

CBSE Class 12 Mathematics Set 3 (65/1/3) Question Paper with Solutions

Time Allowed :3 Hour	Maximum Marks :80	Total Questions :38
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General Instructions

Read the following instructions very carefully and strictly follow them:

1. This question paper contains 38 questions. All questions are compulsory.
2. This question paper is divided into five Sections – A, B, C, D and E.
3. In Section A, Questions no. 1 to 18 are multiple choice questions (MCQs) and questions number 19 and 20 are Assertion-Reason based questions of 1 mark each.
4. In Section B, Questions no. 21 to 25 are very short answer (VSA) type questions, carrying 2 marks each.
5. In Section C, Questions no. 26 to 31 are short answer (SA) type questions, carrying 3 marks each.
6. In Section D, Questions no. 32 to 35 are long answer (LA) type questions carrying 5 marks each.
7. In Section E, Questions no. 36 to 38 are case study based questions carrying 4 marks each.
8. There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
9. Use of calculators is not allowed.

SECTION A

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If $x = at$, $y = \frac{a}{t}$, then $\frac{dy}{dx}$ is:

(1) t^2

(2) $-t^2$

(3) $\frac{1}{t^2}$

(4) $-\frac{1}{t^2}$

Correct Answer: (4) $-\frac{1}{t^2}$

Solution:

Step 1: Find the derivatives of x and y with respect to t . We are given:

$$x = at \quad \text{and} \quad y = \frac{a}{t}.$$

Differentiate x with respect to t :

$$\frac{dx}{dt} = a.$$

Differentiate y with respect to t :

$$\frac{dy}{dt} = -\frac{a}{t^2}.$$

Step 2: Compute $\frac{dy}{dx}$ using the chain rule. The relationship between $\frac{dy}{dx}$, $\frac{dy}{dt}$, and $\frac{dx}{dt}$ is:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Substitute $\frac{dy}{dt} = -\frac{a}{t^2}$ and $\frac{dx}{dt} = a$:

$$\frac{dy}{dx} = \frac{-\frac{a}{t^2}}{a}.$$

Simplify:

$$\frac{dy}{dx} = -\frac{1}{t^2}.$$

Thus, the correct answer is (4) $-\frac{1}{t^2}$.

Quick Tip

When solving problems involving parametric equations, compute the derivatives separately and use $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$. Always check your units and simplify expressions carefully to avoid errors.

2. The solution of the differential equation $\frac{dy}{dx} = \frac{1}{\log y}$ is:

(A) $\log y = x + c$

(B) $y \log y - y = x + c$

(C) $\log y - y = x + c$

(D) $y \log y + y = x + c$

Correct Answer: (B) $y \log y - y = x + c$.

Solution:

The given differential equation is:

$$\frac{dy}{dx} = \frac{1}{\log y}.$$

Step 1: Separate variables. Rearrange to express the equation in terms of y and x :

$$\log y \, dy = dx.$$

Step 2: Integrate both sides. Integrate the left-hand side using integration by parts:

$$\int \log y \, dy = \int dx.$$

For $\int \log y \, dy$, use the integration by parts formula:

$$\int u \, dv = uv - \int v \, du.$$

Let $u = \log y$ and $dv = dy$, so $du = \frac{1}{y} dy$ and $v = y$. Then:

$$\int \log y \, dy = y \log y - \int y \, dy = y \log y - y + C_1,$$

where C_1 is the constant of integration.

For the right-hand side:

$$\int dx = x + C_2.$$

Step 3: Combine results. Equating both sides:

$$y \log y - y = x + C,$$

where $C = C_2 - C_1$.

Hence, the solution is:

$$y \log y - y = x + C.$$

The correct answer is (B) $y \log y - y = x + c$.

Quick Tip

For equations with logarithmic terms, consider separation of variables and use integration by parts for terms like $\int \log y \, dy$. Simplify systematically to derive the solution.

3. The vector with terminal point A (2, -3, 5) and initial point B (3, -4, 7) is:

(A) $\hat{i} - \hat{j} + 2\hat{k}$

(B) $\hat{i} + \hat{j} + 2\hat{k}$

(C) $-\hat{i} - \hat{j} - 2\hat{k}$

(D) $-\hat{i} + \hat{j} - 2\hat{k}$

Correct Answer: (D) $-\hat{i} + \hat{j} - 2\hat{k}$.

Solution:

The formula for the vector from an initial point $B(x_1, y_1, z_1)$ to a terminal point $A(x_2, y_2, z_2)$ is:

$$\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}.$$

Substitute $A(2, -3, 5)$ and $B(3, -4, 7)$:

$$\vec{AB} = (2 - 3)\hat{i} + (-3 - (-4))\hat{j} + (5 - 7)\hat{k}.$$

Simplify each component:

$$\vec{AB} = -\hat{i} + \hat{j} - 2\hat{k}.$$

Hence, the correct answer is:

$$(D) \quad -\hat{i} + \hat{j} - 2\hat{k}.$$

Quick Tip

To find the vector from one point to another, subtract the corresponding coordinates of the initial point from the terminal point. The resulting components give the vector in terms of \hat{i} , \hat{j} , and \hat{k} .

4. The distance of point $P(a, b, c)$ from the y-axis is:

- (A) b
- (B) b^2
- (C) $\sqrt{a^2 + c^2}$
- (D) $a^2 + c^2$

Correct Answer: (C) $\sqrt{a^2 + c^2}$.

Solution:

Approach: Using projection perpendicular to the y-axis.

The y-axis extends in the direction of b , meaning the coordinates a and c define the perpendicular projection of the point $P(a, b, c)$ onto the xz -plane. Hence, the distance of the point $P(a, b, c)$ from the y-axis is the magnitude of the projection onto the xz -plane, which can be calculated using the Euclidean distance:

$$\text{Distance from the y-axis} = \sqrt{a^2 + c^2}.$$

Hence, the correct answer is:

$$(C) \sqrt{a^2 + c^2}.$$

Quick Tip

For calculating the distance from any axis, consider the coordinates in the perpendicular plane. Use the Pythagoras theorem on those coordinates to find the shortest distance.

5. The number of corner points of the feasible region determined by constraints

$x \geq 0, y \geq 0, x + y \geq 4$ is:

- (A) 0
- (B) 1
- (C) 2
- (D) 3

Correct Answer: (C) 2.

Solution:

The constraints are as follows:

$$x \geq 0, \quad y \geq 0, \quad x + y \geq 4.$$

These can be interpreted geometrically: 1. $x \geq 0$: The region lies to the right of the y -axis. 2. $y \geq 0$: The region lies above the x -axis. 3. $x + y \geq 4$: The line passes through points $(4, 0)$ and $(0, 4)$, with the region lying above this line.

Approach: Using intersection points to find corners. The feasible region is formed by the intersection of the above constraints in the first quadrant. The corner points can be calculated as follows: - Intersection of $x \geq 0$ and $x + y = 4$: Substitute $x = 0$ into $x + y = 4$:

$$y = 4 \Rightarrow (0, 4).$$

- Intersection of $y \geq 0$ and $x + y = 4$: Substitute $y = 0$ into $x + y = 4$:

$$x = 4 \Rightarrow (4, 0).$$

The corner points of the feasible region are $(4, 0)$ and $(0, 4)$.

Hence, the number of corner points is:

$$(C) 2.$$

Quick Tip

To find corner points of a feasible region, calculate intersections of the lines or inequalities that form the boundaries, particularly at their extreme values.

6. If matrices A and B are of order 1×3 and 3×1 respectively, then the order of $A'B'$ is:

(A) 1×1

(B) 3×1

(C) 1×3

(D) 3×3

Correct Answer: (D) 3×3 .

Solution:

Step 1: Understanding the transpose operation. The transpose of a matrix changes its dimensions as follows: - For matrix A of order 1×3 , A' will have order 3×1 . - For matrix B of order 3×1 , B' will have order 1×3 .

Step 2: Determining the order of $A'B'$. The product of two matrices A' and B' is defined if

the number of columns of A' equals the number of rows of B' .

$$A' \text{ has order } (3 \times 1), \quad B' \text{ has order } (1 \times 3).$$

Thus, the product $A'B'$ is defined and will have the order of the number of rows of A' and the number of columns of B' :

$$\text{Order of } A'B' = 3 \times 3.$$

Hence, the correct answer is:

$$(D) 3 \times 3.$$

Quick Tip

The order of a matrix product AB is determined by the rows of A and the columns of B . Always check the dimensions carefully, especially after transposing matrices.

7. A relation R defined on a set of human beings as

$R = \{(x, y) : x \text{ is 5 cm shorter than } y\}$ is:

- (1) Reflexive only
- (2) Reflexive and transitive
- (3) Symmetric and transitive
- (4) Neither transitive, nor symmetric, nor reflexive

Correct Answer: (4) Neither transitive, nor symmetric, nor reflexive.

Solution:

Approach: Analyze the relation properties using definitions.

- Reflexive: A relation R is reflexive if $(x, x) \in R$ for all x . In this case, x cannot be 5 cm shorter than itself. Thus, R is not reflexive.

- Symmetric: A relation R is symmetric if $(x, y) \in R$ implies $(y, x) \in R$. If x is 5 cm shorter than y , y cannot simultaneously be 5 cm shorter than x . Thus, R is not symmetric.

- Transitive: A relation R is transitive if $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$. However, if x is 5 cm shorter than y , and y is 5 cm shorter than z , then x is 10 cm shorter than z .

Hence, R is not transitive.

Therefore, the relation R is neither reflexive, symmetric, nor transitive.

Hence, the correct answer is:

(4) Neither transitive, nor symmetric, nor reflexive.

Quick Tip

For any relation, verify reflexive, symmetric, and transitive properties by testing the definitions systematically against given conditions.

8. If a matrix has 36 elements, the number of possible orders it can have, is:

- (1) 13
- (2) 3
- (3) 5
- (4) 9

Correct Answer: (4) 9.

Solution:

Approach: Find the factor pairs of 36.

If a matrix has 36 elements, then the product of the number of rows (m) and the number of columns (n) must equal 36:

$$m \times n = 36.$$

To find the number of possible orders, determine all factor pairs of 36:

$$1 \times 36, 2 \times 18, 3 \times 12, 4 \times 9, 6 \times 6, 9 \times 4, 12 \times 3, 18 \times 2, 36 \times 1.$$

These factor pairs represent all possible combinations of rows and columns for a matrix with 36 elements. Counting these pairs, we find there are 9 possible orders.

Hence, the correct answer is:

(4) 9.

Quick Tip

To find the possible orders of a matrix given the number of elements, calculate all pairs of positive integers whose product equals the total number of elements.

9. Which of the following statements is true for the function:

$$f(x) = \begin{cases} x^2 + 3, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

- (1) $f(x)$ is continuous and differentiable $\forall x \in \mathbb{R}$.
- (2) $f(x)$ is continuous $\forall x \in \mathbb{R}$.
- (3) $f(x)$ is continuous and differentiable $\forall x \in \mathbb{R} - \{0\}$.
- (4) $f(x)$ is discontinuous at infinitely many points.

Correct Answer: (3) $f(x)$ is continuous and differentiable $\forall x \in \mathbb{R} - \{0\}$.

Solution:

Step 1: Check continuity at $x = 0$. For continuity, the left-hand limit, right-hand limit, and function value must be equal.

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (x^2 + 3) = 3.$$

However, $f(0) = 1$. Since $\lim_{x \rightarrow 0} f(x) \neq f(0)$, the function is not continuous at $x = 0$.

Step 2: Check differentiability at $x = 0$. Differentiability requires continuity. As $f(x)$ is not continuous at $x = 0$, it cannot be differentiable at $x = 0$.

Step 3: Check continuity and differentiability for $x \neq 0$. For $x \neq 0$, $f(x) = x^2 + 3$, which is a polynomial function. Polynomials are continuous and differentiable for all x .

Thus, $f(x)$ is continuous and differentiable for all $x \in \mathbb{R} - \{0\}$.

