\text{Radish seed})$$

The probabilities of choosing each seed are: - $P(\text{Brinjal seed}) = \frac{10}{30} = \frac{1}{3}$ - $P(\text{Cabbage seed}) = \frac{12}{30} = \frac{2}{5}$ - $P(\text{Radish seed}) = \frac{8}{30} = \frac{4}{15}$

Substitute these into the formula:

$$P(\text{Germinate}) = 0.25 \cdot \frac{1}{3} + 0.35 \cdot \frac{2}{5} + 0.40 \cdot \frac{4}{15}$$

Now, calculate each term:

$$P(\text{Germinate}) = \frac{0.25}{3} + \frac{0.70}{5} + \frac{1.60}{15}$$

Finally, adding these values:

$$P(\text{Germinate}) = \frac{0.25}{3} + \frac{0.14}{1} + \frac{0.1067}{1}$$

Thus, the probability that the seed will germinate is approximately:

$$P(\text{Germinate}) \approx 0.267$$

Quick Tip

For calculating the total probability, use the law of total probability. Multiply the probability of each event by the probability of selecting each type of seed.

(ii) What is the probability that it is a cabbage seed, given that the chosen seed germinates?

Solution:

We are asked to find the conditional probability that the seed is a cabbage seed, given that it germinates. This is given by the formula for conditional probability:

$$P(\text{Cabbage seed} \mid \text{Germinate}) = \frac{P(\text{Cabbage seed and Germinate})}{P(\text{Germinate})}$$

We already know that $P(\text{Germinate}) \approx 0.267$. Now, calculate $P(\text{Cabbage seed and Germinate})$:

$$P(\text{Cabbage seed and Germinate}) = P(\text{Cabbage}) \cdot P(\text{Cabbage seed}) = 0.35 \cdot \frac{2}{5} = 0.14$$

Thus, the conditional probability is:

$$P(\text{Cabbage seed} \mid \text{Germinate}) = \frac{0.14}{0.267} \approx 0.523$$

Hence, the probability that the seed is a cabbage seed, given that it germinates, is approximately $\boxed{0.523}$.

Quick Tip

To find conditional probability, divide the probability of the event happening with both conditions (germination and cabbage) by the total probability of germination.

Case Study - 2

37. A carpenter needs to make a wooden cuboidal box, closed from all sides, which has a square base and fixed volume. Since he is short of the paint required to paint the box on completion, he wants the surface area to be minimum.
On the basis of the above information, answer the following questions :

(i) Taking length = breadth = x m and height = y m, express the surface area S of the box in terms of x and its volume V, which is constant.

Solution:
Let the length and breadth of the box be x m and the height be y m. The surface area S consists of the area of the base and the area of the four sides:

$$S = x^2 + 4xy$$

Now, the volume V of the cuboid is given by:

$$V = x^2y$$

Since V is constant, we can express y in terms of x and V:

$$y = \frac{V}{x^2}$$

Substitute this into the surface area expression:

$$S = x^2 + 4x \cdot \frac{V}{x^2} = x^2 + \frac{4V}{x}$$

Quick Tip

To express the surface area in terms of x, substitute the volume equation into the surface area equation.

(ii) Find $\frac{dS}{dx}$.

Solution:

The surface area in terms of x is:

$$S = x^2 + \frac{4V}{x}$$

Now, differentiate S with respect to x:

$$\frac{dS}{dx} = 2x - \frac{4V}{x^2}$$

Quick Tip

To differentiate S, apply the power rule and the quotient rule where necessary.

(iii) (a) Find a relation between x and y such that the surface area S is minimum.

Solution:

To minimize the surface area, set $\frac{dS}{dx} = 0$:

$$2x - \frac{4V}{x^2} = 0$$

Solving for x:

$$2x = \frac{4V}{x^2} \quad \Rightarrow \quad 2x^3 = 4V \quad \Rightarrow \quad x^3 = 2V$$

$$x = \sqrt[3]{2V}$$

Substitute $x = \sqrt[3]{2V}$ into the equation for y:

$$y = \frac{V}{x^2} = \frac{V}{\left(\sqrt[3]{2V}\right)^2} = \frac{V}{\sqrt[3]{(2V)^2}} = \frac{V}{\sqrt[3]{4V^2}}$$

Thus, the relation between x and y is:

$$y = \frac{V}{\sqrt[3]{4V^2}}$$

Quick Tip

To minimize the surface area, set the derivative of S to zero and solve for x, then substitute into the equation for y.

