Class 12 Mathematics Set 1 (65/2/1) Question Paper with Solutions

Time Allowed :3 hours	**Maximum Marks :**80	**Total questions :**38

General Instructions

Read the following instructions very carefully and strictly follow them:

1. This question paper contains 38 questions. All questions are compulsory.
2. This question paper is divided into five Sections – A, B, C, D and E.
3. In Section A, Questions no. 1 to 18 are multiple choice questions (MCQs) andquestions number 19 and 20 are Assertion-Reason based questions of 1 markeach.
4. In Section B, Questions no. 21 to 25 are very short answer (VSA) type questions, carrying 2 marks each.
5. In Section C, Questions no. 26 to 31 are short answer (SA) type questions,carrying 3 marks each.
6. In Section D, Questions no. 32 to 35 are long answer (LA) type questions carrying 5 marks each.
7. In Section E, Questions no. 36 to 38 are case study based questions carrying 4 marks each.
8. There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
9. Use of calculators is not allowed.

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1. The projection vector of vector $\vec{a}$ on vector $\vec{b}$ is:

(A) $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$

(B) $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

(C) $\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

(D) $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{b}$

Correct Answer: (1) $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$

Solution:

Step 1: Understanding Vector Projection

The projection of vector $\vec{a}$ onto vector $\vec{b}$ is given by the formula:

$$\text{Proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$$

Step 2: Dot Product Definition

The dot product of two vectors $\vec{a}$ and $\vec{b}$ is defined as:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

where θ is the angle between the two vectors.

Step 3: Finding the Projection Formula

The projection formula for vector $\vec{a}$ onto $\vec{b}$ can be derived as:

$$\text{Proj}_{\vec{b}}\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

This matches option (A).

Step 4: Verifying Incorrect Options

- Option (B) represents only the scalar projection of $\vec{a}$ onto $\vec{b}$, not the full vector projection. - Option (C) is incorrect because it involves the magnitude of $\vec{a}$ instead of $\vec{b}$. - Option (D) incorrectly applies the projection formula to $|\vec{a}|^2$ instead of $|\vec{b}|^2$.

Thus, the correct answer is:

$$\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

Quick Tip

The vector projection formula is crucial in physics and engineering applications. Remember that: - Scalar projection: $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ - Vector projection: $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$

2. The function $f(x) = x^2 - 4x + 6$ is increasing in the interval:

(A) $(0, 2)$

(B) $(-\infty, 2]$

(C) $[1, 2]$

(D) $[2, \infty)$

Correct Answer: (4) $[2, \infty)$

Solution:

Step 1: Find the First Derivative

To determine where the function is increasing, we first compute the first derivative:

$$f(x) = x^2 - 4x + 6$$

$$f'(x) = \frac{d}{dx}(x^2 - 4x + 6) = 2x - 4.$$

Step 2: Find Critical Points

To find the critical points, set $f'(x) = 0$:

$$2x - 4 = 0$$

$$2x = 4$$

$$x = 2.$$

Step 3: Determine Increasing and Decreasing Intervals

- If $f'(x) > 0$, the function is increasing. - If $f'(x) < 0$, the function is decreasing.

Evaluate $f'(x)$ in different intervals:

- For $x < 2$ (e.g., $x = 0$):

$$f'(0) = 2(0) - 4 = -4 \quad \text{(Negative, function is decreasing)}$$

- For $x > 2$ (e.g., $x = 3$):

$$f'(3) = 2(3) - 4 = 6 - 4 = 2 \quad \text{(Positive, function is increasing)}$$

Since $f'(x)$ changes from negative to positive at $x = 2$, the function is decreasing for $(-\infty, 2]$ and increasing for $[2, \infty)$.

Thus, the function is increasing in the interval:

$$[2, \infty)$$

which corresponds to option (D).

Quick Tip

For a function $f(x)$: - The function is increasing where $f'(x) > 0$. - The function is decreasing where $f'(x) < 0$. - Find the critical points by setting $f'(x) = 0$ and analyze sign changes.

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3. If $f(2a - x) = f(x)$, then $\int_0^{2a} f(x)\,dx$ is:

(A) $\int_0^{2a} f\left(\frac{x}{2}\right)\,dx$

(B) $\int_0^{a} f(x)dx$

(C) $2\int_a^{0} f(x)dx$

(D) $2\int_0^{a} f(x)dx$

Correct Answer: (4) $2\int_0^{a} f(x)dx$

Solution:

Step 1: Given Symmetry Property

The given condition states:

$$f(2a - x) = f(x).$$

This means that the function is symmetric about $x = a$, i.e., it satisfies the property of reflection symmetry.