Hence, the correct answer is:

$$(3) f(x) \text{ is continuous and differentiable } \forall x \in \mathbb{R} - \{0\}.$$

Quick Tip

To verify continuity at a point, compare the left-hand limit, right-hand limit, and function value. Differentiability at a point requires the function to be continuous first.

10. Let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b) . Then, this function $f(x)$ is strictly increasing in (a, b) if:

- (1) $f'(x) < 0, \forall x \in (a, b)$
- (2) $f'(x) > 0, \forall x \in (a, b)$

$$(3) f'(x) = 0, \forall x \in (a, b)$$

$$(4) f(x) > 0, \forall x \in (a, b)$$

Correct Answer: (2) $f'(x) > 0, \forall x \in (a, b)$.

Solution:

Step 1: Definition of strictly increasing function. A function $f(x)$ is said to be strictly increasing on an interval (a, b) if, for any $x_1, x_2 \in (a, b)$, where $x_1 < x_2$, we have:

$$f(x_1) < f(x_2).$$

Step 2: Role of the derivative in determining monotonicity. The derivative $f'(x)$ represents the slope of the tangent to the curve at each point x . - If $f'(x) > 0$ for all $x \in (a, b)$, the slope of the tangent is positive, indicating that $f(x)$ is strictly increasing on (a, b) . - Conversely, if $f'(x) < 0$, the function $f(x)$ would be strictly decreasing.

Step 3: Verification of other options. - Option (1): $f'(x) < 0$ implies a strictly decreasing function, not increasing. - Option (3): $f'(x) = 0$ implies a constant function, not strictly increasing. - Option (4): $f(x) > 0$ does not necessarily indicate monotonicity; it only implies the function's values are positive.

Hence, the correct answer is:

$$(2) f'(x) > 0, \forall x \in (a, b).$$

Quick Tip

For strictly increasing or decreasing functions, always analyze the sign of the derivative over the interval. Positive derivative implies strictly increasing, and negative derivative implies strictly decreasing.

11. If

$$\begin{bmatrix} x + y & 2 \\ 5 & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix},$$

then the value of

$$\left(\frac{24}{x} + \frac{24}{y} \right)$$

is:

(1) 7

(2) 6

(3) 8

(4) 18

Correct Answer: (4) 18.

Solution:

From the given matrix equation:

$$\begin{bmatrix} x + y & 2 \\ 5 & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix},$$

we equate the corresponding elements of the matrices: 1. $x + y = 6$, 2. $xy = 8$.

We need to find:

$$\frac{24}{x} + \frac{24}{y}.$$

Step 1: Express the required term in terms of $x + y$ and xy . Using the property of reciprocals:

$$\frac{24}{x} + \frac{24}{y} = 24 \left(\frac{1}{x} + \frac{1}{y} \right).$$

From the relationship of reciprocals:

$$\frac{1}{x} + \frac{1}{y} = \frac{x + y}{xy}.$$

Step 2: Substitute the known values. Substitute $x + y = 6$ and $xy = 8$:

$$\frac{1}{x} + \frac{1}{y} = \frac{6}{8} = \frac{3}{4}.$$

Thus:

$$\frac{24}{x} + \frac{24}{y} = 24 \times \frac{3}{4} = 18.$$

Hence, the correct answer is:

(4) 18.

Quick Tip

When working with matrix equations, equate corresponding elements to derive equations. Use properties of fractions and factorizations to simplify algebraic expressions.

12. If $f(x)$ is an odd function, then:

$$\int_{-\pi/2}^{\pi/2} f(x) \cos^3 x \, dx \text{ equals:}$$

(1) $2 \int_0^{\pi/2} f(x) \cos^3 x \, dx$

(2) 0

(3) $2 \int_0^{\pi/2} f(x) \, dx$

(4) $2 \int_0^{\pi/2} \cos^3 x \, dx$

Correct Answer: (2) 0.

Solution:

Step 1: Understanding the properties of odd functions. If $f(x)$ is an odd function, then $f(-x) = -f(x)$. For symmetric limits of integration, i.e., $[-a, a]$, the integral of an odd function is:

$$\int_{-a}^a f(x) \, dx = 0.$$

Step 2: Analyzing the given integrand. The function $f(x) \cos^3 x$ is the product of $f(x)$ (an odd function) and $\cos^3 x$ (an even function). The product of an odd function and an even function is an odd function.

Step 3: Applying the integral property of odd functions. Since $f(x) \cos^3 x$ is an odd function, its integral over symmetric limits is zero:

$$\int_{-\pi/2}^{\pi/2} f(x) \cos^3 x \, dx = 0.$$

Hence, the correct answer is:

(2) 0.

Quick Tip

The product of an odd function and an even function is odd. The integral of an odd function over symmetric limits is always zero.

13. Let θ be the angle between two unit vectors \hat{a} and \hat{b} such that $\sin \theta = \frac{3}{5}$. Then, $\hat{a} \cdot \hat{b}$ is equal to:

(1) $\pm \frac{3}{5}$

$$(2) \pm \frac{3}{4}$$

$$(3) \pm \frac{4}{5}$$

$$(4) \pm \frac{4}{3}$$

Correct Answer: (3) $\pm \frac{4}{5}$.

Solution:

Step 1: Use the Pythagorean identity. The relationship between $\sin \theta$ and $\cos \theta$ is given by the Pythagorean identity:

$$\sin^2 \theta + \cos^2 \theta = 1.$$

Substitute $\sin \theta = \frac{3}{5}$:

$$\left(\frac{3}{5}\right)^2 + \cos^2 \theta = 1.$$

Simplify:

$$\frac{9}{25} + \cos^2 \theta = 1.$$

Step 2: Solve for $\cos^2 \theta$.

$$\cos^2 \theta = 1 - \frac{9}{25} = \frac{16}{25}.$$

Take the square root:

$$\cos \theta = \pm \frac{4}{5}.$$

Step 3: Calculate the dot product. For unit vectors \hat{a} and \hat{b} , the dot product is given by:

$$\hat{a} \cdot \hat{b} = \cos \theta.$$

Thus:

$$\hat{a} \cdot \hat{b} = \pm \frac{4}{5}.$$

Hence, the correct answer is:

$$(3) \pm \frac{4}{5}.$$

Quick Tip

Use the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ to calculate $\cos \theta$ when $\sin \theta$ is provided.

Remember that the dot product for unit vectors is equal to $\cos \theta$.

14. The integrating factor of the differential equation $(1 - x^2)\frac{dy}{dx} + xy = ax$, $-1 < x < 1$,
is:

(1) $\frac{1}{x^2-1}$

(2) $\frac{1}{\sqrt{x^2-1}}$

(3) $\frac{1}{1-x^2}$

(4) $\frac{1}{\sqrt{1-x^2}}$

Correct Answer: (4) $\frac{1}{\sqrt{1-x^2}}$.

Solution:

The given differential equation is:

$$(1 - x^2)\frac{dy}{dx} + xy = ax.$$

Divide through by $1 - x^2$ to simplify:

$$\frac{dy}{dx} + \frac{xy}{1 - x^2} = \frac{ax}{1 - x^2}.$$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where $P(x) = \frac{x}{1-x^2}$.

Step 1: Finding the integrating factor (IF). The integrating factor is given by:

$$\mathbf{IF} = e^{\int P(x) dx}.$$

Substitute $P(x) = \frac{x}{1-x^2}$:

$$\int P(x) dx = \int \frac{x}{1 - x^2} dx.$$

Let $u = 1 - x^2$, so $du = -2x dx$:

$$\int \frac{x}{1 - x^2} dx = -\frac{1}{2} \int \frac{1}{u} du = -\frac{1}{2} \ln |u| = -\frac{1}{2} \ln |1 - x^2|.$$

Step 2: Simplify the integrating factor. The integrating factor becomes:

$$\mathbf{IF} = e^{-\frac{1}{2} \ln |1-x^2|} = (1 - x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1 - x^2}}.$$

Hence, the correct answer is:

$$(4) \frac{1}{\sqrt{1 - x^2}}.$$

Quick Tip

To find the integrating factor for a linear differential equation, always simplify the equation into the standard form $\frac{dy}{dx} + P(x)y = Q(x)$ and compute $e^{\int P(x) dx}$.

15. If the direction cosines of a line are $\sqrt{3}k, \sqrt{3}k, \sqrt{3}k$, then the value of k is:

- (1) ± 1
- (2) $\pm\sqrt{3}$
- (3) ± 3
- (4) $\pm\frac{1}{3}$

Correct Answer: (4) $\pm\frac{1}{3}$.

Solution:

Step 1: Use the property of direction cosines. The direction cosines l, m, n of a line in 3D space satisfy the equation:

$$l^2 + m^2 + n^2 = 1.$$

Step 2: Substitute the given direction cosines. Here, $l = \sqrt{3}k, m = \sqrt{3}k, n = \sqrt{3}k$.

Substitute these into the equation:

$$(\sqrt{3}k)^2 + (\sqrt{3}k)^2 + (\sqrt{3}k)^2 = 1.$$

Step 3: Simplify the equation.

$$3k^2 + 3k^2 + 3k^2 = 1.$$

$$9k^2 = 1.$$

$$k^2 = \frac{1}{9}.$$

$$k = \pm\frac{1}{3}.$$

Hence, the correct answer is:

$$(4) \pm\frac{1}{3}.$$

Quick Tip

The direction cosines l, m, n are always normalized such that $l^2 + m^2 + n^2 = 1$. Use this property to find unknowns.

16. A linear programming problem deals with the optimization of a/an:

- (1) Logarithmic function
- (2) Linear function
- (3) Quadratic function
- (4) Exponential function

Correct Answer: (2) Linear function.

Solution:

Linear programming is a mathematical method used to optimize a linear objective function, subject to a set of linear equality or inequality constraints. The objective function typically has the form:

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

where c_1, c_2, \dots, c_n are constants, and x_1, x_2, \dots, x_n are decision variables.

This function represents a linear relationship, which is optimized (either maximized or minimized) under given constraints that are also linear in nature.

Hence, the correct answer is:

(2) Linear function.

Quick Tip

Linear programming problems are defined by linear objective functions and linear constraints. They cannot involve nonlinear elements like quadratic or exponential functions.

17. If $P(A|B) = P(A'|B)$, then which of the following statements is true?

- (1) $P(A) = P(A')$
- (2) $P(A) = 2P(B)$
- (3) $P(A \cap B) = \frac{1}{2}P(B)$
- (4) $P(A \cap B) = 2P(B)$

Correct Answer: (3) $P(A \cap B) = \frac{1}{2}P(B)$.

Solution:

We are given that $P(A|B) = P(A'|B)$. Recall the formulas for conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(A'|B) = \frac{P(A' \cap B)}{P(B)}.$$

Since $P(A|B) = P(A'|B)$:

$$\frac{P(A \cap B)}{P(B)} = \frac{P(A' \cap B)}{P(B)}.$$

This implies:

$$P(A \cap B) = P(A' \cap B).$$

From the complementary rule, A and A' are mutually exclusive, and:

$$P(A \cap B) + P(A' \cap B) = P(B).$$

Substituting $P(A \cap B) = P(A' \cap B)$:

$$P(A \cap B) + P(A \cap B) = P(B).$$

$$2P(A \cap B) = P(B).$$

$$P(A \cap B) = \frac{1}{2}P(B).$$

Hence, the correct answer is:

$$(3) P(A \cap B) = \frac{1}{2}P(B).$$

Quick Tip

When $P(A|B) = P(A'|B)$, use the complement property and the fact that the sum of probabilities of complementary events equals the total probability.

18.

$$\begin{vmatrix} x + 1 & x - 1 \\ x^2 + x + 1 & x^2 - x + 1 \end{vmatrix}$$

is equal to:

(1) $2x^3$

(2) 2

(3) 0

(4) $2x^3 - 2$

Correct Answer: (2) 2.