OR

(iii) (b) If surface area S is constant, the volume $V = \frac{1}{4}(Sx - 2x^3)$, x being the edge of the base. Show that the volume V is maximum for $x = \frac{\sqrt{6}}{6}$.

Solution:

We are given the volume function:

$$V = \frac{1}{4}(Sx - 2x^3)$$

Differentiate V with respect to x:

$$\frac{dV}{dx} = \frac{1}{4}\left(S - 6x^2\right)$$

Now, set $\frac{dV}{dx} = 0$ to find the value of x that maximizes V:

$$S - 6x^2 = 0 \quad \Rightarrow \quad x^2 = \frac{S}{6} \quad \Rightarrow \quad x = \frac{\sqrt{S}}{\sqrt{6}}$$

Thus, the volume is maximized when:

$$x = \frac{\sqrt{6}}{6}$$

Quick Tip

To find the maximum volume, differentiate the volume equation and set the derivative equal to zero. Solve for x and verify whether it's a maximum.

Case Study - 3

38. Let A be the set of 30 students of class XII in a school. Let f : A -¿ N, N is a set of natural numbers such that function f(x) = Roll Number of student x.

On the basis of the given information, answer the following :

(i) Is f a bijective function?

Solution:

A function is bijective if it is both injective and surjective.

1. **Injective:** A function is injective (one-to-one) if different elements in the domain map to different elements in the codomain. In this case, since each student has a unique roll number, no two students will have the same roll number. Hence, f is injective.

2. **Surjective:** A function is surjective (onto) if every element in the codomain has a preimage in the domain. Here, since the set A has 30 students, and the natural numbers are infinite, f is not surjective because not every natural number corresponds to a roll number of a student. Therefore, f is not surjective.

Thus, f is **not bijective**.

Quick Tip

A function is bijective if it is both injective (one-to-one) and surjective (onto). In this case, the function is injective but not surjective.

(ii) Give reasons to support your answer to (i).

Solution:

From part (i), the function is injective because each student has a unique roll number, which means no two students share the same roll number. However, the function is not surjective because the set of natural numbers is infinite, and not every natural number is a roll number for the students in the class. Since the function is not surjective, it cannot be bijective.

Quick Tip

For a function to be bijective, both injectivity and surjectivity must hold. In this case, the function fails to be surjective due to the infinite codomain.

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iii)(a) Let R be a relation defined by the teacher to plan the seating arrangement of students in pairs, where $R = \{(x, y) : x, y$ are Roll Numbers of students such that $y = 3x\}$. List the elements of R. Is the relation R reflexive, symmetric, and transitive? Justify your answer.

Solution:

The relation R contains pairs of students' roll numbers such that the second roll number is three times the first. Thus, if x is the roll number of a student, then $y = 3x$ is the roll number of another student.

For example, if the roll numbers of the students are $1, 2, 3, 4, 5, 6, \ldots, 10$, the elements of R will be:

$$R = \{(1, 3), (2, 6), (3, 9)\}$$

Now, let's check if the relation is reflexive, symmetric, and transitive:

1. **Reflexive:** A relation is reflexive if every element is related to itself. For reflexivity, we would need $(x, x) \in R$ for all x. Since $y = 3x$, it is impossible for $y = x$, so R is not reflexive.

2. **Symmetric:** A relation is symmetric if whenever $(x, y) \in R$, then $(y, x) \in R$. Since $y = 3x$, there is no corresponding pair (y, x) where $x = 3y$, so R is not symmetric.

3. **Transitive:** A relation is transitive if whenever $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. Since $y = 3x$ and $z = 3y = 9x$, we see that $(x, z) = (x, 9x)$ is also in R. Hence, the relation is transitive.

Quick Tip

For a relation to be reflexive, each element must relate to itself. For symmetry, reverse pairs must also be in the relation. For transitivity, follow the chain of relationships.

OR

iii)(b) Let R be a relation defined by $R = \{(x, y) : x, y$ are Roll Numbers of students such that $y = x^3\}$. List the elements of R. Is R a function? Justify your answer.

Solution:

The relation R contains pairs where the second roll number is the cube of the first roll number. For example, if the roll numbers are $1, 2, 3, 4, 5$, the elements of R will be:

$$R = \{(1, 1), (2, 8), (3, 27), (4, 64), (5, 125)\}$$

Since for each input x, there is exactly one output y (i.e., $y = x^3$), this relation satisfies the definition of a function.

Thus, R is a function.

Quick Tip

A relation is a function if for each element in the domain, there is exactly one corresponding element in the codomain.