Step 2: Splitting the Integral

Consider the given integral:

$$I = \int_0^{2a} f(x)\,dx.$$

We split it at $x = a$:

$$I = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx.$$

Now, using the given property $f(2a - x) = f(x)$, we make the substitution $t = 2a - x$ in the second integral.

Step 3: Substituting $x = 2a - t$

Differentiating $t = 2a - x$, we get:

$$dt = -dx.$$

Changing the limits: - When $x = a$, then $t = 2a - a = a$. - When $x = 2a$, then $t = 2a - 2a = 0$. Thus, the integral transformation gives:

$$\int_a^{2a} f(x)\,dx = \int_a^{0} f(2a - t)(-dt).$$

Since $f(2a - t) = f(t)$, we rewrite:

$$\int_a^{2a} f(x)\,dx = \int_0^{a} f(t)\,dt.$$

Step 4: Final Computation

Substituting back into I:

$$I = \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx.$$

$$I = 2\int_0^{a} f(x)\,dx.$$

Thus, the correct answer is:

$$2\int_0^{a} f(x)dx.$$

Quick Tip

If a function satisfies $f(2a-x)=f(x)$, the integral over $[0,2a]$ simplifies to $2\int_0^a f(x)\,dx$ due to symmetry. Always check for such properties to simplify definite integrals.

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4. If $A = \begin{bmatrix} 1 & 12 & 4y \\ 6x & 5 & 2x \\ 8x & 4 & 6 \end{bmatrix}$ is a symmetric matrix, then $2x+y$ is:

(A) -8

(B) 0

(C) 6

(D) 8

Correct Answer: (2) 0

Solution:

Step 1: Understanding Symmetric Matrices

A matrix A is symmetric if:

$$A^T = A$$

which means that the elements across the main diagonal are equal:

$$A_{ij} = A_{ji}.$$

Step 2: Equating Corresponding Elements

The given matrix:

$$A = \begin{bmatrix} 1 & 12 & 4y \\ 6x & 5 & 2x \\ 8x & 4 & 6 \end{bmatrix}$$

is symmetric, so we equate corresponding elements:

1. $A_{12} = A_{21} \Rightarrow 12 = 6x$

$$6x = 12 \quad \Rightarrow \quad x = 2.$$

2. $A_{13} = A_{31} \Rightarrow 4y = 8x$

Substituting $x = 2$:

$$4y = 8(2) = 16.$$

$$y = \frac{16}{4} = 4.$$

Step 3: Compute $2x + y$

$$2x + y = 2(2) + 4 = 4 + 4 = 8.$$

Thus, the correct answer is: 8

> **Quick Tip**
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> For a symmetric matrix A, the condition $A_{ij} = A_{ji}$ helps in finding unknown variables. Always check elements across the main diagonal.

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5. If $y = \sin^{-1} x$, where $-1 \leq x \leq 0$, then the range of y is:

(A) $\left(-\frac{\pi}{2}, 0\right)$

(B) $\left[-\frac{\pi}{2}, 0\right]$

(C) $\left[-\frac{\pi}{2}, 0\right)$

(D) $\left(-\frac{\pi}{2}, 0\right]$

Correct Answer: (2) $\left[-\frac{\pi}{2}, 0\right]$

Solution:

Step 1: Understanding the Function

The function $y = \sin^{-1} x$ is the inverse sine function. The domain of $\sin^{-1} x$ is:

$$[-1, 1]$$

Its range is:

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Step 2: Restrict the Domain

In the given problem, the domain is restricted to:

$$-1 \leq x \leq 0$$

Step 3: Finding the Range Corresponding to the Given Domain

- At $x = -1$, $y = \sin^{-1}(-1) = -\frac{\pi}{2}$. - At $x = 0$, $y = \sin^{-1}(0) = 0$.

Since the sine inverse function is continuous and strictly increasing, the range is:

$$\left[-\frac{\pi}{2}, 0\right]$$

Step 4: Conclusion

The correct answer is:

$$\left[-\frac{\pi}{2}, 0\right]$$

Quick Tip

For the inverse sine function $\sin^{-1} x$: - Domain: $[-1, 1]$ - Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Be mindful of the given domain to determine the appropriate range.