Solution:

The determinant of a 2×2 matrix is given by:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For the given matrix:

$$\begin{vmatrix} x + 1 & x - 1 \\ x^2 + x + 1 & x^2 - x + 1 \end{vmatrix},$$

we identify:

$$a = x + 1, \quad b = x - 1, \quad c = x^2 + x + 1, \quad d = x^2 - x + 1.$$

The determinant is:

$$\text{Determinant} = (x + 1)(x^2 - x + 1) - (x - 1)(x^2 + x + 1).$$

Step 1: Expand the first term.

$$(x + 1)(x^2 - x + 1) = x^3 - x^2 + x + x^2 - x + 1 = x^3 + x + 1.$$

Step 2: Expand the second term.

$$(x - 1)(x^2 + x + 1) = x^3 + x^2 + x - x^2 - x - 1 = x^3 - 1.$$

Step 3: Subtract the two terms.

$$\text{Determinant} = (x^3 + x + 1) - (x^3 - 1) = x^3 + x + 1 - x^3 + 1 = 2.$$

Hence, the correct answer is:

$$(2) 2.$$

Quick Tip

To find the determinant of a 2×2 matrix, multiply the diagonal elements and subtract the product of the off-diagonal elements. Simplify carefully to avoid sign errors.

Questions number 19 and 20 are Assertion and Reason based questions. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the codes (A), (B), (C) and (D) as given below.

19. Assertion (A): For the matrix

$$A = \begin{bmatrix} 1 & \cos \theta & 1 \\ -\cos \theta & 1 & \cos \theta \\ -1 & -\cos \theta & 1 \end{bmatrix}, \quad \text{where } \theta \in [0, 2\pi],$$

$$|A| \in [2, 4].$$

Reason (R): $\cos \theta \in [-1, 1], \forall \theta \in [0, 2\pi]$.

(1) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(2) Both Assertion (A) and Reason (R) are true, but Reason (R) is *not* the correct explanation of the Assertion (A).

(3) Assertion (A) is true, but Reason (R) is false.

(4) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (1) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

We calculate the determinant of the matrix A :

$$|A| = \begin{vmatrix} 1 & \cos \theta & 1 \\ -\cos \theta & 1 & \cos \theta \\ -1 & -\cos \theta & 1 \end{vmatrix}.$$

Step 1: Use cofactor expansion along the first row.

$$|A| = 1 \cdot \begin{vmatrix} 1 & \cos \theta \\ -\cos \theta & 1 \end{vmatrix} - \cos \theta \cdot \begin{vmatrix} -\cos \theta & \cos \theta \\ -1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} -\cos \theta & 1 \\ -1 & -\cos \theta \end{vmatrix}.$$

1. Compute the first minor:

$$\begin{vmatrix} 1 & \cos \theta \\ -\cos \theta & 1 \end{vmatrix} = (1)(1) - (-\cos \theta)(\cos \theta) = 1 + \cos^2 \theta.$$

2. Compute the second minor:

$$\begin{vmatrix} -\cos \theta & \cos \theta \\ -1 & 1 \end{vmatrix} = (-\cos \theta)(1) - (\cos \theta)(-1) = -\cos \theta + \cos \theta = 0.$$

3. Compute the third minor:

$$\begin{vmatrix} -\cos \theta & 1 \\ -1 & -\cos \theta \end{vmatrix} = (-\cos \theta)(-\cos \theta) - (1)(-1) = \cos^2 \theta + 1.$$

Step 2: Substitute into the determinant formula.

$$|A| = 1 \cdot (1 + \cos^2 \theta) - \cos \theta \cdot 0 + 1 \cdot (1 + \cos^2 \theta).$$

Simplify:

$$|A| = (1 + \cos^2 \theta) + (1 + \cos^2 \theta) = 2 + 2 \cos^2 \theta.$$

Step 3: Analyze the range of $|A|$. Since $\cos \theta \in [-1, 1]$, we know:

$$\cos^2 \theta \in [0, 1].$$

Thus:

$$|A| = 2 + 2 \cos^2 \theta \in [2, 4].$$

Step 4: Verification of Assertion (A) and Reason (R). - Assertion (A): $|A| \in [2, 4]$ is **true**. -

Reason (R): $\cos \theta \in [-1, 1]$ for all $\theta \in [0, 2\pi]$ is also **true**.

Since the range of $\cos \theta$ explains the range of $|A|$, Reason (R) correctly explains Assertion (A).

Conclusion: Both Assertion (A) and Reason (R) are true, and Reason (R) is the correct explanation of Assertion (A).

Quick Tip

To compute determinants, use cofactor expansion and carefully calculate minors. Leverage trigonometric properties like $\cos \theta \in [-1, 1]$ to determine the range of the result.

20. Assertion (A): A line in space cannot be drawn perpendicular to x , y , and z axes simultaneously.

Reason (R): For any line making angles α, β, γ with the positive directions of x , y , and z axes respectively,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

- (1) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
- (2) Both Assertion (A) and Reason (R) are true, but Reason (R) is *not* the correct explanation of the Assertion (A).
- (3) Assertion (A) is true, but Reason (R) is false.
- (4) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (1) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

A line in three-dimensional space cannot be perpendicular to all three axes simultaneously. For a line to be perpendicular to all three axes, the direction cosines $\cos \alpha, \cos \beta, \cos \gamma$ would have to be zero, which contradicts the fundamental relation of direction cosines:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

This relation guarantees that at least one direction cosine must be non-zero, so the line cannot be perpendicular to the x -axis, y -axis, and z -axis at the same time.

Conclusion: Both Assertion (A) and Reason (R) are true, and Reason (R) correctly explains Assertion (A).

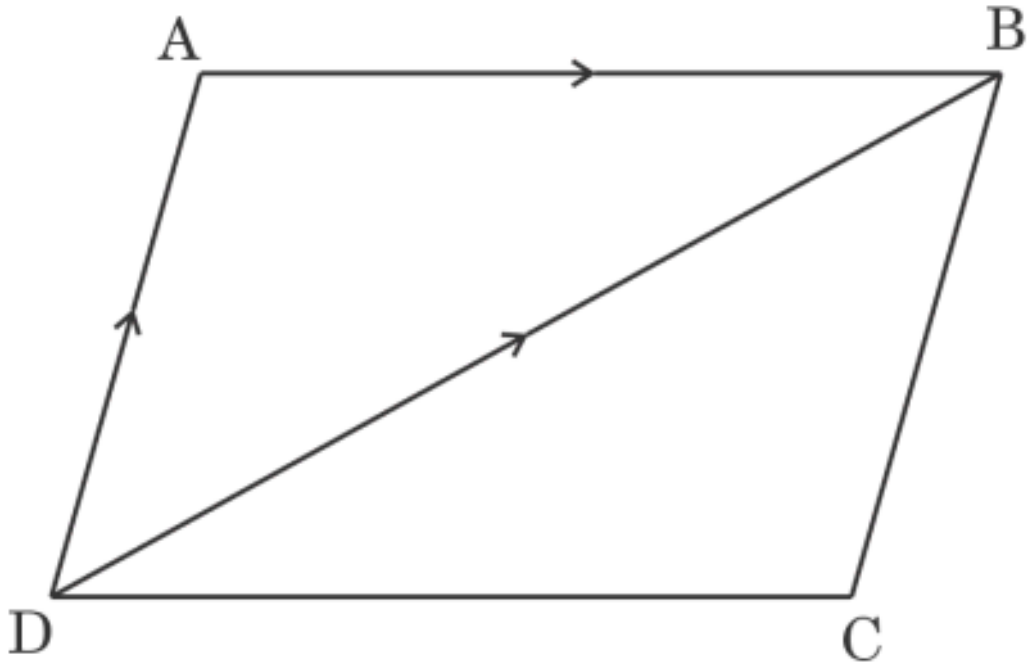
Quick Tip

In 3D geometry, the direction cosines of a line $(\cos \alpha, \cos \beta, \cos \gamma)$ satisfy the fundamental relation $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. This ensures that no line can be perpendicular to all three coordinate axes simultaneously.

SECTION B

This section comprises very short answer (VSA) type questions of 2 marks each.

21. In the given figure, ABCD is a parallelogram. If $\vec{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{DB} = 3\hat{i} - 6\hat{j} + 2\hat{k}$, then find \vec{AD} and hence find the area of parallelogram ABCD.



Solution:

To calculate \vec{AD} , we use the relation:

$$\vec{AD} = \vec{AB} + \vec{DB}.$$

Given $\vec{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{DB} = 3\hat{i} - 6\hat{j} + 2\hat{k}$, we compute:

$$\vec{AD} = (2\hat{i} - 4\hat{j} + 5\hat{k}) + (3\hat{i} - 6\hat{j} + 2\hat{k}).$$

Simplify:

$$\vec{AD} = (2 + 3)\hat{i} + (-4 - 6)\hat{j} + (5 + 2)\hat{k} = 5\hat{i} - 10\hat{j} + 7\hat{k}.$$

The area of the parallelogram is given by the magnitude of the cross product $\vec{AB} \times \vec{AD}$:

$$\text{Area} = |\vec{AB} \times \vec{AD}|.$$

Step 1: Compute the cross product.

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 5 & -10 & 7 \end{vmatrix}.$$

Expand the determinant:

$$\vec{AB} \times \vec{AD} = \hat{i} \begin{vmatrix} -4 & 5 \\ -10 & 7 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 5 \\ 5 & 7 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & -4 \\ 5 & -10 \end{vmatrix}.$$

Step 2: Compute the minors.

$$\hat{i} : (-4)(7) - (5)(-10) = -28 + 50 = 22,$$

$$\hat{j} : (2)(7) - (5)(5) = 14 - 25 = -11,$$

$$\hat{k} : (2)(-10) - (-4)(5) = -20 + 20 = 0.$$

Thus:

$$\vec{AB} \times \vec{AD} = 22\hat{i} + 11\hat{j} + 0\hat{k}.$$

Step 3: Compute the magnitude of the cross product.

$$|\vec{AB} \times \vec{AD}| = \sqrt{22^2 + 11^2 + 0^2} = \sqrt{484 + 121} = \sqrt{605}.$$

Answer: The area of parallelogram ABCD is $\sqrt{605}$.

Quick Tip

To find the area of a parallelogram using vectors, calculate the cross product of two adjacent sides. The magnitude of the resulting vector gives the parallelogram's area.

22. (a) Check the differentiability of the function $f(x) = \lfloor x \rfloor$ at $x = -3$, where $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Solution:

The greatest integer function $f(x) = \lfloor x \rfloor$ gives the greatest integer less than or equal to x . For $x = -3$, we have:

$$f(x) = \lfloor x \rfloor = \begin{cases} -4, & -4 \leq x < -3, \\ -3, & x = -3. \end{cases}$$

To check differentiability, we evaluate the left-hand derivative (LHD) and right-hand derivative (RHD) at $x = -3$:

Step 1: Left-hand derivative (LHD).

$$\text{LHD} = \lim_{h \rightarrow 0^-} \frac{f(-3+h) - f(-3)}{h}.$$

For $h \rightarrow 0^-$, $-3 + h \in (-4, -3)$, so $f(-3 + h) = -4$. Substituting:

$$\text{LHD} = \lim_{h \rightarrow 0^-} \frac{-4 - (-3)}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h}.$$

Since $h \rightarrow 0^-$, the denominator is negative, and $\text{LHD} \rightarrow \infty$.

Step 2: Right-hand derivative (RHD).

$$\text{RHD} = \lim_{h \rightarrow 0^+} \frac{f(-3 + h) - f(-3)}{h}.$$

For $h \rightarrow 0^+$, $-3 + h \in (-3, -2)$, so $f(-3 + h) = -3$. Substituting:

$$\text{RHD} = \lim_{h \rightarrow 0^+} \frac{-3 - (-3)}{h} = \lim_{h \rightarrow 0^+} \frac{0}{h} = 0.$$

Step 3: Conclusion. Since $\text{LHD} \neq \text{RHD}$, the function $f(x) = \lfloor x \rfloor$ is not differentiable at $x = -3$.

Quick Tip

The greatest integer function is not differentiable at integer values of x because it has a discontinuity in the derivative at those points. Always evaluate both LHD and RHD to confirm differentiability.

22.(b) If $x^{1/3} + y^{1/3} = 1$, find $\frac{dy}{dx}$ at the point $(\frac{1}{8}, \frac{1}{8})$.

