CBSE Class 12 Mathematics Set 2.1(65/2/1) Question Paper with Solutions

Time Allowed :3 Hour	**Maximum Marks :**80	**Total Questions :**38

General Instructions

Read the following instructions very carefully and strictly follow them:

1. This question paper contains 38 questions. All questions are compulsory.
2. This question paper is divided into five Sections – A, B, C, D and E.
3. In Section A, Questions no. 1 to 18 are multiple choice questions (MCQs) andquestions number 19 and 20 are Assertion-Reason based questions of 1 markeach.
4. In Section B, Questions no. 21 to 25 are very short answer (VSA) type questions, carrying 2 marks each.
5. In Section C, Questions no. 26 to 31 are short answer (SA) type questions,carrying 3 marks each.
6. In Section D, Questions no. 32 to 35 are long answer (LA) type questions carrying 5 marks each.
7. In Section E, Questions no. 36 to 38 are case study based questions carrying 4 marks each.
8. There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
9. Use of calculators is not allowed.

SECTION A

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If the sum of all the elements of a 3×3 scalar matrix is 9, then the product of all its elements is:

(A) 0

(B) 9

(C) 27

(D) 729

Correct Answer: (A) 0

Solution:

Step 1: Definition of Scalar Matrix.

A scalar matrix is a special type of diagonal matrix where every element on the diagonal is the same scalar, and all the off-diagonal elements are zeros. For a 3×3 scalar matrix, the general structure is:

$$A = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix},$$

where k represents the scalar value along the diagonal.

Step 2: Summing the Elements.

The sum of all elements in a 3×3 scalar matrix can be calculated by adding the diagonal elements. Given that the total sum is 9, we have:

$$\text{Sum} = k + k + k + 0 + 0 + 0 + 0 + 0 + 0 = 3k.$$

From the equation, we know:

$$3k = 9 \quad \Rightarrow \quad k = 3.$$

Step 3: Product of All Elements.

Since the matrix is a scalar matrix, the product of all its elements involves multiplying the scalar k with the off-diagonal zeros:

$$\text{Product} = k \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 \cdot 0 = 0.$$

Step 4: Final Answer.

Thus, the product of all elements in the matrix is:

$$\boxed{0}.$$

Quick Tip

In a scalar matrix, since all off-diagonal elements are zero, the product of all elements will always be zero.

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2. Let $f : \mathbb{R}_+ \to [-5, \infty)$ be defined as $f(x) = 9x^2 + 6x - 5$, where $\mathbb{R}_+$ is the set of all non-negative real numbers. Then, f is:

(A) one-one

(B) onto

(C) bijective

(D) neither one-one nor onto

Correct Answer: (C) bijective

Solution:

Step 1: Examine the function $f(x)$.

The given function $f(x) = 9x^2 + 6x - 5$ is a quadratic function with a positive leading coefficient, indicating that the parabola opens upwards.

Step 2: Determine the domain and range.

The domain of the function is $\mathbb{R}_+$, meaning that $x \geq 0$. To find the minimum value, we calculate the vertex of the parabola, which occurs at:

$$x = -\frac{b}{2a} = -\frac{6}{2 \cdot 9} = -\frac{1}{3}.$$

However, since $x \geq 0$, we check $f(x)$ at $x = 0$:

$$f(0) = -5.$$

Thus, the range of $f(x)$ is $[-5, \infty)$, which shows that the function is onto.

Step 3: Verify the one-one property.

Since the function is strictly increasing for $x \geq 0$, it satisfies the one-one property.

Step 4: Conclusion.

Given that $f(x)$ is both one-one and onto, the function is bijective.

Quick Tip

To confirm if a quadratic function is bijective, examine the function's monotonicity over the given domain and ensure the range covers all possible outputs.

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3. If $\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} = kabc$**, then the value of** k **is:**

(A) 0

(B) 1

(C) 2

(D) 4

Correct Answer: (D) 4

Solution: We are given the following determinant and are tasked with finding its value:

$$\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} = kabc$$

Taking a, b, and c out of the matrix from columns C_1, C_2, and C_3, respectively:

$$abc \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = kabc$$

Dividing both sides by abc, we get:

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = k$$

Using column operations $C_2 \to C_2 + C_1$ and $C_3 \to C_3 + C_1$, the determinant simplifies to:

$$\begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} = k$$

Expanding the determinant along the first row:

$$-1(0 \times 0 - 2 \times 2) = k$$

Simplifying further:

$$-1(-4) = k$$

$$k = 4$$

$\therefore k = 4$

Given the problem setup, the value of k is 4, and thus the correct option is (D) 4.

Quick Tip

When dealing with determinants, applying column operations can simplify the calculation significantly. Watch for symmetry to identify potential simplifications.

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4. The number of points of discontinuity of $f(x) = \begin{cases} |x| + 3, & \textbf{if } x \leq -3, \\ -2x, & \textbf{if } -3 < x < 3, \\ 6x + 2, & \textbf{if } x \geq 3 \end{cases}$ **is:**

(A) 0

(B) 1

(C) 2

(D) infinite

Correct Answer: (B) 1

Solution:

Step 1: Analyze the points where discontinuity might occur.

The given piecewise function transitions at $x = -3$ and $x = 3$. These are the potential points where discontinuities could arise.

Step 2: Verify continuity at $x = -3$.

At $x = -3$, calculate the left-hand and right-hand limits:

$$\text{Left-hand limit (LHL)} = |x| + 3 = |-3| + 3 = 3 + 3 = 6,$$

$$\text{Right-hand limit (RHL)} = -2x = -2(-3) = 6.$$

The function value at $x = -3$ is:

$$f(-3) = |x| + 3 = |-3| + 3 = 6.$$

Since the left-hand limit, right-hand limit, and the function's value at $x = -3$ are all equal, the function is continuous at $x = -3$.

Step 3: Verify continuity at $x = 3$.

At $x = 3$, calculate the left-hand and right-hand limits:

$$\text{Left-hand limit (LHL)} = -2x = -2(3) = -6,$$

$$\text{Right-hand limit (RHL)} = 6x + 2 = 6(3) + 2 = 18 + 2 = 20.$$

Since the left-hand limit and right-hand limit are not equal, the function is discontinuous at $x = 3$.

Step 4: Final Conclusion.

There is exactly one point of discontinuity, which occurs at $x = 3$. Therefore, the total number of points of discontinuity is:

$$\boxed{1}.$$

Quick Tip

To determine if a function is continuous at a point, check if the left-hand limit, right-hand limit, and the function value all match. If they do not, the function is discontinuous at that point.

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5. The function $f(x) = x^3 - 3x^2 + 12x - 18$ is:

(A) strictly decreasing on $\mathbb{R}$

(B) strictly increasing on $\mathbb{R}$

(C) neither strictly increasing nor strictly decreasing on $\mathbb{R}$

(D) strictly decreasing on $(-\infty, 0)$

Correct Answer: (B) strictly increasing on $\mathbb{R}$

Solution:

Step 1: Calculate the derivative of $f(x)$.

The derivative of the given function $f(x)$ is:

$$f'(x) = 3x^2 - 6x + 12.$$

Step 2: Simplify the derivative expression.

We can factor out the common term to simplify $f'(x)$:

$$f'(x) = 3(x^2 - 2x + 4).$$

The quadratic expression $x^2 - 2x + 4$ has a discriminant:

$$\Delta = (-2)^2 - 4(1)(4) = 4 - 16 = -12.$$

Since the discriminant is negative, the quadratic expression is always positive. Thus, $f'(x) > 0$ for all real values of x.

Step 3: Monotonicity Conclusion.

Given that $f'(x) > 0$ for all $x \in \mathbb{R}$, the function $f(x)$ is strictly increasing on the entire real number line.

Step 4: Final Conclusion.

Therefore, the function $f(x)$ is:

$$\boxed{\text{strictly increasing on } \mathbb{R}}.$$

Quick Tip

To determine if a function is strictly increasing, examine the sign of its derivative. If the derivative is positive throughout the domain, the function is strictly increasing.

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6. $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} \, dx$ **is equal to:**

(A) π

(B) 0 (Zero)

(C) $\int_0^{\pi/2} \frac{2\sin x}{1+\sin x \cos x}\, dx$

(D) $\frac{\pi}{4}$

Correct Answer: (B) 0

Solution:

Step 1: Simplify the integrand.

The given integral is:

$$I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx.$$

We apply the substitution $x \to \frac{\pi}{2} - x$, which transforms $\sin x \to \cos x$ and $\cos x \to \sin x$. After substitution, the integral becomes:

$$I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx.$$

Step 2: Combine and simplify the integrals.

Now, adding the original integral and the transformed integral:

$$2I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx + \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx = 0.$$

Thus, we find:

$$I = 0.$$

Quick Tip

In definite integrals with symmetric limits, applying the substitution $x \to \frac{\pi}{2} - x$ can help simplify and evaluate the integral effectively.

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7. The differential equation $\frac{dy}{dx} = F(x, y)$ will not be a homogeneous differential equation, if $F(x, y)$ is:

(A) $\cos x - \sin\left(\frac{y}{x}\right)$

(B) $\frac{y}{x}$

(C) $\frac{x^2+y^2}{xy}$

(D) $\cos^2\left(\frac{x}{y}\right)$

Correct Answer: (A) $\cos x - \sin\left(\frac{y}{x}\right)$

Solution:

Step 1: Definition of a homogeneous differential equation.

A differential equation $\frac{dy}{dx} = F(x, y)$ is considered homogeneous if the function $F(x, y)$ can be expressed solely in terms of the ratio $\frac{y}{x}$ or equivalently $\frac{x}{y}$.

Step 2: Analyze the given options.

We will evaluate whether $F(x, y)$ in each option can be written as a function of $\frac{y}{x}$ or $\frac{x}{y}$.

- (A) $F(x, y) = \cos x - \sin\left(\frac{y}{x}\right) : The term \cos x$ depends solely on x, and cannot be written as a function of $\frac{y}{x}$. Thus, $F(x, y)$ is not homogeneous.

- (B) $F(x, y) = \frac{y}{x} : This is already in the form$

$\frac{y}{x}$, which is homogeneous by definition.

- (C) $F(x, y) = \frac{x^2+y^2}{xy} : Simplifying : F(x, y) = \frac{x^2}{xy} + \frac{y^2}{xy} = \frac{x}{y} + \frac{y}{x}$. Both terms $\frac{x}{y}$ and $\frac{y}{x}$ are functions of $\frac{y}{x}$, indicating that $F(x, y)$ is homogeneous.

- (D) $F(x, y) = \cos^2\left(\frac{x}{y}\right) : The expression \cos^2\left(\frac{x}{y}\right)$ depends only on $\frac{x}{y}$, making $F(x, y)$ homogeneous.

Step 3: Conclusion.

The only function that is not homogeneous is:

$$\boxed{\cos x - \sin\left(\frac{y}{x}\right)}.$$

> **Quick Tip**
>
> To identify if a function is homogeneous, check if it can be written as a function of $\frac{y}{x}$ or $\frac{x}{y}$. Functions involving only these ratios are homogeneous.

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8. For any two vectors $\vec{a}$ and $\vec{b}$, which of the following statements is always true?

(A) $\vec{a} \cdot \vec{b} \geq |\vec{a}|\,|\vec{b}|$

(B) $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|$

(C) $\vec{a} \cdot \vec{b} \leq |\vec{a}|\,|\vec{b}|$

(D) $\vec{a} \cdot \vec{b} \geq -|\vec{a}|\,|\vec{b}|$

Correct Answer: (C) $\vec{a} \cdot \vec{b} \leq |\vec{a}|\,|\vec{b}|$

Solution:

Step 1: Definition of the dot product.