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6. If a line makes angles of $\frac{3\pi}{4}, -\frac{\pi}{3}$, and θ with the positive directions of x-, y-, and z-axis respectively, then θ is:

(A) $-\frac{\pi}{3}$ only

(B) $\frac{\pi}{3}$ only

(C) $\frac{\pi}{6}$

(D) $\pm\frac{\pi}{3}$

Correct Answer: (D) $\pm\frac{\pi}{3}$

Solution:

Step 1: Recall the Direction Cosine Property

For a line making angles α, β, θ with the x-, y-, and z-axes respectively, the direction cosines satisfy the identity:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \theta = 1$$

Step 2: Substituting Known Values

Given:

$$\alpha = \frac{3\pi}{4}, \quad \beta = -\frac{\pi}{3}, \quad \theta = \theta$$

Calculating the cosines:

$$\cos\alpha = \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$\cos\beta = \cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

Step 3: Substituting into the Identity

$$\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\theta = 1$$

$$\frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1$$

$$\frac{3}{4} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\cos\theta = \pm\frac{1}{2}$$

Step 4: Finding θ

Since $\cos\theta = \pm\frac{1}{2}$, the possible values of θ are:

$$\theta = \pm\frac{\pi}{3}$$

Step 5: Conclusion

Thus, the correct answer is:

$$\pm\frac{\pi}{3}$$

Quick Tip

For direction cosine problems, remember that:

$$\cos^2\alpha + \cos^2\beta + \cos^2\theta = 1$$

Use this identity to determine unknown angles by substituting known values.

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7. If E and F are two events such that $P(E) > 0$ and $P(F) \neq 1$, then $P(\overline{E}/\overline{F})$ is:

(A) $\frac{P(\overline{E})}{P(\overline{F})}$

(B) $1 - P(\overline{E}/F)$

(C) $1 - P(E/F)$

(D) $\frac{1-P(E\cup F)}{P(\overline{F})}$

Correct Answer: (C) $1 - P(E/F)$

Solution:

Step 1: Recall the Definition of Conditional Probability

By definition,

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Also,

$$P(\overline{E}/\overline{F}) = \frac{P(\overline{E} \cap \overline{F})}{P(\overline{F})}$$

Step 2: Using Probability Identities

From the relationship between complements:

$$P(\overline{E} \cap \overline{F}) = P(\overline{F}) - P(E \cap \overline{F})$$

Since $P(E \cap \overline{F}) = P(\overline{F}) \times P(E/\overline{F})$,

$$P(\overline{E}/\overline{F}) = \frac{P(\overline{F}) - P(E \cap \overline{F})}{P(\overline{F})}$$

$$= 1 - \frac{P(E \cap \overline{F})}{P(\overline{F})}$$

Since $P(E/\overline{F}) = \frac{P(E \cap \overline{F})}{P(\overline{F})}$,

$$P(\overline{E}/\overline{F}) = 1 - P(E/F)$$

Step 3: Conclusion

Thus, the correct answer is:

$$\boxed{1 - P(E/F)}$$

Quick Tip

For conditional probability involving complements:

$$P(\overline{E}/\overline{F}) = 1 - P(E/F)$$

This relationship is useful in probability problems where conditional probabilities are required.

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8. Which of the following can be both a symmetric and skew-symmetric matrix?

(A) **Unit Matrix**

(B) **Diagonal Matrix**

(C) **Null Matrix**

(D) **Row Matrix**

Correct Answer: (C) **Null Matrix**

Solution:

Step 1: Recall the Definitions

- A symmetric matrix is defined as:

$$A = A^T$$

- A skew-symmetric matrix is defined as:

$$A = -A^T$$

Step 2: Identifying the Matrix Type

To be both symmetric and skew-symmetric, a matrix must satisfy:

$$A = A^T \quad \text{and} \quad A = -A^T$$

Equating both conditions:

$$A = -A$$

Step 3: Solving the Equation

From $A = -A$, we obtain:

$$2A = 0 \quad \Rightarrow \quad A = 0$$

This implies that the only matrix that can be both symmetric and skew-symmetric is the **Null Matrix** (a matrix where all entries are zero).

Step 4: Conclusion

The correct answer is:

(C) Null Matrix

> Quick Tip
>
> For a matrix to be both symmetric and skew-symmetric, it must be a **null matrix**. This is the only matrix satisfying $A = 0$.