Solution:

Rewrite the equation $x^{1/3} + y^{1/3} = 1$ as:

$$y^{1/3} = 1 - x^{1/3}.$$

Step 1: Differentiate explicitly with respect to x . Differentiating both sides:

$$\frac{1}{3}y^{-2/3} \frac{dy}{dx} = -\frac{1}{3}x^{-2/3}.$$

Rearrange to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x^{-2/3}}{y^{-2/3}}.$$

Step 2: Simplify by writing in terms of y/x . From the original equation, $y^{1/3} = 1 - x^{1/3}$, cube both sides to find y :

$$y = (1 - x^{1/3})^3.$$

Substitute $x = \frac{1}{8}$:

$$y = \left(1 - \left(\frac{1}{8}\right)^{1/3}\right)^3 = \left(1 - \frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

Step 3: Use the ratio of exponents to compute $\frac{dy}{dx}$. Substitute $x = \frac{1}{8}$ and $y = \frac{1}{8}$ into

$$\frac{dy}{dx} = -\frac{x^{-2/3}}{y^{-2/3}}:$$

$$x^{-2/3} = \left(\frac{1}{8}\right)^{-2/3} = 4, \quad y^{-2/3} = \left(\frac{1}{8}\right)^{-2/3} = 4.$$

Thus:

$$\frac{dy}{dx} = -\frac{4}{4} = -1.$$

Hence, $\frac{dy}{dx} = -1$ at the point $\left(\frac{1}{8}, \frac{1}{8}\right)$.

Quick Tip

For explicit differentiation, rewrite the equation in terms of one variable (e.g., $y^{1/3} = 1 - x^{1/3}$) to simplify calculations and substitute the given point only at the end for accuracy.

23. Find the local maximum value and local minimum value (whichever exists) for the function $f(x) = 4x^2 + \frac{1}{x}$ ($x \neq 0$).

Solution:

To find the critical points, we differentiate $f(x)$ with respect to x :

$$f'(x) = \frac{d}{dx} \left(4x^2 + \frac{1}{x}\right) = 8x - \frac{1}{x^2}.$$

Set $f'(x) = 0$ to find the critical points:

$$8x - \frac{1}{x^2} = 0 \quad \Rightarrow \quad 8x^3 = 1 \quad \Rightarrow \quad x = \frac{1}{2}.$$

To determine whether the critical point is a local maximum or minimum, evaluate $f'(x)$ around $x = \frac{1}{2}$:

1. For $x < \frac{1}{2}$ (e.g., $x = \frac{1}{3}$):

$$f'(x) = 8x - \frac{1}{x^2} = 8\left(\frac{1}{3}\right) - \frac{1}{\left(\frac{1}{3}\right)^2} = \frac{8}{3} - 9 = -\frac{19}{3} < 0.$$

2. For $x > \frac{1}{2}$ (e.g., $x = 1$):

$$f'(x) = 8x - \frac{1}{x^2} = 8(1) - \frac{1}{1^2} = 8 - 1 = 7 > 0.$$

Since $f'(x)$ changes from negative to positive at $x = \frac{1}{2}$, the function has a local minimum at $x = \frac{1}{2}$.

Next, calculate the value of $f(x)$ at $x = \frac{1}{2}$:

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 + \frac{1}{\frac{1}{2}} = 4\left(\frac{1}{4}\right) + 2 = 1 + 2 = 3.$$

Conclusion: The function $f(x) = 4x^2 + \frac{1}{x}$ has a local minimum value of 3 at $x = \frac{1}{2}$.

Quick Tip

To determine local maxima and minima, the sign of $f'(x)$ around critical points can help classify them without needing the second derivative.

24. (a) Find:

$$\int x\sqrt{1+2x} \, dx$$

Solution:

Let $I = \int x\sqrt{1+2x} \, dx$. Using substitution, let $u = 1 + 2x$. Then:

$$du = 2 \, dx \quad \text{and} \quad x = \frac{u-1}{2}.$$

Substitute these into the integral:

$$I = \int \frac{u-1}{2} \sqrt{u} \cdot \frac{1}{2} \, du = \frac{1}{4} \int (u-1)u^{\frac{1}{2}} \, du.$$

Simplify:

$$I = \frac{1}{4} \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) \, du.$$

Split the integral:

$$I = \frac{1}{4} \left(\int u^{\frac{3}{2}} \, du - \int u^{\frac{1}{2}} \, du \right).$$

Integrate each term:

$$\int u^{\frac{3}{2}} \, du = \frac{2}{5}u^{\frac{5}{2}}, \quad \int u^{\frac{1}{2}} \, du = \frac{2}{3}u^{\frac{3}{2}}.$$

Substitute back into the expression:

$$I = \frac{1}{4} \left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} \right).$$

Simplify:

$$I = \frac{1}{10}u^{\frac{5}{2}} - \frac{1}{6}u^{\frac{3}{2}}.$$

Finally, substitute $u = 1 + 2x$:

$$I = \frac{1}{10}(1 + 2x)^{\frac{5}{2}} - \frac{1}{6}(1 + 2x)^{\frac{3}{2}} + C.$$

Answer:

$$\int x\sqrt{1 + 2x} dx = \frac{1}{10}(1 + 2x)^{\frac{5}{2}} - \frac{1}{6}(1 + 2x)^{\frac{3}{2}} + C.$$

Quick Tip

For integrals involving $x\sqrt{1 + kx}$, substitution $u = 1 + kx$ simplifies the integral. Always rewrite all terms, including dx , in terms of u to avoid errors.

24. (b) Evaluate:

$$\int_0^{\frac{\pi^2}{4}} \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

Solution:

Let $I = \int_0^{\frac{\pi^2}{4}} \frac{\sin \sqrt{x}}{\sqrt{x}} dx$. Using substitution, let $t = \sqrt{x}$. Then:

$$x = t^2, \quad dx = 2t dt, \quad \sqrt{x} = t.$$

Substitute into the integral, and update the limits of integration:

$$\text{When } x = 0, t = \sqrt{0} = 0.$$

$$\text{When } x = \frac{\pi^2}{4}, t = \sqrt{\frac{\pi^2}{4}} = \frac{\pi}{2}.$$

The integral becomes:

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} \cdot 2t dt = 2 \int_0^{\frac{\pi}{2}} \sin t dt.$$

Simplify:

$$I = 2[-\cos t]_0^{\frac{\pi}{2}}.$$

Evaluate the definite integral:

$$I = 2 \left[-\cos \left(\frac{\pi}{2} \right) + \cos(0) \right].$$

Since $\cos \left(\frac{\pi}{2} \right) = 0$ and $\cos(0) = 1$:

$$I = 2[0 + 1] = 2.$$

Answer:

$$\int_0^{\frac{\pi^2}{4}} \frac{\sin \sqrt{x}}{\sqrt{x}} dx = 2.$$

Quick Tip

For integrals involving functions like $\frac{\sin \sqrt{x}}{\sqrt{x}}$, substitution simplifies both the integrand and the limits of integration. Always verify that the limits are correctly updated after substitution.

25. If \vec{a} and \vec{b} are two non-zero vectors such that $(\vec{a} + \vec{b}) \perp \vec{a}$ and $(2\vec{a} + \vec{b}) \perp \vec{b}$, then prove that $|\vec{b}| = \sqrt{2}|\vec{a}|$.

Correct Answer: $|\vec{b}| = \sqrt{2}|\vec{a}|$.

Solution:

Step 1: Apply Condition 1: $(\vec{a} + \vec{b}) \perp \vec{a}$.

$$(\vec{a} + \vec{b}) \cdot \vec{a} = 0 \quad \Rightarrow \quad \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} = 0.$$

Simplify:

$$|\vec{a}|^2 + \vec{a} \cdot \vec{b} = 0 \quad \Rightarrow \quad \vec{a} \cdot \vec{b} = -|\vec{a}|^2.$$

Step 2: Apply Condition 2: $(2\vec{a} + \vec{b}) \perp \vec{b}$.

$$(2\vec{a} + \vec{b}) \cdot \vec{b} = 0 \quad \Rightarrow \quad 2(\vec{a} \cdot \vec{b}) + |\vec{b}|^2 = 0.$$

Substitute $\vec{a} \cdot \vec{b} = -|\vec{a}|^2$:

$$2(-|\vec{a}|^2) + |\vec{b}|^2 = 0 \quad \Rightarrow \quad -2|\vec{a}|^2 + |\vec{b}|^2 = 0.$$

Step 3: Solve for $|\vec{b}|$. Rearrange the equation:

$$|\vec{b}|^2 = 2|\vec{a}|^2 \quad \Rightarrow \quad |\vec{b}| = \sqrt{2}|\vec{a}|.$$

Conclusion: The magnitude of \vec{b} is $|\vec{b}| = \sqrt{2}|\vec{a}|$.

Quick Tip

To solve vector problems, use perpendicularity conditions ($\vec{u} \perp \vec{v} \implies \vec{u} \cdot \vec{v} = 0$) to establish equations, and simplify using dot product properties.

SECTION C

This section comprises short answer (SA) type questions of 3 marks each.

26. Solve the following linear programming problem graphically: Minimise $z = 5x - 2y$

Subject to the constraints:

$$x + 2y \leq 120,$$

$$x + y \geq 60,$$

$$x - 2y \geq 0,$$

$$x \geq 0,$$

$$y \geq 0.$$

Solution:

To solve this linear programming problem graphically, follow these steps:

Step 1: Plot the constraints. Rewrite the constraints into slope-intercept form to plot: 1.

$x + 2y \leq 120$: $y \leq \frac{120-x}{2}$, passing through $(0, 60)$ and $(120, 0)$. 2. $x + y \geq 60$: $y \geq 60 - x$,

passing through $(0, 60)$ and $(60, 0)$. 3. $x - 2y \geq 0$: $y \leq \frac{x}{2}$, passing through $(0, 0)$ and $(120, 60)$.

4. $x \geq 0, y \geq 0$: Restrict the feasible region to the first quadrant.

Step 2: Identify the feasible region. Plot the lines on the coordinate plane and identify the overlapping region that satisfies all the inequalities.

Step 3: Determine the corner points of the feasible region. Find the intersection points of the constraint lines: 1. Intersection of $x + 2y = 120$ and $x + y = 60$:

$$x + 2y = 120, \quad x + y = 60.$$

Subtract the second equation from the first:

$$y = 60, \quad x = 0.$$

Corner point: $(0, 60)$.

2. Intersection of $x + y = 60$ and $x - 2y = 0$:

$$x + y = 60, \quad x - 2y = 0.$$

Solve for $x = 2y$, substitute:

$$2y + y = 60 \implies y = 20, \quad x = 40.$$

Corner point: $(40, 20)$.

3. Intersection of $x - 2y = 0$ and $x + 2y = 120$:

$$x - 2y = 0, \quad x + 2y = 120.$$

Solve for $x = 2y$, substitute:

$$2y + 2y = 120 \implies y = 30, \quad x = 60.$$

Corner point: $(60, 30)$.

Step 4: Evaluate the objective function at each corner point. The objective function is

$z = 5x - 2y$: 1. At $(0, 60)$:

$$z = 5(0) - 2(60) = -120.$$

2. At $(40, 20)$:

$$z = 5(40) - 2(20) = 200 - 40 = 160.$$

3. At $(60, 30)$:

$$z = 5(60) - 2(30) = 300 - 60 = 240.$$

Step 5: Conclusion. The minimum value of $z = 5x - 2y$ occurs at $(0, 60)$ with $z = -120$.

Quick Tip

In graphical linear programming, always verify each corner point of the feasible region by solving pairs of constraint equations. Evaluate the objective function at all corner points to find the optimal solution.

27. E and F are two independent events such that $P(\bar{E}) = 0.6$ and $P(E \cup F) = 0.6$. Find $P(F)$ and $P(\bar{E} \cup \bar{F})$.

Solution:

Step 1: Find $P(E)$.

$$P(E) = 1 - P(\bar{E}) = 1 - 0.6 = 0.4.$$

Step 2: Use the formula for $P(E \cup F)$.