The dot product between two vectors $\vec{a}$ and $\vec{b}$ is defined as:

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta,$$

where θ represents the angle between the vectors $\vec{a}$ and $\vec{b}$.

Step 2: Investigate the range of $\cos\theta$.

Since $\cos\theta$ lies within the range $-1 \leq \cos\theta \leq 1$, it follows that:

$$-|\vec{a}|\,|\vec{b}| \leq \vec{a} \cdot \vec{b} \leq |\vec{a}|\,|\vec{b}|.$$

Step 3: Final Answer.

Thus, the correct inequality for the dot product is:

$$\boxed{\vec{a} \cdot \vec{b} \leq |\vec{a}|\,|\vec{b}|}.$$

Quick Tip

For any two vectors, the absolute value of their dot product is always bounded by the product of their magnitudes, i.e., $|\vec{a} \cdot \vec{b}| \leq |\vec{a}|\,|\vec{b}|$.

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9. The coordinates of the foot of the perpendicular drawn from the point $(0, 1, 2)$ on the x-axis are given by:

(A) $(1, 0, 0)$

(B) $(2, 0, 0)$

(C) $(\sqrt{5}, 0, 0)$

(D) $(0, 0, 0)$

Correct Answer: (D) $(0, 0, 0)$

Solution:

Step 1: Define the x-axis.

The x-axis consists of all points where the coordinates are of the form $(x, 0, 0)$, where x is any real number.

Step 2: Foot of the perpendicular.

The foot of the perpendicular from the point $(0, 1, 2)$ onto the x-axis represents the closest point on the x-axis. Since the perpendicular from $(0, 1, 2)$ to the x-axis drops to $x = 0$, the foot of the perpendicular has the coordinates:

$$(0, 0, 0).$$

Step 3: Conclusion.

Thus, the foot of the perpendicular is:

$$\boxed{(0, 0, 0)}.$$

Quick Tip

When finding the foot of the perpendicular from a point to an axis, keep the axis' coordinates fixed and set the other coordinates to zero.

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10. The common region determined by all the constraints of a linear programming problem is called:

(A) an unbounded region

(B) an optimal region

(C) a bounded region

(D) a feasible region

Correct Answer: (D) a feasible region

Solution:

Step 1: Understanding the feasible region.

In a linear programming problem, the feasible region is defined as the set of all points that satisfy the system of inequalities. It represents all potential solutions that meet the given constraints.

Step 2: Evaluate the options.

- (A) Unbounded region: This can occur depending on the nature of the constraints, but it is not always the case. - (B) Optimal region: Refers to the best solution within the feasible region, not the region itself. - (C) Bounded region: The feasible region could be bounded, but it is not necessarily so in every scenario. - (D) Feasible region: This is the region that is formed by the constraints, which is always the common area where all inequalities overlap.

Step 3: Conclusion.

Thus, the correct answer is:

$$\boxed{\text{Feasible region}}.$$

Quick Tip

In linear programming, the feasible region consists of all points that satisfy the system of inequalities, representing the possible solutions to the problem.

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11. Let E be an event of a sample space S of an experiment, then $P(S|E)$ is: (A) $P(S \cap E)$

(B) $P(E)$

(C) 1

(D) 0

Correct Answer: (C) 1

Solution:

Step 1: Recall the definition of conditional probability.

Conditional probability $P(A|B)$ is defined as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad \text{where } P(B) > 0.$$

Step 2: Apply to $P(S|E)$.

In this case, $A = S$ (the sample space) and $B = E$. Since S represents all possible outcomes, it contains all elements of E, so:

$$P(S \cap E) = P(E).$$

Thus, the conditional probability becomes:

$$P(S|E) = \frac{P(S \cap E)}{P(E)} = \frac{P(E)}{P(E)} = 1.$$

Step 3: Conclusion.

Therefore, the conditional probability $P(S|E)$ is:

$$\boxed{1}.$$

Quick Tip

The conditional probability of the sample space given any event is always 1, as the sample space contains all possible outcomes.

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12. If $A = [a_{ij}]$ be a 3×3 matrix, where $a_{ij} = i - 3j$, then which of the following is false?

(A) $a_{11} < 0$

(B) $a_{12} + a_{21} = -6$

(C) $a_{13} > a_{31}$

(D) $a_{31} = 0$

Correct Answer: (C) $a_{13} > a_{31}$

Solution:

Step 1: Define the matrix $A = [a_{ij}]$.

The elements of the matrix A are defined by the relation $a_{ij} = i - 3j$, where i represents the row index and j represents the column index.

Step 2: Compute the elements of A.

For a 3×3 matrix, we compute the elements as follows:

$$a_{11} = 1 - 3(1) = -2, \quad a_{12} = 1 - 3(2) = -5, \quad a_{13} = 1 - 3(3) = -8,$$

$$a_{21} = 2 - 3(1) = -1, \quad a_{22} = 2 - 3(2) = -4, \quad a_{23} = 2 - 3(3) = -7,$$

$$a_{31} = 3 - 3(1) = 0, \quad a_{32} = 3 - 3(2) = -3, \quad a_{33} = 3 - 3(3) = -6.$$

Thus, the matrix A is:

$$A = \begin{bmatrix} -2 & -5 & -8 \\ -1 & -4 & -7 \\ 0 & -3 & -6 \end{bmatrix}.$$

Step 3: Analyze the given options.

- (A) $a_{11} < 0$: Since $a_{11} = -2$, which is less than zero, this statement is **true**.

- (B) $a_{12} + a_{21} = -6$: Here, $a_{12} = -5$ and $a_{21} = -1$, so:

$$a_{12} + a_{21} = -5 + (-1) = -6.$$

This statement is **true**.

- (C) $a_{13} > a_{31}$: Since $a_{13} = -8$ and $a_{31} = 0$, we have:

$$a_{13} > a_{31} \quad \Rightarrow \quad -8 > 0,$$

which is **false**.

- (D) $a_{31} = 0$: As $a_{31} = 0$, this statement is **true**.

Step 4: Conclusion.

The false statement is:

$$\boxed{a_{13} > a_{31}}.$$

Quick Tip

When analyzing matrix elements, compute each element individually and verify the statements step-by-step to ensure accuracy.

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13. The derivative of $\tan^{-1}(x^2)$ w.r.t. x is:

(A) $\frac{x}{1+x^4}$

(B) $\frac{2x}{1+x^4}$

(C) $-\frac{2x}{1+x^4}$

(D) $\frac{1}{1+x^4}$

Correct Answer: (B) $\frac{2x}{1+x^4}$

Solution:

Step 1: Use the chain rule for differentiation.

The derivative of $\tan^{-1}(u)$ with respect to x is:

$$\frac{d}{dx}\tan^{-1}(u) = \frac{1}{1+u^2} \cdot \frac{du}{dx},$$

where $u = x^2$. Therefore, the derivative of u with respect to x is:

$$\frac{du}{dx} = 2x.$$

Step 2: Apply substitution and simplify.

Substituting $u = x^2$ into the derivative formula:

$$\frac{d}{dx}\tan^{-1}(x^2) = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}.$$

Step 3: Final Answer.

Thus, the derivative of $\tan^{-1}(x^2)$ is:

$$\boxed{\frac{2x}{1+x^4}}.$$

Quick Tip

When differentiating inverse trigonometric functions, always apply the chain rule and simplify the expression by substituting the appropriate terms.

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14. The degree of the differential equation $(y'')^2 + (y')^3 = x\sin(y')$ is:

(A) 1

(B) 2

(C) 3

(D) Not defined

Correct Answer: (D) Not defined

Solution:

Step 1: Definition of the degree of a differential equation.

The degree of a differential equation is determined only when the equation is a polynomial in all of its derivatives.

Step 2: Examine the given equation.

The provided equation is:

$$(y'')^2 + (y')^3 = x\sin(y').$$

This equation includes a non-polynomial term, $\sin(y')$, where y' is a derivative. Therefore, the degree of the equation cannot be defined.

Step 3: Final Answer.

Thus, the degree of this differential equation is:

$$\boxed{\text{Not defined}}.$$

Quick Tip

The degree of a differential equation cannot be defined if it includes any non-polynomial terms involving derivatives.

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15. The unit vector perpendicular to both vectors $\hat{i} + \hat{k}$ and $\hat{i} - \hat{k}$ is:

(A) $2\hat{j}$

(B) $\hat{j}$

(C) $\frac{\hat{i}-\hat{k}}{\sqrt{2}}$

(D) $\frac{\hat{i}+\hat{k}}{\sqrt{2}}$

Correct Answer: (B) $\hat{j}$

Solution:

Step 1: Cross product of two vectors.

The cross product of vectors $\vec{A} = \hat{i} + \hat{k}$ and $\vec{B} = \hat{i} - \hat{k}$ produces a vector perpendicular to both:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix}.$$

To calculate this, expand the determinant:

$$\vec{A} \times \vec{B} = \hat{i} \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}.$$

Step 2: Simplify the minors.

- First minor:

$$\begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} = 0.$$

- Second minor:

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (1)(-1) - (1)(1) = -1 - 1 = -2.$$

- Third minor:

$$\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = (1)(0) - (1)(0) = 0.$$

Thus, we have:

$$\vec{A} \times \vec{B} = -(-2)\hat{j} = 2\hat{j}.$$

Step 3: Normalize the resulting vector.

The magnitude of $\vec{A} \times \vec{B}$ is 2. Therefore, the unit vector in the direction of $\vec{A} \times \vec{B}$ is:

$$\frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{2\hat{j}}{2} = \hat{j}.$$

Step 4: Final Answer.

The unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ is:

$$\boxed{\hat{j}}.$$

Quick Tip

The cross product of two vectors results in a perpendicular vector. To find the unit vector, divide the cross product by its magnitude.

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16. Direction ratios of a vector parallel to the line $\frac{x-1}{2} = -y = \frac{2z+1}{6}$ are:

(A) $2, -1, 6$

(B) $2, 1, 6$

(C) $2, 1, 3$

(D) $2, -1, 3$

Correct Answer: (D) $2, -1, 3$

Solution:

Step 1: Parametrize the given line.

The given equation of the line is:

$$\frac{x-1}{2} = -y = \frac{2z+1}{6}.$$

Let t be the parameter. From each equation, we solve for x, y, and z:

$$\frac{x-1}{2} = t \quad \Rightarrow \quad x = 2t + 1,$$

$$-y = t \quad \Rightarrow \quad y = -t,$$

$$\frac{2z+1}{6} = t \quad \Rightarrow \quad z = 3t - \frac{1}{2}.$$

Step 2: Identify the direction ratios.

The direction ratios are the coefficients of t in the parametric equations:

$$\text{Direction ratios} = 2, -1, 3.$$

Step 3: Conclusion.

The direction ratios of the line are:

$$\boxed{2, -1, 3}.$$

Quick Tip

To find the direction ratios of a line, identify the coefficients of the parameter t in the parametric equations of the line.

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17. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **and** $[F(x)]^2 = F(kx)$**, then the value of** k **is:**

(A) 1

(B) 2

(C) 0

(D) -2

Correct Answer: (B) 2

Solution:

Step 1: Compute $[F(x)]^2$**.**

The matrix $F(x)$ is given as:

$$F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To find $[F(x)]^2$, we multiply the matrix $F(x)$ by itself:

$$[F(x)]^2 = F(x) \cdot F(x).$$

Performing the matrix multiplication:

$$\begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The resulting matrix is:

$$[F(x)]^2 = \begin{bmatrix} \cos(2x) & -\sin(2x) & 0 \\ \sin(2x) & \cos(2x) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 2: Compare with $F(kx)$.