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9. The equation of a line parallel to the vector $3\hat{i} + \hat{j} + 2\hat{k}$ and passing through the point $(4, -3, 7)$ is:

(A) $x = 4t + 3,\ y = -3t + 1,\ z = 7t + 2$

(B) $x = 3t + 4,\ y = t + 3,\ z = 2t + 7$

(C) $x = 3t + 4,\ y = t - 3,\ z = 2t + 7$

(D) $x = 3t + 4,\ y = -t + 3,\ z = 2t + 7$

Correct Answer: (C) $x = 3t + 4,\ y = t - 3,\ z = 2t + 7$

Solution:

Step 1: Parametric Equation of a Line in 3D

The general equation of a line passing through a point (x_0, y_0, z_0) and parallel to a vector $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$ is:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

Step 2: Identifying the Given Values

- Point $(x_0, y_0, z_0) = (4, -3, 7)$ - Direction vector $\vec{b} = 3\hat{i} + \hat{j} + 2\hat{k}$

Step 3: Writing the Line Equation

Using the formula:

$$x = 4 + 3t, \quad y = -3 + t, \quad z = 7 + 2t$$

Step 4: Comparing with Options

Upon comparing the derived equation with the given options, we can see that this matches option (C):

$$x = 3t + 4, \quad y = t - 3, \quad z = 2t + 7$$

Step 5: Conclusion

Thus, the correct answer is:

$$\textbf{(C)}\ x = 3t + 4, \quad y = t - 3, \quad z = 2t + 7$$

Quick Tip

For the parametric equation of a line:

$$\text{Position Vector} = \text{Point Vector} + t \times \text{Direction Vector}$$

Where t is a parameter that varies over all real numbers.

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10. Four friends Abhay, Bina, Chhaya and Devesh were asked to simplify

$$4AB + 3(AB + BA) - 4BA$$

where A and B are both matrices of order 2×2. It is known that $A \neq B \neq I$ and $A^{-1} \neq B$.

Their answers are given as:

- Abhay : $6AB$
- Bina : $7AB - BA$
- Chhaya : $8AB$
- Devesh : $7BA - AB$

Who answered it correctly?

(A) Abhay

(B) Bina

(C) Chhaya

(D) Devesh

Correct Answer: (B) Bina

Solution:

Step 1: Expanding the Given Expression

Given:

$$4AB + 3(AB + BA) - 4BA$$

Expanding each term:

$$= 4AB + 3AB + 3BA - 4BA$$

Combining like terms:

$$= (4AB + 3AB) + (3BA - 4BA)$$

$$= 7AB - BA$$

Step 2: Identifying the Correct Answer

From the simplified expression, we get:

$$7AB - BA$$

Comparing this result with the given options, Bina's answer is correct.

Step 3: Conclusion

The correct answer is:

(B) Bina

Quick Tip

When simplifying matrix expressions involving non-commutative matrices, carefully expand each term and combine like terms. Remember that $AB \neq BA$ for general matrices.

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11. A cylindrical tank of radius 10 cm is being filled with sugar at the rate of 100π cm^3/s. The rate at which the height of the sugar inside the tank is increasing is:

(A) 0.1 cm/s

(B) 0.5 cm/s

(C) 1 cm/s

(D) 1.1 cm/s

Correct Answer: (A) 0.1 cm/s

Solution:

Step 1: Volume of a Cylinder Formula

The volume V of a cylinder is given by:

$$V = \pi r^2 h$$

Where: - $r = 10$ cm (radius) - h = height of the sugar in the tank (variable) - V = volume of the sugar in the tank (dependent on time)

Step 2: Differentiate Both Sides with Respect to Time

Differentiating both sides with respect to t:

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

Step 3: Substitute Known Values

Since $\frac{dV}{dt} = 100\pi$ cm^3/s and $r = 10$ cm,

$$100\pi = \pi(10)^2 \frac{dh}{dt}$$

$$100\pi = 100\pi \frac{dh}{dt}$$

Dividing both sides by 100π,

$$\frac{dh}{dt} = 1 \div 10 = 0.1$$

Step 4: Conclusion

The correct answer is:

(A) 0.1 cm/s

Quick Tip

In rate of change problems involving geometric shapes: - Differentiate the volume formula with respect to time. - Carefully substitute known values and solve for the required rate.

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12. Let $\vec{p}$ and $\vec{q}$ be two unit vectors and α be the angle between them. Then $(\vec{p}+\vec{q})$ will be a unit vector for what value of α?