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

For independent events:

$$P(E \cap F) = P(E) \cdot P(F).$$

Substitute into the formula:

$$0.6 = 0.4 + P(F) - (0.4 \cdot P(F)).$$

Simplify:

$$0.6 = 0.4 + P(F) - 0.4P(F).$$

$$0.2 = P(F)(1 - 0.4).$$

$$0.2 = 0.6P(F).$$

$$P(F) = \frac{0.2}{0.6} = \frac{1}{3}.$$

Step 3: Find $P(\bar{E} \cup \bar{F})$.

$$P(\bar{E} \cup \bar{F}) = P(\bar{E}) + P(\bar{F}) - P(\bar{E} \cap \bar{F}).$$

For independent events:

$$P(\bar{E} \cap \bar{F}) = P(\bar{E}) \cdot P(\bar{F}).$$

Substitute $P(\bar{F}) = 1 - P(F) = 1 - \frac{1}{3} = \frac{2}{3}$:

$$P(\bar{E} \cup \bar{F}) = 0.6 + \frac{2}{3} - (0.6 \cdot \frac{2}{3}).$$

Simplify:

$$P(\bar{E} \cup \bar{F}) = 0.6 + 0.6667 - 0.4 = 0.8667.$$

Conclusion:

$$P(F) = \frac{1}{3}, \quad P(\bar{E} \cup \bar{F}) \approx 0.867.$$

Quick Tip

For problems involving independent events, always use $P(E \cap F) = P(E) \cdot P(F)$ and simplify step by step using the rules for union and intersection probabilities.

28. (a) A relation R on set $A = \{1, 2, 3, 4, 5\}$ is defined as:

$$R = \{(x, y) : |x^2 - y^2| < 8\}.$$

Check whether the relation R is reflexive, symmetric, and transitive.

Solution:

Reflexive: A relation is reflexive if $(x, x) \in R$ for all $x \in A$. For $x \in A$, we need to check $|x^2 - x^2| < 8$:

$$|x^2 - x^2| = 0, \quad \text{and } 0 < 8 \quad \forall x \in A.$$

Thus, $(x, x) \in R$, and the relation is **reflexive**.

Symmetric: A relation is symmetric if $(x, y) \in R$ implies $(y, x) \in R$. Here, $|x^2 - y^2| < 8$ implies $|y^2 - x^2| < 8$, because absolute value satisfies $|x^2 - y^2| = |y^2 - x^2|$. Thus, the relation is **symmetric**.

Transitive: A relation is transitive if $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$. We need to verify whether $|x^2 - z^2| < 8$ when $|x^2 - y^2| < 8$ and $|y^2 - z^2| < 8$.

Consider $x = 1, y = 2, z = 3$:

$$|x^2 - y^2| = |1 - 4| = 3 < 8, \quad |y^2 - z^2| = |4 - 9| = 5 < 8.$$

But:

$$|x^2 - z^2| = |1 - 9| = 8 \not< 8.$$

Hence, the relation is **not transitive**.

Answer: The relation R is reflexive and symmetric, but not transitive.

Quick Tip

For reflexivity, check whether $|x^2 - x^2| = 0$. For symmetry, verify $|x^2 - y^2| = |y^2 - x^2|$. For transitivity, counterexamples can help confirm when the property does not hold.

28. (b) A function f is defined from $R \rightarrow R$ as $f(x) = ax + b$, such that $f(1) = 1$ and $f(2) = 3$. Find the function $f(x)$. Hence, check whether the function $f(x)$ is one-one and onto.

Solution:

From the given conditions, we have two equations:

$$f(1) = a(1) + b = 1 \quad \Rightarrow \quad a + b = 1, \quad (1)$$

$$f(2) = a(2) + b = 3 \quad \Rightarrow \quad 2a + b = 3. \quad (2)$$

Solve equations (1) and (2) simultaneously: - From equation (1):

$$b = 1 - a.$$

- Substitute $b = 1 - a$ into equation (2):

$$2a + (1 - a) = 3 \quad \Rightarrow \quad 2a + 1 - a = 3 \quad \Rightarrow \quad a = 2.$$

- Substitute $a = 2$ into equation (1):

$$2 + b = 1 \quad \Rightarrow \quad b = -1.$$

Thus, the function is:

$$f(x) = 2x - 1.$$

Check if $f(x)$ is one-one (Injective): A function is one-one if distinct inputs produce distinct outputs. For $f(x) = 2x - 1$, consider $f(x_1) = f(x_2)$:

$$2x_1 - 1 = 2x_2 - 1 \quad \Rightarrow \quad x_1 = x_2.$$

Thus, $f(x)$ is one-one.

Check if $f(x)$ is onto (Surjective): A function is onto if for every $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) = y$. For $f(x) = 2x - 1$, solve $y = 2x - 1$ for x :

$$x = \frac{y + 1}{2}.$$

Since x exists for all $y \in \mathbb{R}$, the function is onto.

Answer: The function $f(x) = 2x - 1$ is both one-one and onto.

Quick Tip

For linear functions of the form $f(x) = ax + b$: 1. Use the given points to solve for a and b . 2. Check one-one by verifying that $f(x_1) = f(x_2)$ implies $x_1 = x_2$. 3. Check onto by solving $f(x) = y$ for all $y \in \mathbb{R}$.

29(a). If $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$, prove that $\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}}$.

Solution: We begin by differentiating the given equation implicitly with respect to x :

$$\frac{d}{dx} \left(\sqrt{1 - x^2} + \sqrt{1 - y^2} \right) = \frac{d}{dx} (a(x - y)).$$

Applying the chain rule, we get:

$$\frac{-x}{\sqrt{1-x^2}} + \frac{-y}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = a \left(1 - \frac{dy}{dx} \right).$$

Next, we rearrange the terms:

$$\frac{-x}{\sqrt{1-x^2}} - a = \frac{dy}{dx} \left(a - \frac{y}{\sqrt{1-y^2}} \right).$$

Now, solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{-x}{\sqrt{1-x^2}} - a}{a - \frac{y}{\sqrt{1-y^2}}}.$$

Finally, for the special case where $a = 0$, the expression simplifies to:

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.$$

Conclusion: The result has been proven.

Quick Tip

Use implicit differentiation carefully and isolate $\frac{dy}{dx}$ by simplifying both sides of the equation. Substitutions such as $a = 0$ can help simplify expressions further.

29(b). If $y = (\tan x)^x$, then find $\frac{dy}{dx}$.

Solution:

Let $y = (\tan x)^x$. Take the natural logarithm on both sides:

$$\ln y = \ln((\tan x)^x).$$

Using the logarithmic property $\ln(a^b) = b \ln a$:

$$\ln y = x \ln(\tan x).$$

Differentiate both sides with respect to x using the chain rule:

$$\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} (x \ln(\tan x)).$$

Use the product rule for the right-hand side:

$$\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx}(x) \cdot \ln(\tan x) + x \cdot \frac{d}{dx}(\ln(\tan x)).$$

Simplify:

$$\frac{1}{y} \frac{dy}{dx} = \ln(\tan x) + x \cdot \frac{1}{\tan x} \cdot \sec^2 x.$$

Multiply through by y :

$$\frac{dy}{dx} = y \left(\ln(\tan x) + x \cdot \frac{\sec^2 x}{\tan x} \right).$$

Substitute $y = (\tan x)^x$:

$$\frac{dy}{dx} = (\tan x)^x \left(\ln(\tan x) + x \cdot \frac{\sec^2 x}{\tan x} \right).$$

Final Answer:

$$\frac{dy}{dx} = (\tan x)^x \left(\ln(\tan x) + x \cdot \frac{\sec^2 x}{\tan x} \right).$$

Quick Tip

For functions in the form $y = f(x)^{g(x)}$, take the natural logarithm to simplify the power and use the product rule for differentiation. Substitute back the original function at the end.

30(a). Find:

$$\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx.$$

Solution:

Step 1: Decompose into partial fractions. We rewrite the integrand as:

$$\frac{x^2}{(x^2 + 4)(x^2 + 9)} = \frac{A}{x^2 + 4} + \frac{B}{x^2 + 9}.$$

Multiply through by $(x^2 + 4)(x^2 + 9)$:

$$x^2 = A(x^2 + 9) + B(x^2 + 4).$$

Expand and simplify:

$$x^2 = Ax^2 + 9A + Bx^2 + 4B.$$

Combine terms:

$$x^2 = (A + B)x^2 + (9A + 4B).$$

Equating coefficients:

$$A + B = 1, \quad 9A + 4B = 0. \tag{1}$$

Step 2: Solve for A and B. From $A + B = 1$, we get $B = 1 - A$. Substitute $B = 1 - A$ into $9A + 4B = 0$:

$$9A + 4(1 - A) = 0.$$

$$9A + 4 - 4A = 0 \Rightarrow 5A = -4 \Rightarrow A = -\frac{4}{5}.$$

Substitute $A = -\frac{4}{5}$ into $B = 1 - A$:

$$B = 1 - \left(-\frac{4}{5}\right) = 1 + \frac{4}{5} = \frac{9}{5}.$$

Thus, the partial fraction decomposition is:

$$\frac{x^2}{(x^2 + 4)(x^2 + 9)} = \frac{-\frac{4}{5}}{x^2 + 4} + \frac{\frac{9}{5}}{x^2 + 9}.$$

Step 3: Integrate each term.

$$\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx = -\frac{4}{5} \int \frac{1}{x^2 + 4} dx + \frac{9}{5} \int \frac{1}{x^2 + 9} dx.$$

Use the standard integral formula:

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right).$$

Substitute $a = 2$ for the first term and $a = 3$ for the second term:

$$\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx = -\frac{4}{5} \cdot \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + \frac{9}{5} \cdot \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right).$$

Simplify:

$$\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx = -\frac{2}{5} \tan^{-1} \left(\frac{x}{2} \right) + \frac{3}{5} \tan^{-1} \left(\frac{x}{3} \right) + C.$$

Final Answer:

$$\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx = -\frac{2}{5} \tan^{-1} \left(\frac{x}{2} \right) + \frac{3}{5} \tan^{-1} \left(\frac{x}{3} \right) + C.$$

Quick Tip

For integrals involving partial fractions, decompose the rational expression, solve for coefficients, and use standard integral formulas like $\int \frac{1}{x^2 + a^2} dx$.

30(b). Evaluate:

$$\int_1^3 (|x - 1| + |x - 2| + |x - 3|) dx.$$

Solution:

The given integral involves absolute values, so split the integration range based on the points where the arguments of the absolute values change sign: $x = 1$, $x = 2$, and $x = 3$.

Step 1: Break the integral into intervals.

$$\int_1^3 (|x-1| + |x-2| + |x-3|) dx = \int_1^2 (x-1 + 2-x + 3-x) dx + \int_2^3 (x-1 + x-2 + 3-x) dx.$$

Simplify the terms in each interval: - For $x \in [1, 2]$:

$$|x-1| = x-1, \quad |x-2| = 2-x, \quad |x-3| = 3-x.$$

$$|x-1| + |x-2| + |x-3| = x-1 + 2-x + 3-x = 4-x.$$

- For $x \in [2, 3]$:

$$|x-1| = x-1, \quad |x-2| = x-2, \quad |x-3| = 3-x.$$

$$|x-1| + |x-2| + |x-3| = x-1 + x-2 + 3-x = x.$$

Step 2: Evaluate the integrals.

$$\int_1^3 (|x-1| + |x-2| + |x-3|) dx = \int_1^2 (4-x) dx + \int_2^3 x dx.$$

- First integral:

$$\begin{aligned} \int_1^2 (4-x) dx &= \left[4x - \frac{x^2}{2} \right]_1^2 = \left(4(2) - \frac{2^2}{2} \right) - \left(4(1) - \frac{1^2}{2} \right) \\ &= (8-2) - (4-0.5) = 6-3.5 = 2.5. \end{aligned}$$

- Second integral:

$$\int_2^3 x dx = \left[\frac{x^2}{2} \right]_2^3 = \frac{3^2}{2} - \frac{2^2}{2} = \frac{9}{2} - \frac{4}{2} = \frac{5}{2}.$$

Step 3: Add the results.

$$\int_1^3 (|x-1| + |x-2| + |x-3|) dx = 2.5 + 2.5 = 5.$$

Final Answer:

$$\int_1^3 (|x-1| + |x-2| + |x-3|) dx = 5.$$

Quick Tip

For integrals involving absolute values, split the integral at points where the arguments of the absolute values change sign, simplify for each interval, and evaluate separately.