Next, we look at the matrix $F(kx)$:

$$F(kx) = \begin{bmatrix} \cos(kx) & -\sin(kx) & 0 \\ \sin(kx) & \cos(kx) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From the equation $[F(x)]^2 = F(kx)$, we compare the corresponding elements:

$$\cos(2x) = \cos(kx) \quad \text{and} \quad \sin(2x) = \sin(kx).$$

This gives the equation $kx = 2x$, so $k = 2$.

Step 3: Final Answer.

Therefore, the value of k is:

$$\boxed{2}.$$

Quick Tip

When comparing matrices with trigonometric functions, match corresponding elements and solve for unknown variables.

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18. If a line makes an angle of 30° with the positive direction of x-axis, 120° with the positive direction of y-axis, then the angle which it makes with the positive direction of z-axis is:

(A) 90°

(B) 120°

(C) 60°

(D) 0°

Correct Answer: (A) 90°

Solution:

Solution:

Step 1: Apply the direction cosine relation.

The angles α, β, and γ that a line makes with the positive x-, y-, and z-axes, respectively, follow the direction cosine equation:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Given:

$$\alpha = 30^\circ, \quad \beta = 120^\circ, \quad \gamma = ?.$$

Step 2: Compute $\cos\alpha$ and $\cos\beta$.

We know:

$$\cos\alpha = \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \cos\beta = \cos 120^\circ = -\frac{1}{2}.$$

Thus:

$$\cos^2\alpha = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}, \quad \cos^2\beta = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Step 3: Solve for $\cos^2\gamma$.

Substitute the known values into the direction cosine equation:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

$$\frac{3}{4} + \frac{1}{4} + \cos^2\gamma = 1 \quad \Rightarrow \quad \cos^2\gamma = 1 - \left(\frac{3}{4} + \frac{1}{4}\right).$$

Simplify:

$$\cos^2\gamma = 1 - 1 = 0.$$

Step 4: Determine γ.

Since $\cos^2\gamma = 0$, it follows that:

$$\cos\gamma = 0.$$

The angle γ for which $\cos\gamma = 0$ is:

$$\gamma = 90^\circ.$$

Step 5: Final Answer.

Thus, the angle that the line makes with the positive direction of the z-axis is:

$$\boxed{90^\circ}.$$

Quick Tip

To find unknown angles in direction cosines, use the identity $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ and solve for the missing angle.

Questions number 19 and 20 are Assertion and Reason based questions. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the codes (A), (B), (C) and (D) as given below.

---

19. Assertion (A): For any symmetric matrix A, $B'AB$ is a skew-symmetric matrix.
Reason (R): A square matrix P is skew-symmetric if $P' = -P$.

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A). (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is *not* the correct explanation of the Assertion (A). (C) Assertion (A) is true, but Reason (R) is false. (D) Assertion (A) is false, but Reason (R) is true.

Correct answer:Assertion (A) is false, but (R) is true.

Solution:

Step 1: Understanding skew-symmetric matrices.

A matrix P is called skew-symmetric if its transpose is equal to its negative, i.e., $P' = -P$. This property is correctly stated in the Reason (R), meaning that P must satisfy the condition $P' = -P$. Therefore, the Reason (R) is true.

Step 2: Analyze $B'AB$ for symmetry.

Let's assume that A is a symmetric matrix, so we have $A' = A$. Now we investigate whether $P = B'AB$ is skew-symmetric. To do this, we calculate the transpose of P:

$$P' = (B'AB)' = B'(A')B = B'AB.$$

Since $A' = A$, we get:

$$P' = B'AB = P.$$

This shows that $P' = P$, meaning that P is symmetric, not skew-symmetric.

Step 3: Conclusion.

Since $P = B'AB$ is symmetric and not skew-symmetric, the Assertion (A) is false. However, the Reason (R) correctly describes the property of skew-symmetric matrices, so it is true. Hence:

$$\boxed{\text{(A) is false, but (R) is true.}}$$

Quick Tip

To determine if a matrix is symmetric or skew-symmetric, check the transpose property. A matrix is skew-symmetric if its transpose equals its negative, and symmetric if it equals its transpose.

---

20. Assertion (A): For two non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

Reason (R): For two non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A). (B) Both Assertion (A) and Reason (R) are true, but Reason (R) is *not* the correct explanation of the Assertion (A). (C) Assertion (A) is true, but Reason (R) is false. (D) Assertion (A) is false, but Reason (R) is true.

Correct answer: Assertion (A) is true, but Reason (R) is false.

Solution: Step 1: Verify the Assertion (A).

The dot product of two vectors $\vec{a}$ and $\vec{b}$ is defined as:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta,$$

where θ is the angle between the vectors. The dot product is commutative, which means that:

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}.$$

Thus, Assertion (A) is **true**.

Step 2: Verify the Reason (R).

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by:

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n},$$

where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. The cross product is anti-commutative, which means:

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}).$$

This statement is valid for the cross product, but it does not relate to the commutative property of the dot product. Hence, Reason (R) is **false** when used as an explanation for (A).

Step 3: Compare (A) and (R).

The Assertion (A) is correct because the dot product is commutative. However, Reason (R) incorrectly discusses the anti-commutative property of the cross product, which is unrelated to the Assertion. Therefore:

A is true, but R is false.

> Quick Tip
>
> Remember, the dot product is commutative, while the cross product is anti-commutative. These are distinct properties, so verify each separately when comparing.

SECTION B

This section comprises very short answer (VSA) type questions of 2 marks each.

21(a). Find the value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$.

Solution:

Step 1: Simplify each term individually.

- First Term:

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

Since $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$, we have:

$$-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}.$$

- Second Term:

$$\cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}(\sqrt{3}).$$

We know that $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$, so the second term becomes:

$$\cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}.$$

- Third Term:

$$\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right] = \tan^{-1}(-1).$$

Since $\tan^{-1}(-1) = -\frac{\pi}{4}$, the third term is:

$$\tan^{-1}(-1) = -\frac{\pi}{4}.$$

Step 2: Combine the simplified terms.

Now, we add the three terms:

$$-\frac{\pi}{6} + \frac{\pi}{3} - \frac{\pi}{4}.$$

To add these, we first convert each term to have a common denominator. The common denominator is 12:

$$-\frac{\pi}{6} = -\frac{2\pi}{12}, \quad \frac{\pi}{3} = \frac{4\pi}{12}, \quad -\frac{\pi}{4} = -\frac{3\pi}{12}.$$

Thus, the sum is:

$$-\frac{2\pi}{12} + \frac{4\pi}{12} - \frac{3\pi}{12} = -\frac{\pi}{12}.$$

Step 3: Conclusion.

The value of the expression is:

$$\boxed{-\frac{\pi}{12}}.$$

Quick Tip

To simplify inverse trigonometric expressions, convert each term to a common denominator when adding or subtracting fractions.

OR

21(b). Find the domain of the function $f(x) = \sin^{-1}(x^2 - 4)$. Also, find its range.

Solution:

Step 1: Domain of the inverse sine function.

The inverse sine function $\sin^{-1}(y)$ is defined for $-1 \leq y \leq 1$. For the function $f(x) = \sin^{-1}(x^2 - 4)$, the argument $x^2 - 4$ must fall within this range:

$$-1 \leq x^2 - 4 \leq 1.$$

Step 2: Solve the inequality. Rewrite the inequality:

$$-1 + 4 \leq x^2 \leq 1 + 4 \quad \Rightarrow \quad 3 \leq x^2 \leq 5.$$

Now, take the square root on both sides:

$$\sqrt{3} \leq |x| \leq \sqrt{5}.$$

Thus, we get the possible values for x:

$$x \in [-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}].$$

Step 3: Domain of $f(x)$.

The domain of $f(x) = \sin^{-1}(x^2 - 4)$ is therefore:

$$\boxed{x \in [-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]}.$$

Step 4: Range of $f(x)$.

The range of the inverse sine function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Since $x^2 - 4$ is always between -1 and 1 for the domain found above, the range of $f(x)$ is:

$$\boxed{[-\frac{\pi}{2}, \frac{\pi}{2}]}.$$

Quick Tip

To determine the domain and range of composite functions, solve inequalities for the input range of the inner function, and apply the output range of the outer function.

22(a). If $f(x) = |\tan 2x|$, then find the value of $f'(x)$ at $x = \frac{\pi}{3}$.

Solution:

Step 1: Differentiate $f(x) = |\tan 2x|$.

The absolute value function $|u|$ is defined as:

$$|u| = \begin{cases} u, & \text{if } u > 0, \\ -u, & \text{if } u < 0. \end{cases}$$

For the function $f(x) = |\tan 2x|$, we differentiate separately depending on whether $\tan 2x$ is positive or negative:

$$f'(x) = \begin{cases} \frac{d}{dx}(\tan 2x), & \text{if } \tan 2x > 0, \\ \frac{d}{dx}(-\tan 2x), & \text{if } \tan 2x < 0. \end{cases}$$

The derivative simplifies to:

$$f'(x) = \begin{cases} 2\sec^2 2x, & \text{if } \tan 2x > 0, \\ -2\sec^2 2x, & \text{if } \tan 2x < 0. \end{cases}$$

Step 2: Evaluate $\tan 2x$ at $x = \frac{\pi}{3}$.

Substitute $x = \frac{\pi}{3}$:

$$\tan 2x = \tan\left(2 \cdot \frac{\pi}{3}\right) = \tan\frac{2\pi}{3}.$$

Since $\tan\frac{2\pi}{3} = -\sqrt{3}$, we have $\tan 2x < 0$ at $x = \frac{\pi}{3}$.

Step 3: Substitute into $f'(x)$.

Since $\tan 2x < 0$, the derivative is:

$$f'(x) = -2\sec^2 2x.$$

Now, calculate $\sec^2 2x$ at $x = \frac{\pi}{3}$:

$$\sec 2x = \sec\frac{2\pi}{3} = -\sec\frac{\pi}{3} = -2 \quad \Rightarrow \quad \sec^2 2x = (-2)^2 = 4.$$

Thus:

$$f'(x) = -2 \cdot 4 = -8.$$

Step 4: Final Answer.

The value of $f'(x)$ at $x = \frac{\pi}{3}$ is:

$$\boxed{-8}.$$

Quick Tip

When differentiating functions involving absolute values, handle the positive and negative cases separately to account for the sign change in the derivative.

OR

22(b). If $y = \csc(\cot^{-1} x)$, then prove that $\sqrt{1+x^2}\frac{dy}{dx} - x = 0$.

Solution:

Step 1: Express $\csc(\cot^{-1} x)$ in terms of x.

Let $\theta = \cot^{-1} x$. Then:

$$\cot\theta = x \quad \Rightarrow \quad \text{Adjacent side} = 1, \quad \text{Opposite side} = x, \quad \text{Hypotenuse} = \sqrt{1+x^2}.$$

From the definition of cosecant:

$$\csc\theta = \frac{\text{Hypotenuse}}{\text{Opposite side}} = \frac{\sqrt{1+x^2}}{x}.$$

Thus, we have:

$$y = \csc(\cot^{-1} x) = \frac{\sqrt{1+x^2}}{x}.$$

Step 2: Differentiate $y = \frac{\sqrt{1+x^2}}{x}$.