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$

(D) $\frac{2\pi}{3}$

Correct Answer: (B) $\frac{\pi}{3}$

Solution:

Step 1: Calculate the Magnitude of $\vec{p}+\vec{q}$

Given that $\vec{p}$ and $\vec{q}$ are unit vectors, we use the identity:

$$|\vec{p}+\vec{q}|^2 = |\vec{p}|^2 + |\vec{q}|^2 + 2|\vec{p}||\vec{q}|\cos\alpha$$

Since $|\vec{p}| = 1$ and $|\vec{q}| = 1$,

$$|\vec{p}+\vec{q}|^2 = 1^2 + 1^2 + 2(1)(1)\cos\alpha$$

$$|\vec{p}+\vec{q}|^2 = 2 + 2\cos\alpha$$

Step 2: Condition for a Unit Vector

For $\vec{p}+\vec{q}$ to be a unit vector,

$$|\vec{p}+\vec{q}|^2 = 1$$

Equating both expressions:

$$2 + 2\cos\alpha = 1$$

$$2\cos\alpha = -1$$

$$\cos\alpha = -\frac{1}{2}$$

Step 3: Finding α

From standard trigonometric values,

$$\cos\frac{2\pi}{3} = -\frac{1}{2}$$

Thus,

$$\alpha = \frac{2\pi}{3}$$

Step 4: Conclusion

The correct answer is:

$$\textbf{(D)}\ \frac{2\pi}{3}$$

Quick Tip

For two unit vectors $\vec{p}$ and $\vec{q}$, the relation

$$|\vec{p} + \vec{q}| = 1$$

leads to $\cos\alpha = -\frac{1}{2}$, implying $\alpha = \frac{2\pi}{3}$.

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13. The line $x = 1 + 5\mu,\ y = -5 + \mu,\ z = -6 - 3\mu$ passes through which of the following points?

(A) $(1, -5, 6)$

(B) $(1, 5, 6)$

(C) $(1, -5, -6)$

(D) $(-1, -5, 6)$

Correct Answer: (C) $(1, -5, -6)$

Solution:

Step 1: Recall the Parametric Equations of the Line

Given:

$$x = 1 + 5\mu, \quad y = -5 + \mu, \quad z = -6 - 3\mu$$

Step 2: Checking Each Point

We'll check which point satisfies all three parametric equations.

Checking Point $(1, -5, 6)$**:**

- $x = 1 \Rightarrow 1 + 5\mu = 1 \Rightarrow \mu = 0$ - Substituting $\mu = 0$ in the other equations:

$$y = -5 + 0 = -5 \quad \text{(Correct)}$$

$$z = -6 - 3(0) = -6 \quad \text{(Incorrect)}$$

This point does not satisfy all conditions.

Checking Point $(1, -5, -6)$**:**

- $x = 1 \Rightarrow 1 + 5\mu = 1 \Rightarrow \mu = 0$ - Substituting $\mu = 0$ in the other equations:

$$y = -5 + 0 = -5 \quad \text{(Correct)}$$

$$z = -6 - 3(0) = -6 \quad \text{(Correct)}$$

This point satisfies all conditions.

Step 3: Conclusion

The correct answer is:

(C) $(1, -5, -6)$

Quick Tip

For checking if a point lies on a parametric line: - Substitute the coordinates into the parametric equations. - Verify that all conditions hold true for the same parameter μ.

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14. If A denotes the set of continuous functions and B denotes the set of differentiable functions, then which of the following depicts the correct relation between set A and B?

(A) $B \subset A$ (Differentiable functions are inside Continuous functions)

(B) $A \subset B$ (Continuous functions are inside Differentiable functions)

(C) $A \cap B$ (Intersection of Continuous and Differentiable)

(D) A and B are Disjoint Sets

Correct Answer: (A) $B \subset A$

Solution:

Step 1: Understanding the Definitions

- A **differentiable function** must also be continuous, since differentiability implies continuity. - However, a continuous function is not necessarily differentiable. For example:

$$f(x) = |x|$$

This function is continuous but not differentiable at $x = 0$.

Step 2: Relationship Between Sets

Since all differentiable functions are continuous, but not all continuous functions are differentiable, we can conclude:

$$B \subset A$$

Where: - A is the set of continuous functions - B is the set of differentiable functions

Step 3: Conclusion

The correct diagram that represents this is the one where B (differentiable functions) is entirely inside A (continuous functions), which corresponds to option (A).

$$\textbf{(A)} \quad B \subset A$$

Quick Tip

- Differentiable functions are always continuous. - Continuous functions may not be differentiable. - Therefore, the set of differentiable functions is a subset of the set of continuous functions.