31. Solve the following differential equation:

$$(\tan^{-1} y - x) dy = (1 + y^2) dx.$$

Solution:

Rearrange the given equation to separate the variables x and y :

$$\frac{\tan^{-1} y - x}{1 + y^2} dy = dx.$$

Rewrite as:

$$\frac{\tan^{-1} y}{1 + y^2} dy - x dx = 0.$$

Step 1: Break the equation into simpler components. Rewrite the left-hand side and rearrange the terms for easier integration:

$$\frac{\tan^{-1} y}{1 + y^2} dy = x dx.$$

Step 2: Integrate the components separately. - For the y -dependent term:

$$\int \frac{\tan^{-1} y}{1 + y^2} dy.$$

Let $u = \tan^{-1} y$, so $du = \frac{1}{1+y^2} dy$:

$$\int \frac{\tan^{-1} y}{1 + y^2} dy = \int u du = \frac{u^2}{2} = \frac{(\tan^{-1} y)^2}{2}.$$

- For the x -dependent term:

$$\int x dx = \frac{x^2}{2}.$$

Step 3: Combine the results and introduce the constant of integration. The equation becomes:

$$\frac{(\tan^{-1} y)^2}{2} = \frac{x^2}{2} + C,$$

where C is the constant of integration.

Multiply through by 2 to eliminate the fraction:

$$(\tan^{-1} y)^2 = x^2 + 2C.$$

Let $2C = C_1$, so the solution becomes:

$$(\tan^{-1} y)^2 - x^2 = C_1.$$

Final Solution:

$$(\tan^{-1} y)^2 - x^2 = C_1,$$

where C_1 is the constant of integration.

Quick Tip

For differential equations involving trigonometric functions, simplify using substitutions where necessary. Identify terms that can be integrated directly and handle complex expressions step by step.

SECTION D

This section comprises long answer type questions (LA) of 5 marks each.

32. Find the equation of a line l_2 which is the mirror image of the line l_1 with respect to line $l : \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$, given that line l_1 passes through the point $P(1, 6, 3)$ and is parallel to line l .

Solution:

1. Parametric form of the given line l : The given line l is:

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}.$$

Its parametric equations are:

$$x = t, \quad y = 1 + 2t, \quad z = 2 + 3t,$$

where t is the parameter.

2. Direction ratios of line l : The direction ratios (DRs) of line l are $(1, 2, 3)$.

3. Equation of line l_1 : Line l_1 passes through the point $P(1, 6, 3)$ and is parallel to l , so its DRs are also $(1, 2, 3)$. The equation of l_1 is:

$$\frac{x-1}{1} = \frac{y-6}{2} = \frac{z-3}{3}.$$

4. Find the foot of the perpendicular from $P(1, 6, 3)$ to l : Let the foot of the perpendicular be $Q(t, 1 + 2t, 2 + 3t)$ on line l . The line joining $P(1, 6, 3)$ to $Q(t, 1 + 2t, 2 + 3t)$ is perpendicular to line l . The DRs of PQ are $(t - 1, (1 + 2t) - 6, (2 + 3t) - 3) = (t - 1, 2t - 5, 3t - 1)$.

Using the condition for perpendicularity of two lines, $DR_{PQ} \cdot DR_l = 0$:

$$(t - 1)(1) + (2t - 5)(2) + (3t - 1)(3) = 0.$$

Simplify:

$$t - 1 + 4t - 10 + 9t - 3 = 0.$$

$$14t - 14 = 0 \quad \Rightarrow \quad t = 1.$$

Substitute $t = 1$ into $Q(t, 1 + 2t, 2 + 3t)$:

$$Q(1, 1 + 2(1), 2 + 3(1)) = Q(1, 3, 5).$$

5. Reflection of $P(1, 6, 3)$ about $Q(1, 3, 5)$: The reflection point $P'(x', y', z')$ of $P(x_1, y_1, z_1)$ across $Q(x_2, y_2, z_2)$ is:

$$x' = 2x_2 - x_1, \quad y' = 2y_2 - y_1, \quad z' = 2z_2 - z_1.$$

Substitute $P(1, 6, 3)$ and $Q(1, 3, 5)$:

$$x' = 2(1) - 1 = 1, \quad y' = 2(3) - 6 = 0, \quad z' = 2(5) - 3 = 7.$$

Thus, the reflected point is $P'(1, 0, 7)$.

6. Equation of line l_2 : Line l_2 passes through $P'(1, 0, 7)$ and is parallel to l , so its DRs are $(1, 2, 3)$. The equation of l_2 is:

$$\frac{x - 1}{1} = \frac{y}{2} = \frac{z - 7}{3}.$$

Final Solution:

$$\frac{x - 1}{1} = \frac{y}{2} = \frac{z - 7}{3}.$$

Quick Tip

To find the mirror image of a line, first calculate the foot of the perpendicular using parametric equations, then reflect the given point across this foot and write the equation of the reflected line.

33. (a) If $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$, find A^{-1} and use it to solve the following system of

equations:

$$x - 2y = 10, \quad 2x - y - z = 8, \quad -2y + z = 7.$$

Solution: Given a system of linear equations $AX = B$, where:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & -1 \\ -4 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix},$$

we are tasked with finding the solution vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Step 1: Determining if A^{-1} Exists

First, calculate the determinant of A :

$$|A| = 1(0 \cdot 3 - (-1)(2)) - 2(2 \cdot 3 - (-1)(-4)) + (-1)(2 \cdot 2 - 0 \cdot (-4)),$$

$$|A| = 1(2) - 2(6 - 4) - 1(4),$$

$$|A| = 1 \neq 0.$$

Since $|A| \neq 0$, the matrix A is invertible, and A^{-1} exists.

Step 2: Finding A^{-1}

The adjugate ($\text{Adj } A$) is calculated as:

$$\text{Adj } A = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}.$$

The inverse of A is given by:

$$A^{-1} = \frac{1}{|A|} \text{Adj } A = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}.$$

Step 3: Solving $AX = B$

The solution to the system is obtained using the formula $X = A^{-1}B$:

$$X = A^{-1}B = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}.$$

Perform the matrix multiplication:

$$X = \begin{bmatrix} (-3)(10) + (2)(8) + (2)(7) \\ (-2)(10) + (1)(8) + (1)(7) \\ (-4)(10) + (2)(8) + (3)(7) \end{bmatrix},$$
$$X = \begin{bmatrix} -30 + 16 + 14 \\ -20 + 8 + 7 \\ -40 + 16 + 21 \end{bmatrix},$$
$$X = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}.$$

Final Answer:

$$x = 0, \quad y = -5, \quad z = -3.$$

Quick Tip

When solving linear equations using matrices, verify $\det(A) \neq 0$ to ensure the inverse exists. Compute $\text{adj}(A)$ accurately, as errors in cofactors can lead to incorrect results.

33. (b) If $A = \begin{bmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{bmatrix}$, find the value of $(a + x) - (b + y)$.

Solution:

Step 1: Use the inverse matrix property. The relationship between A and its inverse is given by:

$$A \cdot A^{-1} = I_3,$$

where I_3 is the identity matrix.

Step 2: Compute $A \cdot A^{-1}$ row-by-row.

$$\begin{bmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Analyze the rows of the product one by one.

Step 3: Row 1 of the product. The first row of A multiplied with the columns of A^{-1} gives:

$$[-1(1) + a(-8) + 2(b)] = 1, \quad [-1(-1) + a(7) + 2(y)] = 0, \quad [-1(1) + a(-5) + 2(3)] = 0.$$

Simplify: 1. $-1 - 8a + 2b = 1 \implies -8a + 2b = 2 \implies 4a - b = -1$. 2.

$$1 + 7a + 2y = 0 \implies 7a + 2y = -1. \quad 3. \quad -1 - 5a + 6 = 0 \implies -5a = -5 \implies a = 1.$$

Step 4: Row 2 of the product. The second row of A multiplied with the columns of A^{-1} gives:

$$[1(1) + 2(-8) + x(b)] = 0, \quad [1(-1) + 2(7) + x(y)] = 1, \quad [1(1) + 2(-5) + x(3)] = 0.$$

Simplify: 1. $1 - 16 + xb = 0 \implies xb = 15$. 2. $-1 + 14 + xy = 1 \implies xy = -12$. 3.

$$1 - 10 + 3x = 0 \implies 3x = 9 \implies x = 3.$$

Step 5: Row 3 of the product. The third row of A multiplied with the columns of A^{-1} gives:

$$[3(1) + 1(-8) + 1(b)] = 0, \quad [3(-1) + 1(7) + 1(y)] = 0, \quad [3(1) + 1(-5) + 1(3)] = 1.$$

Simplify: 1. $3 - 8 + b = 0 \implies b = 5$. 2. $-3 + 7 + y = 0 \implies y = -4$.

Step 6: Compute $(a + x) - (b + y)$. Substitute $a = 1$, $x = 3$, $b = 5$, and $y = -4$:

$$(a + x) - (b + y) = (1 + 3) - (5 + (-4)).$$

Simplify:

$$(a + x) - (b + y) = 4 - (5 - 4) = 4 - 1 = 3.$$

Final Answer:

$$(a + x) - (b + y) = 3.$$

Quick Tip

When solving for unknowns in matrix equations, carefully compute row-wise products and equate the resulting elements to the identity matrix to form equations for the unknowns.

34(a). Find:

$$\int \frac{(3 \cos x - 2) \sin x}{5 - \sin^2 x - 4 \cos x} dx.$$

Solution:

Step 1: Simplify the denominator.

Let the denominator $D = 5 - \sin^2 x - 4 \cos x$. Using $\sin^2 x = 1 - \cos^2 x$, substitute this into D :

$$D = 5 - (1 - \cos^2 x) - 4 \cos x = 5 - 1 + \cos^2 x - 4 \cos x = \cos^2 x - 4 \cos x + 4.$$

Thus, the integral becomes:

$$\int \frac{(3 \cos x - 2) \sin x}{\cos^2 x - 4 \cos x + 4} dx.$$

Step 2: Substitution.

Let $u = \cos x$, so $du = -\sin x dx$. Substitute into the integral:

$$\int \frac{(3 \cos x - 2) \sin x}{\cos^2 x - 4 \cos x + 4} dx = - \int \frac{3u - 2}{u^2 - 4u + 4} du.$$

Factor the denominator:

$$u^2 - 4u + 4 = (u - 2)^2.$$

The integral becomes:

$$- \int \frac{3u - 2}{(u - 2)^2} du.$$

Step 3: Simplify the fraction.

Split the numerator $3u - 2$ as:

$$3u - 2 = 3(u - 2) + 4.$$

Thus, the integral becomes:

$$- \int \frac{3(u - 2)}{(u - 2)^2} du - \int \frac{4}{(u - 2)^2} du.$$

Simplify each term: - For the first term:

$$- \int \frac{3(u - 2)}{(u - 2)^2} du = - \int \frac{3}{u - 2} du = -3 \ln |u - 2|.$$

- For the second term:

$$- \int \frac{4}{(u - 2)^2} du = - \int 4(u - 2)^{-2} du.$$

Using the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$ for $n = -2$:

$$- \int 4(u - 2)^{-2} du = \frac{4}{u - 2}.$$

Step 4: Substitute back $u = \cos x$.

Combine the results:

$$\int \frac{(3 \cos x - 2) \sin x}{5 - \sin^2 x - 4 \cos x} dx = -3 \ln |\cos x - 2| + \frac{4}{\cos x - 2} + C.$$

Final Answer:

$$\int \frac{(3 \cos x - 2) \sin x}{5 - \sin^2 x - 4 \cos x} dx = -3 \ln |\cos x - 2| + \frac{4}{\cos x - 2} + C.$$

Quick Tip

When integrating functions involving trigonometric terms, use substitutions like $u = \cos x$ or $u = \sin x$ to simplify the integrand. Factorize the denominator to break the fraction into simpler terms.