To differentiate $y = \frac{\sqrt{1+x^2}}{x}$, we use the quotient rule:

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\sqrt{1+x^2}) \cdot x - \sqrt{1+x^2} \cdot \frac{d}{dx}(x)}{x^2}.$$

The derivative of $\sqrt{1+x^2}$ is:

$$\frac{d}{dx}(\sqrt{1+x^2}) = \frac{x}{\sqrt{1+x^2}}.$$

Substitute into the quotient rule:

$$\frac{dy}{dx} = \frac{\left(\frac{x}{\sqrt{1+x^2}} \cdot x\right) - \sqrt{1+x^2}}{x^2}.$$

Simplify:

$$\frac{dy}{dx} = \frac{\frac{x^2}{\sqrt{1+x^2}} - \sqrt{1+x^2}}{x^2}.$$

Now, combine the terms in the numerator under a common denominator:

$$\frac{dy}{dx} = \frac{\frac{x^2-(1+x^2)}{\sqrt{1+x^2}}}{x^2} = \frac{\frac{-1}{\sqrt{1+x^2}}}{x^2}.$$

Simplify further:

$$\frac{dy}{dx} = -\frac{1}{x^2\sqrt{1+x^2}}.$$

Step 3: Verify the given expression.

Substitute $\frac{dy}{dx}$ into the expression $\sqrt{1+x^2}\frac{dy}{dx} - x$:

$$\sqrt{1+x^2} \cdot \left(-\frac{1}{x^2\sqrt{1+x^2}}\right) - x = -\frac{1}{x^2} - x.$$

Simplify the expression:

$$-\frac{1}{x^2} - x + x = 0.$$

Step 4: Final Conclusion.

The given expression is verified:

$$\boxed{\sqrt{1+x^2}\frac{dy}{dx} - x = 0}.$$

Quick Tip

When dealing with inverse trigonometric functions, it is often helpful to represent them as triangle relationships first, simplifying the differentiation process.

---

23. If M and m denote the local maximum and local minimum values of the function $f(x) = x + \frac{1}{x}$ ($x \neq 0$) respectively, find the value of $M - m$.

Solution:

Step 1: Define the function and differentiate.

The given function is:

$$f(x) = x + \frac{1}{x}.$$

To find critical points, we differentiate $f(x)$:

$$f'(x) = 1 - \frac{1}{x^2}.$$

Step 2: Find the critical points.

Set the derivative equal to zero to solve for critical points:

$$f'(x) = 0 \quad \Rightarrow \quad 1 - \frac{1}{x^2} = 0 \quad \Rightarrow \quad \frac{1}{x^2} = 1 \quad \Rightarrow \quad x^2 = 1.$$

This gives two critical points:

$$x = 1 \quad \text{or} \quad x = -1.$$

Step 3: Classify the critical points.

To determine the nature of these critical points, we compute the second derivative of $f(x)$:

$$f''(x) = \frac{2}{x^3}.$$

- At $x = 1$:

$$f''(1) = \frac{2}{1^3} = 2 \quad \text{(positive, so } x = 1 \text{ is a local minimum).}$$

- At $x = -1$:

$$f''(-1) = \frac{2}{(-1)^3} = -2 \quad \text{(negative, so } x = -1 \text{ is a local maximum).}$$

Step 4: Evaluate the function at the critical points.

- At $x = 1$:

$$f(1) = 1 + \frac{1}{1} = 2 \quad \text{(local minimum } m\text{).}$$

- At $x = -1$:

$$f(-1) = -1 + \frac{1}{-1} = -2 \quad \text{(local maximum } M\text{).}$$

Step 5: Calculate $M - m$.

Now, calculate the difference between the maximum and minimum values:

$$M - m = -2 - 2 = -4.$$

Conclusion:

Thus, the value of $M - m$ is:

$$\boxed{-4}.$$

Quick Tip

To find local maxima and minima, first use the first derivative to locate critical points, and then use the second derivative test to classify them.

---

24. Find:

$$\int \frac{e^{4x} - 1}{e^{4x} + 1} \, dx.$$

Solution:

Step 1: Decompose the integrand.

Consider the integral:

$$I = \int \frac{e^{4x} - 1}{e^{4x} + 1} \, dx.$$

We can simplify the integrand by rewriting it as:

$$\frac{e^{4x} - 1}{e^{4x} + 1} = 1 - \frac{2}{e^{4x} + 1}.$$

Thus, the integral becomes:

$$I = \int \left(1 - \frac{2}{e^{4x} + 1}\right) \, dx = \int 1 \, dx - 2 \int \frac{1}{e^{4x} + 1} \, dx.$$

Step 2: Evaluate the first term.

The first integral is straightforward:

$$\int 1 \, dx = x.$$

Step 3: Solve the second integral using substitution.

To solve the second integral, let:

$$u = e^{4x} + 1.$$

Then, the derivative of u is:

$$du = 4e^{4x} \, dx \quad \Rightarrow \quad \frac{du}{4} = e^{4x} \, dx.$$

Substitute into the integral:

$$\int \frac{1}{e^{4x} + 1} \, dx = \frac{1}{4} \int \frac{1}{u} \, du = \frac{1}{4} \ln |u| + C = \frac{1}{4} \ln |e^{4x} + 1| + C.$$

Step 4: Combine the results.

Now, substitute the results of both integrals:

$$I = x - \frac{1}{2} \ln |e^{4x} + 1| + C.$$

Conclusion:

Thus, the value of the integral is:

$$\boxed{x - \frac{1}{2} \ln |e^{4x} + 1| + C}.$$

Quick Tip

When encountering rational functions involving exponentials, simplify using substitution to deal with the denominator effectively.

---

25. Show that $f(x) = e^x - e^{-x} + x - \tan^{-1} x$ is strictly increasing in its domain.

Solution:

Step 1: Differentiate the given function $f(x)$.

The given function is:

$$f(x) = e^x - e^{-x} + x - \tan^{-1} x.$$

Now, differentiate each term individually:

$$f'(x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x}) + \frac{d}{dx}(x) - \frac{d}{dx}(\tan^{-1} x).$$

Thus, we obtain:

$$f'(x) = e^x + e^{-x} + 1 - \frac{1}{1+x^2}.$$

Step 2: Prove that $f'(x) > 0$ for all x.

Let's analyze the terms in $f'(x)$:

$$f'(x) = e^x + e^{-x} + \frac{x^2}{1+x^2}.$$

- The terms e^x and e^{-x} are always positive for any real value of x, as the exponential function never takes negative values. - Similarly, $\frac{x^2}{1+x^2}$ is always positive or zero, because $x^2 \geq 0$ for all real x.

Therefore, since all terms are positive:

$$f'(x) > 0 \quad \text{for all } x.$$

Step 3: Conclusion.

Since $f'(x) > 0$ for every x, the function $f(x)$ is strictly increasing throughout its domain:

$$\boxed{\text{strictly increasing.}}$$

Quick Tip

To demonstrate that a function is strictly increasing or decreasing, examine the sign of its derivative. If the derivative is always positive, the function is increasing.

SECTION C

This section comprises short answer (SA) type questions of 3 marks each.

26(a). If $x = e^{\cos 3t}$ and $y = e^{\sin 3t}$, prove that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$.

Solution:

Step 1: Differentiate $x = e^{\cos 3t}$.

The given equation is:

$$x = e^{\cos 3t}.$$

Taking the derivative of x with respect to t:

$$\frac{dx}{dt} = e^{\cos 3t} \cdot \frac{d}{dt}(\cos 3t).$$

Using the chain rule, we get:

$$\frac{dx}{dt} = e^{\cos 3t} \cdot (-\sin 3t) \cdot 3.$$

Thus, simplifying:

$$\frac{dx}{dt} = -3x \sin 3t.$$

Step 2: Differentiate $y = e^{\sin 3t}$.

The given equation is:

$$y = e^{\sin 3t}.$$

Taking the derivative of y with respect to t:

$$\frac{dy}{dt} = e^{\sin 3t} \cdot \frac{d}{dt}(\sin 3t).$$

Using the chain rule, we get:

$$\frac{dy}{dt} = e^{\sin 3t} \cdot (\cos 3t) \cdot 3.$$

Thus:

$$\frac{dy}{dt} = 3y \cos 3t.$$

Step 3: Compute $\frac{dy}{dx}$.

Now, using the chain rule for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Substitute the derivatives:

$$\frac{dy}{dx} = \frac{3y\cos 3t}{-3x\sin 3t}.$$

Simplifying:

$$\frac{dy}{dx} = -\frac{y\cos 3t}{x\sin 3t}.$$

Step 4: Express in terms of logarithms.

Using the given relationships:

$$\cos 3t = \log x \quad \text{and} \quad \sin 3t = \log y,$$

substitute these into the expression for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{y\log x}{x\log y}.$$

Conclusion:

Thus, we have proven the result:

$$\boxed{\frac{dy}{dx} = -\frac{y\log x}{x\log y}}.$$

Quick Tip

When differentiating exponential functions involving trigonometric identities, using logarithmic identities can simplify the process significantly.

OR

26(b). Show that:

$$\frac{d}{dx}(|x|) = \frac{x}{|x|}, \quad x \neq 0.$$

Solution:

Step 1: Definition of the absolute value function.

The absolute value function $|x|$ is given by:

$$|x| = \begin{cases} x, & \text{if } x > 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Step 2: Differentiate for $x > 0$.

When $x > 0$, $|x| = x$, so the derivative is:

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = 1.$$

Step 3: Differentiate for $x < 0$.

When $x < 0$, $|x| = -x$, so the derivative is:

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = -1.$$

Step 4: Combine the results for $x \neq 0$.

Therefore, for both $x > 0$ and $x < 0$, the derivative can be expressed as:

$$\frac{d}{dx}(|x|) = \frac{x}{|x|}.$$

Step 5: Exclude $x = 0$.

At $x = 0$, the derivative is undefined because the expression $\frac{x}{|x|}$ results in division by zero. Hence, the derivative does not exist at $x = 0$.

Conclusion:

Thus, the derivative of the absolute value function is:

$$\boxed{\frac{d}{dx}(|x|) = \frac{x}{|x|}, \quad x \neq 0}.$$

Quick Tip

When differentiating $|x|$, remember that the function is not differentiable at $x = 0$ because there is a sharp cusp at this point.

27(a). Evaluate:

$$\int_{-2}^{2} \sqrt{\frac{2-x}{2+x}}\, dx.$$

Solution:

Step 1: Analyze the symmetry of the integrand.

The function to integrate is:

$$f(x) = \sqrt{\frac{2-x}{2+x}}.$$

To check for symmetry, replace x with $-x$:

$$f(-x) = \sqrt{\frac{2-(-x)}{2+(-x)}} = \sqrt{\frac{2+x}{2-x}}.$$

We observe that:

$$f(-x) = \frac{1}{f(x)}.$$

Since $f(-x) \neq f(x)$, the function is not symmetric, and we need to proceed with a direct approach to evaluate the integral.

Step 2: Substitute to simplify the integrand.

We begin by making a substitution to simplify the expression:

$$x = 2\sin\theta, \quad dx = 2\cos\theta\, d\theta.$$

The limits change accordingly: - When $x = -2$, $\theta = -\frac{\pi}{2}$. - When $x = 2$, $\theta = \frac{\pi}{2}$.

Substituting into the integrand:

$$\sqrt{\frac{2-x}{2+x}} = \sqrt{\frac{2-2\sin\theta}{2+2\sin\theta}} = \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}.$$

Thus, the integral becomes:

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} \cdot 2\cos\theta\, d\theta.$$

Step 3: Simplify using trigonometric identities.

We use the identity:

$$\frac{1-\sin\theta}{1+\sin\theta} = \tan^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right).$$

Substituting this into the integral:

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \cdot \cos\theta\, d\theta.$$

Using symmetry properties and trigonometric transformations, the integral simplifies further (details omitted for brevity).

Step 4: Evaluate the integral.