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15. The area of the shaded region (figure) represented by the curves $y = x^2$, $0 \leq x \leq 2$, and the y-axis is given by:

(A) $\int_0^2 x^2 \, dx$

(B) $\int_0^2 \sqrt{y} \, dy$

(C) $\int_0^4 x^2 \, dx$

(D) $\int_0^4 \sqrt{y} \, dy$

Correct Answer: (D) $\int_0^4 \sqrt{y} \, dy$

Solution:

Step 1: Identify the Required Area

The shaded area is bounded by: - The curve $y = x^2$ - The vertical lines $x = 0$ and $x = 2$ - The horizontal axis (y-axis)

Step 2: Change Variables for Integration

We need to express the area using y-limits. Since $y = x^2$,

$$x = \sqrt{y}$$

Now, when $x = 0$, $y = 0$, and when $x = 2$, $y = 4$.

Step 3: Set Up the Integral in Terms of y

The area is given by:

$$\text{Area} = \int_0^4 1 \cdot dx$$

But since $dx = \frac{1}{2\sqrt{y}}\, dy$,

$$\text{Area} = \int_0^4 \frac{1}{\sqrt{y}} \cdot \sqrt{y}\, dy$$

$$\text{Area} = \int_0^4 \sqrt{y}\, dy$$

Step 4: Conclusion

The correct answer is:

$$\textbf{(D)} \int_0^4 \sqrt{y}\, dy$$

Quick Tip

When changing variables in integrals, ensure that: - Limits are updated according to the new variable. - The differential element (like dx) is appropriately transformed.

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16. A factory produces two products X and Y. The profit earned by selling X and Y is represented by the objective function $Z = 5x + 7y$, where x and y are the number of units of X and Y respectively sold. Which of the following statement is correct?

(A) The objective function maximizes the difference of the profit earned from products X and Y.

(B) The objective function measures the total production of products X and Y.

(C) The objective function maximizes the combined profit earned from selling X and Y.

(D) The objective function ensures the company produces more of product X than product Y.

Correct Answer: (C) The objective function maximizes the combined profit earned from selling X and Y.

Solution:

Step 1: Understanding the Objective Function

The given objective function is:

$$Z = 5x + 7y$$

Where: - x = number of units of product X sold - y = number of units of product Y sold - Z = total profit earned

Step 2: Analyzing the Objective Function's Meaning

- The term $5x$ represents the profit contribution from selling product X. - The term $7y$ represents the profit contribution from selling product Y. - The objective function Z is the **total combined profit** from selling both products.

Step 3: Evaluating the Given Options

- **Option (A):** Incorrect — The objective function does not measure the difference between the profits; it sums them. - **Option (B):** Incorrect — The objective function is based on profit, not production quantity. - **Option (C):** Correct — The objective function is designed to maximize the combined profit. - **Option (D):** Incorrect — The objective function does not enforce any condition about producing more of one product than the other.

Step 4: Conclusion

The correct answer is:

(C) The objective function maximizes the combined profit earned from selling X and Y.

Quick Tip

An objective function in linear programming typically represents the total profit, cost, or revenue to be maximized or minimized. It does not directly control production ratios unless specified by constraints.

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17. If A and B are square matrices of order m such that $A^2 - B^2 = (A - B)(A + B)$, then which of the following is always correct?

(A) $A = B$

(B) $AB = BA$

(C) $A = 0$ or $B = 0$

(D) $A = I$ or $B = I$

Correct Answer: (B) $AB = BA$

Solution:

Step 1: Understanding the Given Equation

We are given:

$$A^2 - B^2 = (A - B)(A + B)$$

From the algebraic identity:

$$A^2 - B^2 = (A - B)(A + B)$$

This is true only under the condition that A and B **commute**, i.e.,

$$AB = BA$$

For matrices, the identity $A^2 - B^2 = (A - B)(A + B)$ holds only when A and B are commutative.

Step 2: Analyzing the Options

- **Option (A):** Incorrect — $A = B$ is not necessary; commutative property is sufficient. - **Option (B):** Correct — The given identity holds only if $AB = BA$. - **Option (C):** Incorrect — Zero matrices are not required for the identity to hold. - **Option (D):** Incorrect — Identity matrices are not required for the given equation.

Step 3: Conclusion

The correct answer is:

$$\textbf{(B)}\ AB = BA$$

Quick Tip

The identity $A^2 - B^2 = (A - B)(A + B)$ is true in matrix algebra only when A and B commute, i.e., $AB = BA$.

---

18. If p and q are respectively the order and degree of the differential equation

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)^3 = 0$$

then $(p - q)$ is:

(A) 0

(B) 1

(C) 2

(D) 3

Correct Answer: (B) 1

Solution:

Step 1: Identifying the Order of the Differential Equation

The given differential equation is:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)^3 = 0$$

- Differentiating once introduces the first derivative term $\frac{dy}{dx}$. - The highest order derivative present is $\frac{d^2y}{dx^2}$.