34 (b). Evaluate:

$$\int_{-2}^2 \frac{x^3 + |x| + 1}{x^2 + 4|x| + 4} dx.$$

Solution: Let's break down the provided solution into a step-by-step explanation with careful attention to detail:

Integral Expression

We start with the following expression:

$$\int_{-2}^2 \frac{x^3}{x^2 + 4|x| + 4} dx + \int_{-2}^2 \frac{|x| + 1}{x^2 + 4|x| + 4} dx$$

We will denote these two integrals as I_1 and I_2 :

$$I_1 = \int_{-2}^2 \frac{x^3}{x^2 + 4|x| + 4} dx \quad \text{and} \quad I_2 = \int_{-2}^2 \frac{|x| + 1}{x^2 + 4|x| + 4} dx$$

Now, we analyze the integrals separately.

Step 1: Evaluating I_1

We begin by focusing on I_1 :

$$I_1 = \int_{-2}^2 \frac{x^3}{x^2 + 4|x| + 4} dx$$

Notice that the integrand involves x^3 , and x^3 is an odd function. Recall that an **odd function** is defined as a function $f(x)$ where:

$$f(-x) = -f(x)$$

For odd functions, the integral over a symmetric interval $[-a, a]$ is zero because the contributions from $x = -a$ to $x = 0$ cancel out the contributions from $x = 0$ to $x = a$. Therefore, I_1 simplifies as follows:

$$I_1 = 0 \quad (\text{since } \frac{x^3}{x^2 + 4|x| + 4} \text{ is an odd function})$$

Step 2: Evaluating I_2

Next, we turn to I_2 :

$$I_2 = \int_{-2}^2 \frac{|x| + 1}{x^2 + 4|x| + 4} dx$$

In this case, the integrand involves $|x|$, and the expression $\frac{|x|+1}{x^2+4|x|+4}$ is an **even function** because $\frac{|x|+1}{x^2+4|x|+4}$ is symmetric in x . That is, for even functions, $f(x) = f(-x)$.

We can exploit the symmetry of the integral by transforming it as follows:

$$I_2 = 2 \int_0^2 \frac{x + 1}{x^2 + 4x + 4} dx$$

This transformation effectively reduces the limits of integration from $[-2, 2]$ to $[0, 2]$, and the factor of 2 accounts for the symmetry.

Next, we simplify the integrand:

$$x^2 + 4x + 4 = (x + 2)^2$$

Thus, the integral becomes:

$$I_2 = 2 \int_0^2 \frac{x + 1}{(x + 2)^2} dx$$

Step 3: Substituting for Simplicity

To simplify the integration, we perform the substitution:

$$x + 2 = t \quad \text{so that} \quad dx = dt$$

When $x = 0$, $t = 2$, and when $x = 2$, $t = 4$. Therefore, the limits of integration change from $x = 0$ to $x = 2$ to $t = 2$ to $t = 4$. Substituting into the integral, we get:

$$I_2 = 2 \int_2^4 \frac{t - 1}{t^2} dt$$

Now, split the integrand:

$$I_2 = 2 \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2} \right) dt$$

Step 4: Solving the Integral

Now we solve the two integrals separately:

$$I_2 = 2 \left[\int_2^4 \frac{1}{t} dt - \int_2^4 \frac{1}{t^2} dt \right]$$

For the first integral, we know that:

$$\int \frac{1}{t} dt = \log |t|$$

For the second integral:

$$\int \frac{1}{t^2} dt = -\frac{1}{t}$$

Thus, we have:

$$I_2 = 2 \left[\log |t| - \frac{1}{t} \right]_2^4$$

Now, evaluate at the limits $t = 4$ and $t = 2$:

$$I_2 = 2 \left[\log 4 - \frac{1}{4} - \left(\log 2 - \frac{1}{2} \right) \right]$$

Simplify:

$$I_2 = 2 \left[\log 4 - \log 2 + \frac{1}{2} - \frac{1}{4} \right]$$

Using $\log 4 = 2 \log 2$, we get:

$$I_2 = 2 \left[2 \log 2 - \log 2 + \frac{1}{2} - \frac{1}{4} \right]$$

Simplifying further:

$$I_2 = 2 \left[\log 2 + \frac{1}{2} - \frac{1}{4} \right]$$

$$I_2 = 2 \log 2 + \frac{1}{2}$$

Final Answer

Thus, the total integral becomes:

$$I_1 + I_2 = 0 + \left(2 \log 2 - \frac{1}{2} \right)$$

So, the final result is:

$$I_1 + I_2 = 2 \log 2 - \frac{1}{2}$$

This is the value of the original expression.

Quick Tip

For integrals involving absolute values and piecewise functions, break the integral into intervals where the function behavior changes and solve each part separately.

35. Using integration, find the area of the ellipse:

$$\frac{x^2}{16} + \frac{y^2}{4} = 1,$$

included between the lines $x = -2$ **and** $x = 2$.

Solution: We are tasked with finding the area of a given curve.

Step 1: Defining the Integral

The given equation is:

$$\text{Area} = 4 \int_0^2 y \, dx$$

The factor of 4 comes from symmetry: the curve described is symmetric with respect to the x-axis, so we only need to compute the area from 0 to 2 and then multiply it by 4.

Step 2: Expressing y as a Function of x

Next, we need to express y in terms of x . The function y is given by:

$$y = \sqrt{4^2 - x^2}$$

This is the equation of a semicircle with radius 4, centered at the origin.

Thus, the area integral becomes:

$$\text{Area} = 4 \int_0^2 \sqrt{4^2 - x^2} \, dx$$

Step 3: Simplifying the Integral

The expression $4^2 - x^2$ under the square root suggests that we are dealing with a standard integral form related to the area of a circle. To simplify, we divide the integral by 2:

$$\text{Area} = 4 \left[\frac{1}{2} \int_0^2 \sqrt{4^2 - x^2} \, dx \right]$$

Now, we focus on solving the integral inside the brackets. This integral is of the form of a standard trigonometric substitution.

Step 4: Trigonometric Substitution

Let's perform a trigonometric substitution. Set:

$$x = 4 \sin \theta$$

Then:

$$dx = 4 \cos \theta \, d\theta$$

Also, note that the limits of integration change: - When $x = 0$, $\theta = 0$ - When $x = 2$,

$$\theta = \sin^{-1} \left(\frac{2}{4} \right) = \sin^{-1} \left(\frac{1}{2} \right) = \frac{\pi}{6}$$

Substituting into the integral:

$$\int_0^2 \sqrt{4^2 - x^2} dx = \int_0^{\frac{\pi}{6}} \sqrt{16 - 16 \sin^2 \theta} \cdot 4 \cos \theta d\theta$$

Simplify the square root:

$$\sqrt{16 - 16 \sin^2 \theta} = \sqrt{16 \cos^2 \theta} = 4 \cos \theta$$

Thus, the integral becomes:

$$\int_0^{\frac{\pi}{6}} 4 \cos \theta \cdot 4 \cos \theta d\theta = 16 \int_0^{\frac{\pi}{6}} \cos^2 \theta d\theta$$

Step 5: Using the Trigonometric Identity

We can now simplify $\cos^2 \theta$ using the identity:

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

Thus, the integral becomes:

$$16 \int_0^{\frac{\pi}{6}} \frac{1 + \cos(2\theta)}{2} d\theta$$

Now, split the integral into two parts:

$$8 \int_0^{\frac{\pi}{6}} 1 d\theta + 8 \int_0^{\frac{\pi}{6}} \cos(2\theta) d\theta$$

The first part:

$$8 \int_0^{\frac{\pi}{6}} 1 d\theta = 8 \cdot \frac{\pi}{6} = \frac{4\pi}{3}$$

The second part:

$$8 \int_0^{\frac{\pi}{6}} \cos(2\theta) d\theta = 8 \left[\frac{\sin(2\theta)}{2} \right]_0^{\frac{\pi}{6}} = 4 \left[\sin\left(\frac{\pi}{3}\right) - \sin(0) \right] = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

Thus, the total integral is:

$$\frac{4\pi}{3} + 2\sqrt{3}$$

Step 6: Final Calculation

Finally, multiply the integral by 4 (since we divided it by 2 earlier), and we get:

$$\text{Area} = 4 \left[\frac{4\pi}{3} + 2\sqrt{3} \right] = \frac{16\pi}{3} + 8\sqrt{3}$$

This is the final result for the area under the curve.

Summary

The area under the curve is:

$$4\sqrt{3} + \frac{8\pi}{3}$$

This completes the solution with detailed explanations at each step.

Quick Tip

For finding areas of symmetric regions like ellipses, reduce the limits of integration by using symmetry. Use substitutions to simplify complex square root terms.

SECTION E

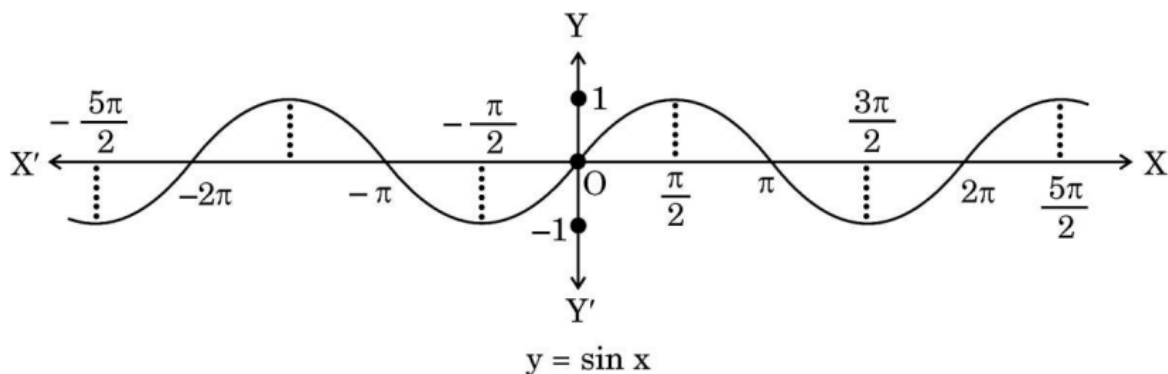
This section comprises 3 case study based questions of 4 marks each.

Case Study - 1

36. If a function $f : X \rightarrow Y$ defined as $f(x) = y$ is one-one and onto, then we can define a unique function $g : Y \rightarrow X$ such that $g(y) = x$, where $x \in X$ and $y = f(x)$, $y \in Y$.

Function g is called the inverse of function f .

The domain of sine function is \mathbb{R} and function $\text{sine} : \mathbb{R} \rightarrow \mathbb{R}$ is neither one-one nor onto. The following graph shows the sine function.



Let sine function be defined from set A to $[-1, 1]$ such that inverse of sine function exists, i.e., $\sin^{-1} x$ is defined from $[-1, 1]$ to A .

On the basis of the above information, answer the following questions:

If A is the interval other than principal value branch, give an example of one such interval.

If $\sin^{-1}(x)$ is defined from $[-1, 1]$ to its principal value branch, find the value of $\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1)$.

Draw the graph of $\sin^{-1} x$ from $[-1, 1]$ to its principal value branch.

OR Find the domain and range of $f(x) = 2 \sin^{-1}(1 - x)$.

Solution: (i) Interval other than the principal value branch: The principal value branch of the sine function is the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. An alternative interval where the sine function is one-one and onto $[-1, 1]$ is:

$$A = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

(ii) Compute $\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1)$:

Step 1: Evaluate $\sin^{-1}\left(-\frac{1}{2}\right)$: The angle θ such that $\sin \theta = -\frac{1}{2}$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is:

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

Step 2: Evaluate $\sin^{-1}(1)$: The angle θ such that $\sin \theta = 1$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is:

$$\sin^{-1}(1) = \frac{\pi}{2}.$$

Step 3: Subtract the values:

$$\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1) = -\frac{\pi}{6} - \frac{\pi}{2}.$$

Simplify:

$$-\frac{\pi}{6} - \frac{\pi}{2} = -\frac{\pi}{6} - \frac{3\pi}{6} = -\frac{4\pi}{6} = -\frac{2\pi}{3}.$$

(iii) (a) Graph of $\sin^{-1}(x)$: The graph of $y = \sin^{-1}(x)$ maps $[-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$. It is the reflection of $y = \sin x$ (restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$) across the line $y = x$.