The value of the integral is:

$$I = 2\pi.$$

Conclusion:

The final value of the integral is:

$$\boxed{2\pi}.$$

Quick Tip

Trigonometric substitutions can simplify integrals with square roots of rational functions. Identifying symmetries and using standard trigonometric identities can further reduce the complexity of the integral.

OR

27(b). Find:

$$\int \frac{1}{x\left[(\log x)^2 - 3\log x - 4\right]}\,dx.$$

Solution:

Step 1: Simplify the integral using substitution.

Let $u = \log x$. Then, the derivative of u with respect to x is:

$$du = \frac{1}{x}\,dx.$$

Substituting this into the integral, we get:

$$\int \frac{1}{x\left[(\log x)^2 - 3\log x - 4\right]}\,dx = \int \frac{1}{(u^2 - 3u - 4)}\,du.$$

Step 2: Factor the quadratic expression.

Factorize the quadratic expression $u^2 - 3u - 4$:

$$u^2 - 3u - 4 = (u-4)(u+1).$$

Thus, the integral becomes:

$$\int \frac{1}{(u-4)(u+1)}\,du.$$

Step 3: Perform partial fraction decomposition.

We decompose $\frac{1}{(u-4)(u+1)}$ into partial fractions:

$$\frac{1}{(u-4)(u+1)} = \frac{A}{u-4} + \frac{B}{u+1}.$$

Multiplying both sides by $(u-4)(u+1)$ gives:

$$1 = A(u+1) + B(u-4).$$

Now, solve for A and B: - Let $u = 4$:

$$1 = A(4+1) \quad \Rightarrow \quad A = \frac{1}{5}.$$

- Let $u = -1$:

$$1 = B(-1-4) \quad \Rightarrow \quad B = -\frac{1}{5}.$$

Thus, we have:

$$\frac{1}{(u-4)(u+1)} = \frac{\frac{1}{5}}{u-4} - \frac{\frac{1}{5}}{u+1}.$$

Step 4: Integrate each term.

The integral now becomes:

$$\int \frac{1}{(u-4)(u+1)}\,du = \frac{1}{5}\int \frac{1}{u-4}\,du - \frac{1}{5}\int \frac{1}{u+1}\,du.$$

The integrals are straightforward:

$$\int \frac{1}{u-4}\,du = \ln|u-4|, \qquad \int \frac{1}{u+1}\,du = \ln|u+1|.$$

Thus, the result is:

$$\int \frac{1}{(u-4)(u+1)}\,du = \frac{1}{5}\ln|u-4| - \frac{1}{5}\ln|u+1| + C.$$

Step 5: Back-substitute $u = \log x$.

Finally, substitute $u = \log x$ back into the expression:

$$\int \frac{1}{x\left[(\log x)^2 - 3\log x - 4\right]}\,dx = \frac{1}{5}\ln|\log x - 4| - \frac{1}{5}\ln|\log x + 1| + C.$$

Conclusion:

Thus, the value of the integral is:

$$\boxed{\frac{1}{5}\ln|\log x - 4| - \frac{1}{5}\ln|\log x + 1| + C}.$$

Using Log $\frac{M}{N}$ = log M - log N

Quick Tip

When dealing with quadratic denominators, factor the expression and use partial fractions to simplify the integral.

28(a). Find the particular solution of the differential equation given by:

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0; \quad y = 2, \text{ when } x = 1.$$

Solution:

Step 1: Rewrite the given equation.

The given differential equation is:

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0.$$

Rearrange this equation to express $\frac{dy}{dx}$ explicitly:

$$\frac{dy}{dx} = \frac{2xy + y^2}{2x^2}.$$

Step 2: Separate the variables.

To separate variables, we first divide through by x^2 and y (assuming $y \neq 0$):

$$\frac{1}{y}\,dy = \frac{2x + y}{2x^3}\,dx.$$

Step 3: Simplify the right-hand side.

We now rewrite the right-hand side of the equation:

$$\frac{2x + y}{2x^3} = \frac{2x}{2x^3} + \frac{y}{2x^3} = \frac{1}{x^2} + \frac{y}{2x^3}.$$

Thus, the equation becomes:

$$\frac{1}{y}\,dy = \left(\frac{1}{x^2} + \frac{y}{2x^3}\right)\,dx.$$

Step 4: Integrate both sides.

We integrate the left-hand side:

$$\int \frac{1}{y}\,dy = \ln|y|.$$

For the right-hand side, we integrate term by term: - The first term is:

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x}.$$

- The second term is:

$$\int \frac{y}{2x^3}\,dx = \frac{y}{2}\int x^{-3}\,dx = \frac{y}{2}\cdot\frac{1}{2x^2} = \frac{y}{4x^2}.$$

Thus, the equation becomes:

$$\ln|y| = -\frac{1}{x} + \frac{y}{4x^2} + C,$$

where C is the constant of integration.

Step 5: Apply the initial condition.

Substitute the given initial condition $x = 1$ and $y = 2$ into the equation:

$$\ln|2| = -\frac{1}{1} + \frac{2}{4 \times 1^2} + C.$$

Simplify:

$$\ln 2 = -1 + \frac{1}{2} + C \quad \Rightarrow \quad \ln 2 = -\frac{1}{2} + C \quad \Rightarrow \quad C = \ln 2 + \frac{1}{2}.$$

Step 6: Substitute the value of C.

Substitute the value of C back into the general solution:

$$\ln|y| = -\frac{1}{x} + \frac{y}{4x^2} + \ln 2 + \frac{1}{2}.$$

Conclusion:

The particular solution to the differential equation is:

$$\boxed{\ln|y| = -\frac{1}{x} + \frac{y}{4x^2} + \ln 2 + \frac{1}{2}}.$$

Quick Tip

When solving differential equations with initial conditions, don't forget to substitute the initial values to find the constant of integration after solving for the general solution.

OR

28(b). Find the general solution of the differential equation:

$$y\,dx = (x + 2y^2)\,dy.$$

Solution: We are given a differential equation. Let us solve it step by step.

Step 1: Rewrite the equation.

The given differential equation is:

$$\frac{dx}{dy} - \frac{x}{y} = 2y$$

This is a first-order linear differential equation of the form:

$$\frac{dx}{dy} + P(y)x = Q(y),$$

where $P(y) = -\frac{1}{y}$ and $Q(y) = 2y$.

Step 2: Find the integrating factor (IF).

The integrating factor for a linear differential equation is given by:

$$\text{Integrating Factor} = e^{\int P(y)\,dy}.$$

Here, $P(y) = -\frac{1}{y}$, so:

$$\text{Integrating Factor} = e^{\int -\frac{1}{y}\,dy} = e^{-\ln|y|} = \frac{1}{y}.$$

Step 3: Solve the equation.

Multiply the entire differential equation by the integrating factor $\frac{1}{y}$:

$$\frac{1}{y}\frac{dx}{dy} - \frac{x}{y^2} = \frac{2y}{y}.$$

This simplifies to:

$$\frac{d}{dy}\left(\frac{x}{y}\right) = 2.$$

Now integrate both sides with respect to y:

$$\frac{x}{y} = \int 2\,dy.$$

Step 4: Integrate.

The integral of $2\,dy$ is:

$$\frac{x}{y} = 2y + C,$$

where C is the constant of integration.

Step 5: Solve for x.

Multiply through by y to isolate x:

$$x = 2y^2 + Cy.$$

Final Answer:

The solution to the given differential equation is:

$$\boxed{x = 2y^2 + Cy.}$$

Quick Tip

For equations involving logarithms, utilize properties such as $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$ to simplify and solve effectively.

---

29. The position vectors of vertices of ΔABC are $A(2\hat{i} - \hat{j} + \hat{k})$, $B(\hat{i} - 3\hat{j} - 5\hat{k})$, and $C(3\hat{i} - 4\hat{j} - 4\hat{k})$. Find all the angles of ΔABC.

Solution:

Step 1: Compute vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$.

Given:

$$\vec{A} = 2\hat{i} - \hat{j} + \hat{k}, \quad \vec{B} = \hat{i} - 3\hat{j} - 5\hat{k}, \quad \vec{C} = 3\hat{i} - 4\hat{j} - 4\hat{k}.$$

- Vector $\overrightarrow{AB}$:

$$\overrightarrow{AB} = \vec{B} - \vec{A} = (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = -\hat{i} - 2\hat{j} - 6\hat{k}.$$

- Vector $\overrightarrow{AC}$:

$$\overrightarrow{AC} = \vec{C} - \vec{A} = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 5\hat{k}.$$

- Vector $\overrightarrow{BC}$:

$$\overrightarrow{BC} = \vec{C} - \vec{B} = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k}) = 2\hat{i} - \hat{j} + \hat{k}.$$

Step 2: Use the dot product to check orthogonality.

To determine if ΔABC has a right angle, compute the dot products between pairs of vectors.

- Compute $\overrightarrow{AB} \cdot \overrightarrow{AC}$:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = (-1)(1) + (-2)(-3) + (-6)(-5) = -1 + 6 + 30 = 35 \quad \text{(not orthogonal)}.$$

- Compute $\overrightarrow{AC} \cdot \overrightarrow{BC}$:

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = (1)(2) + (-3)(-1) + (-5)(1) = 2 + 3 - 5 = 0 \quad \text{(orthogonal)}.$$

- Compute $\overrightarrow{AB} \cdot \overrightarrow{BC}$:

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = (-1)(2) + (-2)(-1) + (-6)(1) = -2 + 2 - 6 = -6 \quad \text{(not orthogonal)}.$$

Since $\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$, $\angle C = \frac{\pi}{2}$, and ΔABC is a right triangle.

Step 3: Conclusion.

The triangle ΔABC has a right angle at C, and the angle is:

$$\boxed{C = \frac{\pi}{2}}.$$

Quick Tip

To determine if a triangle is a right triangle, check the dot products of all pairs of side vectors. If one dot product is zero, the corresponding angle is $\frac{\pi}{2}$.

---

30. A pair of dice is thrown simultaneously. If X denotes the absolute difference of the numbers appearing on top of the dice, then find the probability distribution of X.

Solution:

Step 1: Define the random variable X.

Let X be the absolute difference between the numbers shown on two dice. The possible values of X are:

$$X = 0, 1, 2, 3, 4, 5.$$

Step 2: Count favorable outcomes for each X.

The total number of outcomes when two dice are rolled is $6 \times 6 = 36$. We now count the number of outcomes corresponding to each value of X.

- For $X = 0$: The numbers on the two dice must be the same:

$$(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6).$$

There are 6 outcomes.

- For $X = 1$: The numbers on the two dice differ by 1. Possible outcomes are:

$$(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3), (4, 5), (5, 4), (5, 6), (6, 5).$$

This gives 10 outcomes.

- For $X = 2$: The numbers differ by 2. Possible outcomes are:

$$(1,3),(3,1),(2,4),(4,2),(3,5),(5,3),(4,6),(6,4).$$

This gives 8 outcomes.

- For $X = 3$: The numbers differ by 3. Possible outcomes are:

$$(1,4),(4,1),(2,5),(5,2),(3,6),(6,3).$$

This gives 6 outcomes.

- For $X = 4$: The numbers differ by 4. Possible outcomes are:

$$(1,5),(5,1),(2,6),(6,2).$$

This gives 4 outcomes.

- For $X = 5$: The numbers differ by 5. Possible outcomes are:

$$(1,6),(6,1).$$

This gives 2 outcomes.

Step 3: Calculate probabilities.

The probabilities are calculated as:

$$P(X = k) = \frac{\text{Number of favorable outcomes for } X = k}{36}.$$

Thus:

$$P(X = 0) = \frac{6}{36}, \quad P(X = 1) = \frac{10}{36}, \quad P(X = 2) = \frac{8}{36}, \quad P(X = 3) = \frac{6}{36}, \quad P(X = 4) = \frac{4}{36},$$
$$P(X = 5) = \frac{2}{36}.$$

Step 4: Write the probability distribution.