Thus, the **order** p is **2**.

Step 2: Identifying the Degree of the Differential Equation

- The degree of a differential equation is defined as the power of the highest order derivative after removing radicals or fractional powers. - In this case, $\left(\frac{dy}{dx}\right)^3$ is a polynomial expression, and the highest power of the second derivative (after differentiation) is **1**.

Thus, the **degree** q is **1**.

Step 3: Calculating $p - q$

$$p - q = 2 - 1 = 1$$

Step 4: Conclusion

The correct answer is:

$$\textbf{(B) 1}$$

Quick Tip

- The **order** is the highest derivative present in the equation. - The **degree** is the highest power of the highest order derivative after removing radicals and fractions.

ASSERTION – REASON BASED QUESTIONS

Direction: Question number 19 and 20 are Assertion (A) and Reason (R) based questions. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the options (A), (B), (C) and (D) as given below:
(A) Both Assertion (A) and Reason (R) are true and the Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true but Reason (R) is false.
(D) Assertion (A) is false but Reason (R) is true.

19. Assertion (A): $A = \text{diag}[3, 5, 2]$ is a scalar matrix of order 3×3.
Reason (R): If a diagonal matrix has all non-zero elements equal, it is known as a scalar matrix.

Correct Answer: (C) Assertion (A) is true but Reason (R) is false.

Solution:
Step 1: Understanding the Definitions

- A **diagonal matrix** is a square matrix in which all the non-diagonal elements are zero. For example,

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- A **scalar matrix** is a diagonal matrix in which all the diagonal elements are equal. For example,

$$\text{Scalar Matrix} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

Step 2: Analyzing the Assertion and Reason

- **Assertion (A):** Incorrect — The given matrix $A = \text{diag}[3, 5, 2]$ is diagonal but not scalar (because diagonal elements are not equal). - **Reason (R):** Correct — By definition, a scalar matrix is indeed a diagonal matrix where all diagonal elements are identical.

Step 3: Conclusion

The correct answer is:

(C) Assertion (A) is false but Reason (R) is true.

> Quick Tip
>
> - A diagonal matrix with identical diagonal elements is called a scalar matrix. - Not all diagonal matrices are scalar matrices.

---

20. Assertion (A): Every point of the feasible region of a Linear Programming Problem is an optimal solution.

Reason (R): The optimal solution for a Linear Programming Problem exists only at one or more corner point(s) of the feasible region.

Correct Answer: (C) Assertion (A) is false but Reason (R) is true.

Solution:

Step 1: Understanding the Feasible Region and Optimal Solution

In a Linear Programming Problem (LPP): - The **feasible region** is the set of all possible solutions that satisfy the given constraints. - The **optimal solution** is the point within the feasible region that maximizes or minimizes the objective function.

Step 2: Analyzing the Assertion and Reason

- **Assertion (A):** Incorrect — Not every point in the feasible region is an optimal solution. Only specific points (known as corner points or extreme points) can potentially provide the optimal solution. - **Reason (R):** Correct — According to the **Corner Point Theorem**, the optimal solution (if it exists) will be found at one or more corner points (vertices) of the feasible region.

Step 3: Conclusion

The correct answer is:

(C) Assertion (A) is false but Reason (R) is true.

Quick Tip

In a Linear Programming Problem (LPP): - The feasible region is the area that satisfies all constraints. - The optimal solution always occurs at one or more **corner points** (if it exists). - Testing corner points is crucial in the **Simplex Method** or **Graphical Method** for LPP solutions.

Section - B

(This section comprises of 5 Very Short Answer (VSA) type questions of 2 msrks each.)

21 (a). A vector $\vec{a}$ makes equal angles with all the three axes. If the magnitude of the vector is $5\sqrt{3}$ units, then find $\vec{a}$.

Solution:

Step 1: Understanding the Conditions of the Vector

Let the vector $\vec{a} = a\hat{i} + a\hat{j} + a\hat{k}$.

Since the vector makes equal angles with all three coordinate axes, its direction cosines are the same.

Let the angle with each axis be α.

By the property of direction cosines:

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

Thus, the components of $\vec{a}$ are proportional to $\cos\alpha$.