(iii) (b) Domain and range of $f(x) = 2 \sin^{-1}(1 - x)$:

Step 1: Domain of $f(x)$: For $\sin^{-1}(1 - x)$ to be defined, $1 - x$ must lie in $[-1, 1]$:

$$-1 \leq 1 - x \leq 1.$$

Simplify:

$$-1 - 1 \leq -x \leq 1 - 1 \implies -2 \leq -x \leq 0.$$

Multiplying through by -1 (and reversing the inequality):

$$0 \leq x \leq 2.$$

Thus, the domain is:

$$x \in [0, 2].$$

Step 2: Range of $f(x)$: The range of $\sin^{-1}(x)$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Scaling by 2, the range of $f(x)$ becomes:

$$f(x) \in \left[2 \cdot -\frac{\pi}{2}, 2 \cdot \frac{\pi}{2}\right] = [-\pi, \pi].$$

Final Answers: 1. An example of an interval other than the principal value branch is:

$$A = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

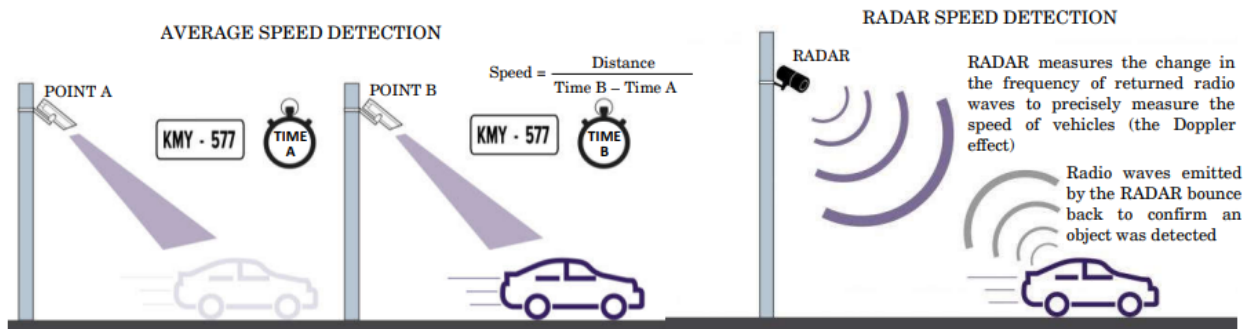
2. $\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1) = -\frac{2\pi}{3}$. 3. For $f(x) = 2\sin^{-1}(1-x)$: - Domain: $[0, 2]$, - Range: $[-\pi, \pi]$.

Quick Tip

To handle inverse trigonometric functions, remember their restricted domains and ranges. For transformations like scaling and shifts, carefully adjust the domain and range accordingly.

Case Study - 2

37. The traffic police has installed Over Speed Violation Detection (OSVD) system at various locations in a city. These cameras can capture a speeding vehicle from a distance of 300 m and even function in the dark.



A camera is installed on a pole at the height of 5 m. It detects a car travelling away from the pole at the speed of 20 m/s. At any point, x m away from the base of the pole, the angle of elevation of the speed camera from the car C is θ .

On the basis of the above information, answer the following questions:

(i) Express θ in terms of the height of the camera installed on the pole and x .

(ii) Find $\frac{d\theta}{dx}$.

(iii) (a) Find the rate of change of angle of elevation with respect to time at an instant when the car is 50 m away from the pole.

(iii) (b) If the rate of change of angle of elevation with respect to time of another car at a distance of 50 m from the base of the pole is $\frac{3}{101}$ rad/s, then find the speed of the car.

Solution: A camera is installed on a pole at a height of 5 m. It tracks a car traveling away from the pole at a speed of 20 m/s. The car is at a horizontal distance of x m from the base of the pole, and the angle of elevation to the car is θ .

(i) Express θ in terms of x : Using the triangle formed by the height of the pole and the horizontal distance x :

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{x}.$$

Thus:

$$\theta = \tan^{-1} \left(\frac{5}{x} \right).$$

(ii) Find $\frac{d\theta}{dx}$: Differentiate $\theta = \tan^{-1} \left(\frac{5}{x} \right)$ with respect to x :

$$\frac{d\theta}{dx} = \frac{1}{1 + \left(\frac{5}{x} \right)^2} \cdot \frac{d}{dx} \left(\frac{5}{x} \right).$$

Simplify:

$$\frac{d}{dx} \left(\frac{5}{x} \right) = -\frac{5}{x^2}.$$

Substitute:

$$\frac{d\theta}{dx} = \frac{1}{1 + \frac{25}{x^2}} \cdot \left(-\frac{5}{x^2} \right).$$

Simplify further:

$$\frac{d\theta}{dx} = \frac{-5}{x^2 + 25}.$$

(iii) (a) Rate of change of angle of elevation ($\frac{d\theta}{dt}$) when $x = 50$: The car's speed is

$\frac{dx}{dt} = 20$ m/s. The rate of change of θ with respect to time is:

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}.$$

From (ii), $\frac{d\theta}{dx} = \frac{-5}{x^2+25}$. At $x = 50$:

$$\frac{d\theta}{dx} = \frac{-5}{50^2 + 25} = \frac{-5}{2525} = \frac{-1}{505}.$$

Substitute:

$$\frac{d\theta}{dt} = \frac{-1}{505} \cdot 20 = \frac{-20}{505} = \frac{-4}{101} \text{ rad/s.}$$

(iii) (b) Find the speed of the car when $\frac{d\theta}{dt} = \frac{3}{101}$: Let the car's speed be $\frac{dx}{dt} = v$. From the relationship:

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}.$$

Substitute $\frac{d\theta}{dt} = \frac{3}{101}$ and $\frac{d\theta}{dx} = \frac{-1}{505}$:

$$\frac{3}{101} = \frac{-1}{505} \cdot v.$$

Solve for v :

$$v = \frac{3}{101} \cdot 505 = \frac{1515}{101} = 15 \text{ m/s.}$$

Final Answers: 1. $\theta = \tan^{-1} \left(\frac{5}{x} \right)$, 2. $\frac{d\theta}{dx} = \frac{-5}{x^2+25}$, 3. (a) $\frac{d\theta}{dt} = \frac{-4}{101}$ rad/s, (b) Speed of the car: 15 m/s.

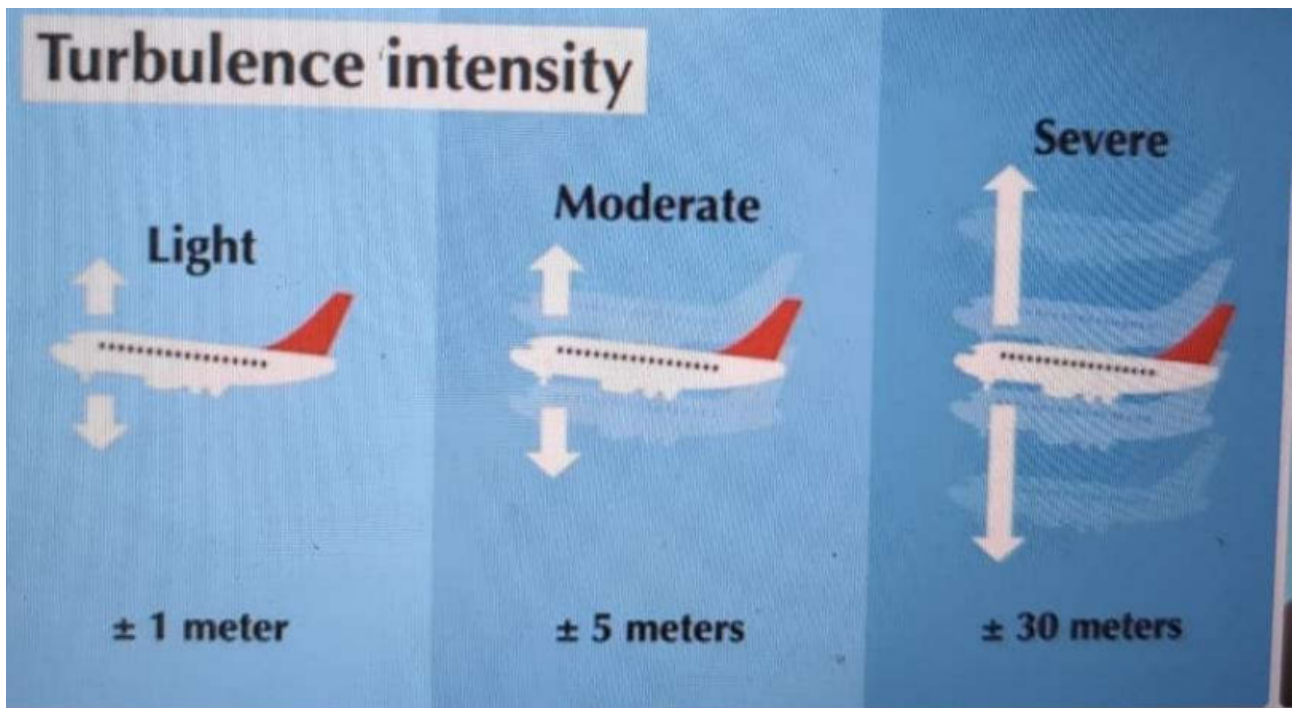
Quick Tip

When solving related rates problems, clearly express all quantities in terms of relevant variables. Differentiate carefully using the chain rule, and substitute known values only after simplifying.

Case Study - 3

38. According to recent research, air turbulence has increased in various regions around the world due to climate change. Turbulence makes flights bumpy and often delays the flights.

Assume that an airplane observes severe turbulence, moderate turbulence or light turbulence with equal probabilities. Further, the chance of an airplane reaching late to the destination are 55%, 37% and 17% due to severe, moderate and light turbulence respectively.



On the basis of the above information, answer the following questions:

- (i) Find the probability that an airplane reached its destination
- (ii) If the airplane reached its destination late, find the probability that it was due to moderate turbulence.

Solution: (i) Probability of the airplane reaching its destination late:

From the law of total probability:

$$P(\text{Late}) = P(\text{Late}|\text{Severe})P(\text{Severe}) + P(\text{Late}|\text{Moderate})P(\text{Moderate}) + P(\text{Late}|\text{Light})P(\text{Light}).$$

Substitute the given probabilities:

$$P(\text{Late}) = \left(0.55 \cdot \frac{1}{3}\right) + \left(0.37 \cdot \frac{1}{3}\right) + \left(0.17 \cdot \frac{1}{3}\right).$$

Simplify:

$$P(\text{Late}) = \frac{0.55 + 0.37 + 0.17}{3} = \frac{1.09}{3}.$$

Thus:

$$P(\text{Late}) = 0.3633 \text{ (approximately).}$$

(ii) Probability of moderate turbulence given the airplane reached late:

Using Bayes' theorem:

$$P(\text{Moderate}|\text{Late}) = \frac{P(\text{Late}|\text{Moderate})P(\text{Moderate})}{P(\text{Late})}.$$

Substitute the values:

$$P(\text{Moderate}|\text{Late}) = \frac{(0.37 \cdot \frac{1}{3})}{0.3633}.$$

Simplify:

$$P(\text{Moderate}|\text{Late}) = \frac{0.37}{3 \cdot 0.3633} = \frac{0.37}{1.09}.$$

Thus:

$$P(\text{Moderate}|\text{Late}) = 0.3394 \text{ (approximately).}$$

Final Answers: 1. The probability that the airplane reached its destination late is:

$$P(\text{Late}) = 0.3633.$$

2. The probability that the airplane was late due to moderate turbulence is:

$$P(\text{Moderate}|\text{Late}) = 0.3394.$$

Quick Tip

When working with probabilities involving multiple conditions, use the law of total probability to compute overall probabilities and Bayes' theorem to determine conditional probabilities. Double-check calculations for clarity and consistency.