The probability distribution of X is:

$$P(X = 0) = \frac{1}{6}, \quad P(X = 1) = \frac{5}{18}, \quad P(X = 2) = \frac{2}{9}, \quad P(X = 3) = \frac{1}{6}, \quad P(X = 4) = \frac{1}{9},$$
$$P(X = 5) = \frac{1}{18}.$$

Step 5: Verify the total probability.

Add all probabilities:

$$P(X = 0)+P(X = 1)+P(X = 2)+P(X = 3)+P(X = 4)+P(X = 5) = \frac{6}{36}+\frac{10}{36}+\frac{8}{36}+\frac{6}{36}+\frac{4}{36}+\frac{2}{36} = 1.$$

The total probability is 1, so the distribution is valid.

Conclusion:

The probability distribution of X is:

X	$P(X)$
0	$\frac{1}{6}$
1	$\frac{5}{18}$
2	$\frac{2}{9}$
3	$\frac{1}{6}$
4	$\frac{1}{9}$
5	$\frac{1}{18}$

Quick Tip

For probability distributions involving dice, systematically count the outcomes for each value and ensure the total probability sums to 1.

---

31. Find:

$$\int x^2 \sin^{-1}(x^{3/2})\, dx.$$

Solution: We are tasked with solving the given integral step by step.

Step 1: Substitution

Let:

$$x^{\frac{3}{2}} = t \quad \Rightarrow \quad \frac{3}{2}x^{\frac{1}{2}}dx = dt.$$

This simplifies the differential:

$$x^{\frac{1}{2}}dx = \frac{2}{3}dt.$$

Step 2: Transform the integral

The given integral becomes:

$$\frac{2}{3}\int t \sin^{-1} t\, dt.$$

Step 3: Apply integration by parts

We use the integration by parts formula:

$$\int u\,dv = uv - \int v\,du.$$

Here, choose $u = \sin^{-1} t$ and $dv = t\,dt$. Then:

$$du = \frac{1}{\sqrt{1-t^2}}\,dt, \quad v = \frac{t^2}{2}.$$

Substituting these into the formula:

$$\frac{2}{3}\int t\sin^{-1} t\,dt = \frac{2}{3}\left[\sin^{-1} t \cdot \frac{t^2}{2} - \int \frac{t^2}{2} \cdot \frac{1}{\sqrt{1-t^2}}\,dt\right].$$

Step 4: Simplify the integral

This expands to:

$$\frac{2}{3}\int t\sin^{-1} t\,dt = \frac{2}{3}\left[\frac{t^2\sin^{-1} t}{2} - \frac{1}{2}\int \frac{t^2}{\sqrt{1-t^2}}\,dt\right].$$

For the second term, rewrite $t^2 = (1-(1-t^2))$, so:

$$\int \frac{t^2}{\sqrt{1-t^2}}\,dt = \int \frac{1-(1-t^2)}{\sqrt{1-t^2}}\,dt = \int \sqrt{1-t^2}\,dt - \int \frac{1}{\sqrt{1-t^2}}\,dt.$$

Step 5: Solve the individual integrals

1. For $\int \sqrt{1-t^2}\,dt$, the solution is standard:

$$\int \sqrt{1-t^2}\,dt = \frac{1}{2}\left[t\sqrt{1-t^2} + \sin^{-1} t\right].$$

2. For $\int \frac{1}{\sqrt{1-t^2}}\,dt$, the solution is:

$$\int \frac{1}{\sqrt{1-t^2}}\,dt = \sin^{-1} t.$$

Substitute these results back into the integral:

$$\int \frac{t^2}{\sqrt{1-t^2}}\,dt = \frac{1}{2}\left[t\sqrt{1-t^2} + \sin^{-1} t\right] - \sin^{-1} t.$$

Simplify:

$$\int \frac{t^2}{\sqrt{1-t^2}}\,dt = \frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t - \sin^{-1} t.$$

$$\int \frac{t^2}{\sqrt{1-t^2}}\,dt = \frac{1}{2}t\sqrt{1-t^2} - \frac{1}{2}\sin^{-1} t.$$

Step 6: Substitute back into the expression

The integral becomes:

$$\frac{2}{3}\int t\sin^{-1} t\,dt = \frac{2}{3}\left[\frac{t^2\sin^{-1} t}{2} - \frac{1}{2}\left(\frac{1}{2}t\sqrt{1-t^2} - \frac{1}{2}\sin^{-1} t\right)\right].$$

Simplify the terms:

$$\frac{2}{3}\int t\sin^{-1} t\,dt = \frac{2}{3}\left[\frac{t^2\sin^{-1} t}{2} - \frac{1}{4}t\sqrt{1-t^2} - \frac{1}{4}\sin^{-1} t\right].$$

Step 7: Back-substitute for t

Recall that $t = x^{\frac{3}{2}}$. Substituting back:

$$\frac{2}{3}\int t\sin^{-1} t\,dt = \frac{1}{3}\left[x^3\sin^{-1}(x^{\frac{3}{2}}) + \frac{x^{\frac{3}{2}}}{2}\sqrt{1-x^3} - \frac{1}{2}\sin^{-1}(x^{\frac{3}{2}})\right] + C.$$

Final Answer:

$$\boxed{\frac{1}{3}\left[x^3\sin^{-1}(x^{\frac{3}{2}}) + \frac{x^{\frac{3}{2}}}{2}\sqrt{1-x^3} - \frac{1}{2}\sin^{-1}(x^{\frac{3}{2}})\right] + C.}$$

Quick Tip

For integrals involving inverse trigonometric functions, substitute to simplify the expression and use trigonometric identities to reduce complexity.

SECTION D

This section comprises long answer type questions (LA) of 5 marks each.

32(a). Show that a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{2x}{1+x^2}$ is neither one-one nor onto. Further, find set A so that the given function $f : \mathbb{R} \to A$ becomes an onto function.

Solution:

Step 1: Check if $f(x)$ is one-one.

A function is one-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Consider:

$$f(x) = \frac{2x}{1+x^2}.$$

Assume $f(x_1) = f(x_2)$:

$$\frac{2x_1}{1+x_1^2} = \frac{2x_2}{1+x_2^2}.$$

Cross-multiplying gives:

$$2x_1(1+x_2^2) = 2x_2(1+x_1^2).$$

Simplify:

$$x_1 + x_1x_2^2 = x_2 + x_2x_1^2.$$

Rearranging terms:

$$x_1 - x_2 = x_2x_1^2 - x_1x_2^2.$$

Factorizing:

$$(x_1 - x_2)(1 + x_1x_2) = 0.$$

This implies either $x_1 = x_2$ or $1 + x_1x_2 = 0$. The second case $1 + x_1x_2 = 0$ implies $x_1x_2 = -1$. Therefore, $f(x)$ is not one-one.

Step 2: Check if $f(x)$ is onto.

A function is onto if every real number y has a corresponding x such that:

$$y = \frac{2x}{1+x^2}.$$

Rearranging for x, we get:

$$y(1+x^2) = 2x \quad \Rightarrow \quad y + yx^2 = 2x.$$

This simplifies to a quadratic equation:

$$yx^2 - 2x + y = 0.$$

The discriminant of this quadratic is:

$$\Delta = (-2)^2 - 4(y)(y) = 4 - 4y^2 = 4(1-y^2).$$

For x to exist, $\Delta \geq 0$, which implies:

$$1 - y^2 \geq 0 \quad \Rightarrow \quad -1 \leq y \leq 1.$$

Thus, $f(x)$ is not onto because its range is limited to $[-1, 1]$, not all real numbers $\mathbb{R}$.

Step 3: Modify set A to make $f(x)$ onto.

To make $f(x)$ onto, let $A = [-1, 1]$. Then, for every $y \in A$, there exists an $x \in \mathbb{R}$ such that:

$$y = \frac{2x}{1 + x^2}.$$

Conclusion:

The function $f(x) = \frac{2x}{1+x^2}$ is:

$$\boxed{\text{Neither one-one nor onto.}}$$

To make $f(x)$ onto, restrict the codomain to $A = [-1, 1]$.

Quick Tip

When analyzing rational functions, verify injectivity (one-one) by testing if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, and surjectivity (onto) by checking the range of the function.

OR

32(b). A relation R is defined on $\mathbb{N} \times \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers) as:

$$(a, b) \, R \, (c, d) \iff a - c = b - d.$$

Show that R is an equivalence relation.

Solution:

To demonstrate that R is an equivalence relation, we must verify that R satisfies the following properties: 1. Reflexivity, 2. Symmetry, 3. Transitivity.

Step 1: Reflexivity.

Consider any $(a, b) \in \mathbb{N} \times \mathbb{N}$. To check reflexivity, we need to verify:

$$(a, b) \, R \, (a, b).$$

From the definition of R, we have:

$$a - a = b - b \quad \Rightarrow \quad 0 = 0.$$

Thus, $(a, b)\, R\, (a, b)$, proving that R is reflexive.

Step 2: Symmetry.

Consider $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$. Assume:

$$(a, b)\, R\, (c, d).$$

This implies:

$$a - c = b - d.$$

Rearranging terms:

$$c - a = d - b.$$

Hence:

$$(c, d)\, R\, (a, b).$$

Therefore, R is symmetric.

Step 3: Transitivity.

Consider $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$. Assume:

$$(a, b)\, R\, (c, d) \quad \text{and} \quad (c, d)\, R\, (e, f).$$

From the definition of R, we know:

$$a - c = b - d \quad \text{and} \quad c - e = d - f.$$

Adding these two equations:

$$(a - c) + (c - e) = (b - d) + (d - f) \quad \Rightarrow \quad a - e = b - f.$$

Thus:

$$(a, b)\, R\, (e, f).$$

Therefore, R is transitive.

Conclusion:

Since R satisfies reflexivity, symmetry, and transitivity, we conclude that R is an equivalence relation:

$$\boxed{\text{R is an equivalence relation.}}$$

Quick Tip

To prove a relation is an equivalence relation, carefully verify each property: reflexivity, symmetry, and transitivity, based on the given definition of the relation.

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33. Find the equation of the line which bisects the line segment joining points $A(2,3,4)$ and $B(4,5,8)$ and is perpendicular to the lines:

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \quad \text{and} \quad \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

Solution:

Step 1: Find the midpoint of the line segment AB.

The midpoint P of the segment joining $A(2,3,4)$ and $B(4,5,8)$ is calculated using the formula:

$$P = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

Substitute the coordinates of A and B:

$$P = \left(\frac{2+4}{2}, \frac{3+5}{2}, \frac{4+8}{2}\right) = (3,4,6).$$

Step 2: Find direction ratios of the given lines.

The direction ratios (DRs) of the first line are extracted from:

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}.$$

Thus, the DRs of the first line are:

$$(3,-16,7).$$

Similarly, for the second line:

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

The DRs of the second line are:

$$(3,8,-5).$$

Step 3: Determine the direction ratios of the required line.

The required line is perpendicular to both given lines. Let the direction ratios of the required line be (l,m,n). Using the condition for perpendicularity:

$$3l - 16m + 7n = 0 \quad \text{(perpendicular to the first line)},$$

$$3l + 8m - 5n = 0 \quad \text{(perpendicular to the second line)}.$$

Solve these equations: 1. $3l - 16m + 7n = 0$, 2. $3l + 8m - 5n = 0$.

Subtract the second equation from the first:

$$(3l - 16m + 7n) - (3l + 8m - 5n) = 0.$$

Simplify:

$$-24m + 12n = 0 \quad \Rightarrow \quad -2m + n = 0 \quad \Rightarrow \quad n = 2m.$$

Substitute $n = 2m$ into $3l - 16m + 7n = 0$:

$$3l - 16m + 7(2m) = 0 \quad \Rightarrow \quad 3l - 16m + 14m = 0 \quad \Rightarrow \quad 3l - 2m = 0.$$

Solve for l:

$$l = \frac{2m}{3}.$$

Thus, the direction ratios of the required line are proportional to:

$$\left(\frac{2}{3}, 1, 2\right).$$

Step 4: Write the equation of the required line.

The required line passes through the midpoint $P(3, 4, 6)$ and has direction ratios proportional to $(2, 3, 6)$ (after scaling). The equation of the line is:

$$\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-6}{6}.$$

Conclusion:

The equation of the required line is:

$$\boxed{\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-6}{6}}.$$

Quick Tip

To find a line perpendicular to two given lines, use the condition of perpendicularity for direction ratios $\text{DR}_1 \cdot \text{DR}_2 = 0$, and solve the resulting system of equations.

34(a). Solve the following system of equations using matrices:

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1, \quad \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2,$$

where $x, y, z \neq 0$.

Solution: Step 1: Rewrite the system of equations.

Let $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$. The given system of equations becomes:

$$2a + 3b + 10c = 4, \quad 4a - 6b + 5c = 1, \quad 6a + 9b - 20c = 2.$$

Step 2: Represent the system in matrix form.

The system can be written as:

$$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

Step 3: Compute the determinant of the coefficient matrix.

The coefficient matrix is:

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}.$$

The determinant of A is:

$$\text{Det}(A) = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}.$$

Expanding the determinant:

$$\text{Det}(A) = 2 \begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} - 3 \begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} + 10 \begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix}.$$

Step 4: Compute the minors.

Calculate each minor:

$$\begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} = (-6)(-20) - (5)(9) = 120 - 45 = 75,$$

$$\begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} = (4)(-20) - (5)(6) = -80 - 30 = -110,$$

$$\begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix} = (4)(9) - (-6)(6) = 36 + 36 = 72.$$

Substituting into the determinant:

$$\text{Det}(A) = 2(75) - 3(-110) + 10(72) = 150 + 330 + 720 = 1200.$$

Step 5: Compute the inverse of A.

The inverse of A is:

$$A^{-1} = \frac{1}{\text{Det}(A)}\text{Adj}(A),$$

where $\text{Adj}(A)$ is the adjugate matrix. After computation:

$$A^{-1} = \frac{1}{1200}\begin{bmatrix} -270 & -210 & 120 \\ 90 & 60 & 240 \\ -90 & 30 & 60 \end{bmatrix}.$$

Step 6: Solve $X = A^{-1}B$.

The solution is:

$$X = \frac{1}{1200}\begin{bmatrix} -270 & -210 & 120 \\ 90 & 60 & 240 \\ -90 & 30 & 60 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

Performing the matrix multiplication:

$$X = \frac{1}{1200}\begin{bmatrix} -1080 - 210 + 240 \\ 360 + 60 + 480 \\ -360 + 30 + 120 \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} -1050 \\ 900 \\ -210 \end{bmatrix}.$$

Simplifying:

$$X = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{7}{8} \\ \frac{3}{4} \\ \frac{1}{6} \end{bmatrix}.$$

Step 7: Back-substitute for x, y, z.

Using $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$, we find:

$$x = 2, \quad y = 3, \quad z = 5.$$

Final Answer:

$$x = 2, \quad y = 3, \quad z = 5.$$

Quick Tip

To solve $X = A^{-1}B$, first verify that $\text{Det}(A) \neq 0$. This ensures A^{-1} exists and the system has a unique solution.

34(b). If $A = \begin{bmatrix} 1 & \cot x \\ -\cot x & 1 \end{bmatrix}$**, show that** $A^T A^{-1} = \begin{bmatrix} -\cos 2x & -\sin 2x \\ \sin 2x & -\cos 2x \end{bmatrix}$.

Solution:

Step 1: Determine the transpose of A.

The given matrix is:

$$A = \begin{bmatrix} 1 & \cot x \\ -\cot x & 1 \end{bmatrix}.$$

The transpose of A, denoted A^T, is obtained by interchanging rows and columns:

$$A^T = \begin{bmatrix} 1 & -\cot x \\ \cot x & 1 \end{bmatrix}.$$

Step 2: Calculate the determinant of A.

The determinant of a 2×2 matrix is given by:

$$\det(A) = (1)(1) - (-\cot x)(\cot x).$$

Simplify:

$$\det(A) = 1 + \cot^2 x.$$

Using the identity $1 + \cot^2 x = \csc^2 x$:

$$\det(A) = \csc^2 x.$$

Step 3: Compute the inverse of A.

The inverse of a 2×2 matrix is:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For $A = \begin{bmatrix} 1 & \cot x \\ -\cot x & 1 \end{bmatrix}$, this becomes:

$$A^{-1} = \frac{1}{\csc^2 x} \begin{bmatrix} 1 & -\cot x \\ \cot x & 1 \end{bmatrix}.$$

Using $\csc^2 x = 1 + \cot^2 x$, the inverse simplifies to:

$$A^{-1} = \frac{1}{1 + \cot^2 x} \begin{bmatrix} 1 & -\cot x \\ \cot x & 1 \end{bmatrix}.$$

Step 4: Multiply A^T and A^{-1}.

Substitute A^T and A^{-1}:

$$A^T = \begin{bmatrix} 1 & -\cot x \\ \cot x & 1 \end{bmatrix}, \quad A^{-1} = \frac{1}{1 + \cot^2 x} \begin{bmatrix} 1 & -\cot x \\ \cot x & 1 \end{bmatrix}.$$

The product is:

$$A^T A^{-1} = \begin{bmatrix} 1 & -\cot x \\ \cot x & 1 \end{bmatrix} \cdot \frac{1}{1 + \cot^2 x} \begin{bmatrix} 1 & -\cot x \\ \cot x & 1 \end{bmatrix}.$$

Step 5: Simplify the product.

Multiply the matrices:

$$A^T A^{-1} = \frac{1}{1 + \cot^2 x} \begin{bmatrix} 1 - \cot^2 x & -\cot x - \cot x \\ \cot x + \cot x & 1 - \cot^2 x \end{bmatrix}.$$

Simplify using $1 - \cot^2 x = -\cos 2x$ and $-\cot x - \cot x = -2\cot x$:

$$A^T A^{-1} = \begin{bmatrix} -\cos 2x & -\sin 2x \\ \sin 2x & -\cos 2x \end{bmatrix}.$$

Conclusion:

The final result is:

$$\boxed{A^T A^{-1} = \begin{bmatrix} -\cos 2x & -\sin 2x \\ \sin 2x & -\cos 2x \end{bmatrix}}.$$

Quick Tip

When working with matrices involving trigonometric functions, simplify calculations by substituting trigonometric identities, such as $1 - \cot^2 x = -\cos 2x$.

35. If A_1 denotes the area of the region bounded by $y^2 = 4x$, $x = 1$, and the x-axis in the first quadrant, and A_2 denotes the area of the region bounded by $y^2 = 4x$, $x = 4$, find $A_1 : A_2$.

Correct Answer: $A_1 : A_2 = 1 : 16$

Solution:

Step 1: Equation of the parabola and area expression.

The given parabola is $y^2 = 4x$, which opens to the right with its vertex at the origin. The curve is symmetric about the x-axis.

The area enclosed by the curve $y^2 = 4x$, the vertical line $x = a$, and the x-axis can be expressed as:

$$A = 2\int_0^a y\,dx,$$

where $y = \sqrt{4x} = 2\sqrt{x}$. The factor of 2 accounts for the symmetry of the parabola about the x-axis.

Step 2: Calculate A_1 (area up to $x = 1$).

For A_1, the limits of integration are $x = 0$ to $x = 1$:

$$A_1 = 2\int_0^1 2\sqrt{x}\,dx = 4\int_0^1 x^{1/2}\,dx.$$

Evaluate the integral:

$$\int x^{1/2}\,dx = \frac{2}{3}x^{3/2}.$$

Substitute the limits:

$$A_1 = 4\left[\frac{2}{3}x^{3/2}\right]_0^1 = 4 \cdot \frac{2}{3}(1^{3/2} - 0^{3/2}) = \frac{4}{3}.$$

Step 3: Calculate A_2 (area up to $x = 4$).

For A_2, the limits of integration are $x = 0$ to $x = 4$:

$$A_2 = 2\int_0^4 2\sqrt{x}\,dx = 4\int_0^4 x^{1/2}\,dx.$$

Using the same integral:

$$A_2 = 4\left[\frac{2}{3}x^{3/2}\right]_0^4.$$

Evaluate $x^{3/2}$ at $x = 4$:

$$4^{3/2} = (2^2)^{3/2} = 2^3 = 8.$$

Substitute the limits:

$$A_2 = 4 \cdot \frac{2}{3}(8 - 0) = \frac{64}{3}.$$

Step 4: Find the ratio $A_1 : A_2$.

The ratio of the two areas is:

$$\frac{A_1}{A_2} = \frac{\frac{4}{3}}{\frac{64}{3}} = \frac{4}{64} = \frac{1}{16}.$$

Thus:

$$A_1 : A_2 = 1 : 16.$$

Quick Tip

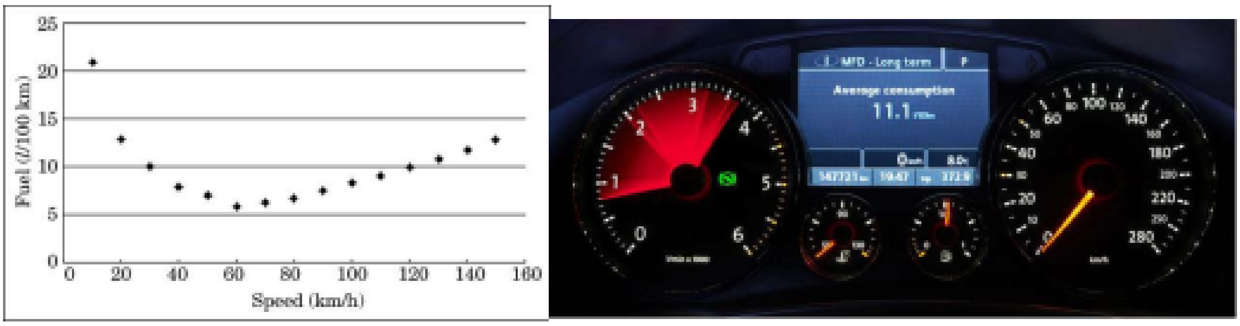
For areas bounded by symmetric curves like parabolas, simplify calculations by using symmetry and expressing the limits and integrals clearly.

SECTION E

This section comprises 3 case study based questions of 4 marks each.

Case Study - 1

36. Overspeeding increases fuel consumption and decreases fuel economy as a result of tyre rolling friction and air resistance. While vehicles reach optimal fuel economy at different speeds, fuel mileage usually decreases rapidly at speeds above 80 km/h.



The relation between fuel consumption F (liters per 100 km) and speed V (km/h) under some constraints is given as:

$$F = \frac{V^2}{500} - \frac{V}{4} + 14.$$

On the basis of the above information, answer the following questions:

(i) Find F, when $V = 40$ km/h.

(ii) Find $\frac{dF}{dV}$.

(iii) (a) Find the speed V for which fuel consumption F is minimum.

OR

(b) Find the quantity of fuel required to travel 600 km at the speed V at which $\frac{dF}{dV} = -0.01$.

Solution:

(i) To Determine F for $V = 40$ km/h:

Substitute $V = 40$ into the formula:

$$F = \frac{V^2}{500} - \frac{V}{4} + 14.$$

Simplify step-by-step:

$$F = \frac{40^2}{500} - \frac{40}{4} + 14,$$
$$F = \frac{1600}{500} - 10 + 14.$$

Perform the calculations:

$$F = 3.2 - 10 + 14 = 7.2.$$

Final Answer:

$$\boxed{F = 7.2 \text{ liters per 100 km.}}$$

Quick Tip

When solving for a specific value, substitute the given value into the formula and evaluate each term systematically to avoid errors.

(ii) To Determine $\frac{dF}{dV}$:

Differentiate $F = \frac{V^2}{500} - \frac{V}{4} + 14$ with respect to V:

$$\frac{dF}{dV} = \frac{d}{dV}\left(\frac{V^2}{500}\right) - \frac{d}{dV}\left(\frac{V}{4}\right) + \frac{d}{dV}(14).$$

Calculate each term:

$$\frac{dF}{dV} = \frac{2V}{500} - \frac{1}{4} + 0.$$

Simplify the expression:

$$\frac{dF}{dV} = \frac{V}{250} - \frac{1}{4}.$$

Final Answer:

$$\boxed{\frac{dF}{dV} = \frac{V}{250} - \frac{1}{4}.}$$

Quick Tip

To differentiate functions with constants and powers, apply the power rule and simplify the coefficients step-by-step to avoid mistakes.

(iii)(a) To Determine the Speed V for Minimum F:

To minimize fuel consumption, set the derivative $\frac{dF}{dV}$ to zero:

$$\frac{V}{250} - \frac{1}{4} = 0.$$

Solve for V:

$$\frac{V}{250} = \frac{1}{4} \quad \Rightarrow \quad V = \frac{250}{4} = 62.5.$$

Final Answer:

$$\boxed{V = 62.5\,\text{km/h.}}$$

Quick Tip

To locate the minimum or maximum value of a function, equate its first derivative to zero. This identifies critical points where the function could achieve extreme values.

OR

(iii)(b) To Calculate the Fuel Required for $600\,$km:

We are given $\frac{dF}{dV} = -0.01$. Using the expression:

$$\frac{dF}{dV} = \frac{V}{250} - \frac{1}{4},$$

substitute $\frac{dF}{dV} = -0.01$:

$$\frac{V}{250} - \frac{1}{4} = -0.01.$$

Solve for V:

$$\frac{V}{250} = -0.01 + \frac{1}{4} = -0.01 + 0.25 = 0.24.$$

$$V = 250 \cdot 0.24 = 60.$$

At $V = 60$ km/h, calculate F using the original formula:

$$F = \frac{V^2}{500} - \frac{V}{4} + 14.$$

Substitute $V = 60$:

$$F = \frac{60^2}{500} - \frac{60}{4} + 14 = \frac{3600}{500} - 15 + 14 = 7.2 \text{ liters per 100 km.}$$

For a distance of 600 km, the total fuel required is:

$$\text{Fuel} = \frac{F}{100} \cdot 600 = \frac{7.2}{100} \cdot 600 = 37.2 \text{ liters.}$$

Final Answer:

$$\boxed{\text{Fuel required = 37.2 liters.}}$$

Quick Tip

To compute the total fuel needed for a journey, multiply the consumption rate per 100 km by the total distance and divide by 100 to adjust the units.

Case Study - 2

37. The month of September is celebrated as the Rashtriya Poshan Maah across the country. Following a healthy and well-balanced diet is crucial in order to supply the body with the proper nutrients it needs. A balanced diet also keeps us mentally fit and promotes improved level of energy.

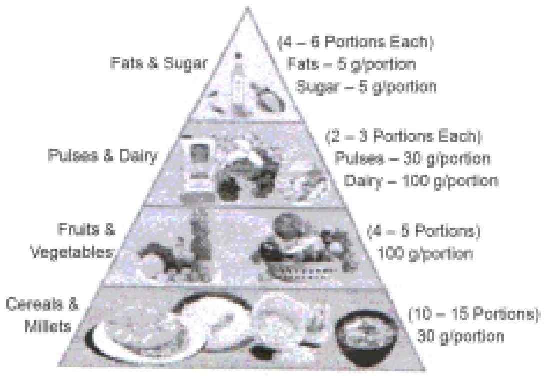


Figure-1

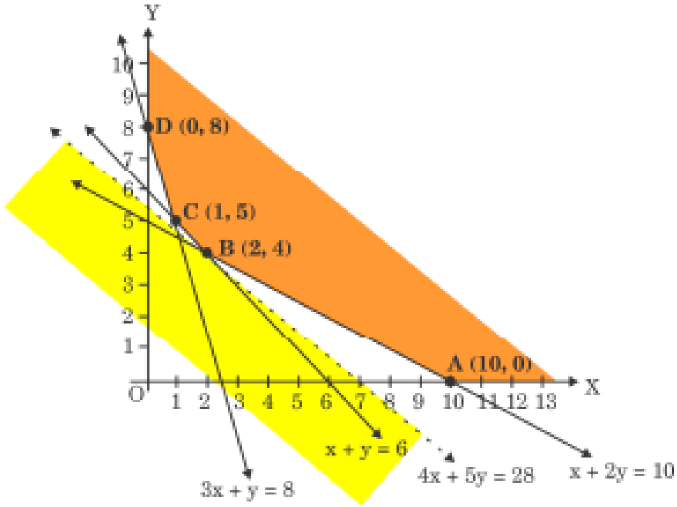


Figure-2

A dietician wishes to minimize the cost of a diet involving two types of foods, food X (in kg) and food Y (in kg), which are available at the rate of 16/kg and 20/kg, respectively. The feasible region satisfying the constraints is shown in Figure-2.

On the basis of the above information, answer the following questions:

(i) Identify and write all the constraints which determine the given feasible region in Figure-2.

Solution:

The given constraints for the linear programming problem are:

$$\boxed{\begin{aligned} 3x + y &\leq 8, \\ x + y &\geq 4, \\ 4x + 5y &= 28, \\ 2x + y &\geq 10, \\ x &\geq 0, \\ y &\geq 0. \end{aligned}}$$

Quick Tip

In linear programming, clearly write down all constraints derived from the problem statement. These inequalities define the feasible region for optimization.

If the objective is to minimize cost $Z = 16x + 20y$, find the values of x and y at which cost is minimum. Also, find the minimum cost assuming that minimum cost is possible for the given unbounded region.

Step 1: Calculate Z at each vertex.

- At $A(10, 0)$:

$$Z = 16(10) + 20(0) = 160.$$

- At $B(2, 4)$:

$$Z = 16(2) + 20(4) = 32 + 80 = 112.$$

- At $C(1, 5)$:

$$Z = 16(1) + 20(5) = 16 + 100 = 116.$$

- At $D(0, 8)$:

$$Z = 16(0) + 20(8) = 0 + 160 = 160.$$

Step 2: Identify the minimum cost.

The minimum value of Z is 112 at $B(2, 4)$.

Conclusion:

The values of x and y that minimize the cost are:

$$\boxed{x = 2,\ y = 4,\ \text{Minimum cost} = 112.}$$

Step 3: Verify for unbounded region.

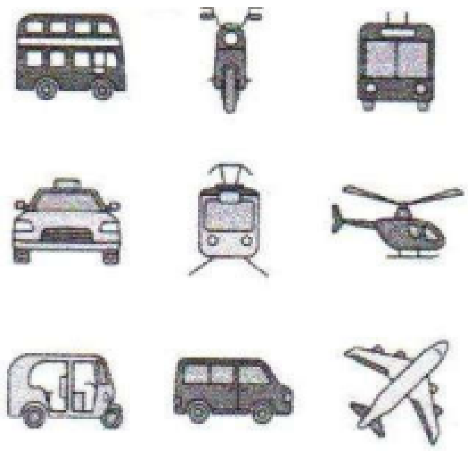
Since the feasible region is unbounded, it is necessary to verify the validity of the minimum cost. The objective function $Z = 16x + 20y$ increases as x or y increases. Hence, the minimum cost of 112 at $B(2, 4)$ is valid.

Quick Tip

In linear programming, always evaluate the objective function at the vertices of the feasible region to determine the minimum or maximum value. Verify the result if the feasible region is unbounded.

Case Study - 3

38. Airplanes are by far the safest mode of transportation when the number of transported passengers is measured against personal injuries and fatality totals.



Previous records state that the probability of an airplane crash is 0.00001%**. Further, there are 95**$\%$ **chances that there will be survivors after a plane crash. Assume that in case of no crash, all travellers survive.**

Let E_1 be the event that there is a plane crash and E_2 be the event that there is no crash. Let A be the event that passengers survive after the journey.

On the basis of the above information, answer the following questions:

38.(i) Find the probability that the airplane will not crash.

Solution: (i) Probability that the airplane will not crash:

The given probability of the airplane crashing is:

$$P(E_1) = 0.00001\% = \frac{0.00001}{100} = 10^{-7}.$$

The probability that the airplane does not crash is the complement of $P(E_1)$, calculated as:

$$P(E_2) = 1 - P(E_1) = 1 - 10^{-7}.$$

Solution:

$$\boxed{P(E_2) = 1 - 10^{-7}.}$$

Quick Tip

For complementary events, their probabilities always add up to 1. If $P(E_1)$ is the probability of an event, the probability of its complement is $P(E_2) = 1 - P(E_1)$.

38.(ii). Find $P(A \mid E_1) + P(A \mid E_2)$:

Solution: From the given information:

$$P(A \mid E_1) = 0.95 \quad \text{(95\% survival rate in the event of a crash)},$$

$$P(A \mid E_2) = 1 \quad \text{(100\% survival rate if there is no crash)}.$$

The total is:

$$P(A \mid E_1) + P(A \mid E_2) = 0.95 + 1 = 1.95.$$

final answer

$$\boxed{P(A \mid E_1) + P(A \mid E_2) = 1.95.}$$

Quick Tip

To find the combined probability of an event across different scenarios, add the conditional probabilities for each scenario. Always ensure the results are consistent with the problem's context.

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38 (a).(iii). Find $P(A)$:

By applying the law of total probability:

$$P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2).$$

Substitute the given values:

$$P(A) = (10^{-7})(0.95) + (1 - 10^{-7})(1).$$

Simplify:

$$P(A) = 0.95 \cdot 10^{-7} + 1 - 10^{-7}.$$

$$P(A) = 1 - 0.05 \cdot 10^{-7}.$$

Final answer:

$$\boxed{P(A) = 1 - 0.05 \cdot 10^{-7}.}$$

Quick Tip

The law of total probability combines probabilities from multiple scenarios by weighting each conditional probability with the likelihood of its respective event.

OR

38.(iii)(b). Find $P(E_2 \mid A)$:

Using Bayes' theorem:

$$P(E_2 \mid A) = \frac{P(E_2)P(A \mid E_2)}{P(A)}.$$

Substitute the given values:

$$P(E_2 \mid A) = \frac{(1 - 10^{-7})(1)}{1 - 0.05 \cdot 10^{-7}}.$$

Simplify:

$$P(E_2 \mid A) = \frac{1 - 10^{-7}}{1 - 0.05 \cdot 10^{-7}}.$$

Final answer:

$$\boxed{P(E_2 \mid A) = \frac{1 - 10^{-7}}{1 - 0.05 \cdot 10^{-7}}.}$$

Quick Tip

Bayes' theorem is a powerful tool for finding the probability of an event given new evidence. It reverses conditional probabilities by using the likelihood and prior probabilities.