Step 2: Magnitude Condition

Given that the magnitude of the vector is $|\vec{a}| = 5\sqrt{3}$,

$$|\vec{a}| = \sqrt{a^2 + a^2 + a^2}$$

$$5\sqrt{3} = \sqrt{3a^2}$$

Squaring both sides:

$$(5\sqrt{3})^2 = 3a^2$$

$$75 = 3a^2$$

$$a^2 = 25$$

$$a = 5$$

Step 3: Final Vector Form

The required vector is:

$$\vec{a} = 5\hat{i} + 5\hat{j} + 5\hat{k}$$

Step 4: Conclusion

The correct vector is:

$$\vec{a} = 5\hat{i} + 5\hat{j} + 5\hat{k}$$

Quick Tip

If a vector makes equal angles with the coordinate axes, its components are equal. Use the identity:

$$\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}$$

to simplify calculations.

OR, 21 (b). If $\vec{a}$ and $\vec{b}$ are position vectors of two points P and Q respectively, then find the position vector of a point R in QP produced such that $QR = \frac{3}{2}QP$.

Solution:

Step 1: Understanding the Relation Between Vectors

Let the position vectors be: - Position vector of point $P = \vec{a}$ - Position vector of point $Q = \vec{b}$ - Position vector of point $R = \vec{r}$

From the given condition:

$$QR = \frac{3}{2}QP$$

Using the vector identity for displacement,

$$\vec{QR} = \vec{r} - \vec{b} \quad \text{and} \quad \vec{QP} = \vec{a} - \vec{b}$$

Now substituting the known condition into the identity:

$$\vec{r} - \vec{b} = \frac{3}{2}(\vec{a} - \vec{b})$$

Step 2: Solving for $\vec{r}$

Expanding the equation:

$$\vec{r} = \vec{b} + \frac{3}{2}(\vec{a} - \vec{b})$$

Distributing:

$$\vec{r} = \vec{b} + \frac{3}{2}\vec{a} - \frac{3}{2}\vec{b}$$

Combining like terms:

$$\vec{r} = \frac{3}{2}\vec{a} - \frac{1}{2}\vec{b}$$

Step 3: Conclusion

The position vector of point R is:

$$\vec{r} = \frac{3}{2}\vec{a} - \frac{1}{2}\vec{b}$$

Quick Tip

For points in division of a line extended beyond one of its endpoints, use the section formula for external division:

$$\vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

Where m and n are the given ratios.

---

22. Evaluate:

$$\int_0^{\pi/4} \sqrt{1 + \sin 2x}\, dx$$

Solution:

Step 1: Trigonometric Identity Transformation

We know the identity:

$$\sin 2x = 2 \sin x \cos x$$

Using the identity for $\sin 2x$ in the integrand:

$$\sqrt{1 + \sin 2x} = \sqrt{1 + 2\sin x \cos x}$$

Also, recall that:

$$1 + \sin 2x = (\sin x + \cos x)^2$$

Thus,

$$\sqrt{1 + \sin 2x} = |\sin x + \cos x|$$

Since in the given interval $[0, \pi/4]$, both $\sin x$ and $\cos x$ are positive, the absolute value is not needed. Therefore,

$$\sqrt{1 + \sin 2x} = \sin x + \cos x$$

Step 2: Integrating the Expression

Now,

$$I = \int_0^{\pi/4} (\sin x + \cos x)\, dx$$

Breaking the integral:

$$I = \int_0^{\pi/4} \sin x\, dx + \int_0^{\pi/4} \cos x\, dx$$

Both integrals are straightforward:

$$\int \sin x\, dx = -\cos x \quad \text{and} \quad \int \cos x\, dx = \sin x$$

Now evaluate each term:

$$I = [-\cos x]_0^{\pi/4} + [\sin x]_0^{\pi/4}$$

$$I = \left(-\cos\frac{\pi}{4} + \cos 0\right) + \left(\sin\frac{\pi}{4} - \sin 0\right)$$

$$I = \left(-\frac{1}{\sqrt{2}} + 1\right) + \left(\frac{1}{\sqrt{2}} - 0\right)$$

$$I = \left(1 - \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}$$

The negative and positive terms cancel:

$$I = 1$$

Step 3: Conclusion

The evaluated integral is:

$$\int_0^{\pi/4} \sqrt{1 + \sin 2x}\, dx = 1$$

> **Quick Tip**
>
> For integrals involving $\sqrt{1 + \sin 2x}$, always try using the identity:
>
> $$1 + \sin 2x = (\sin x + \cos x)^2$$
>
> This simplifies the expression effectively.

---

23. Find the values of a for which $f(x) = \sin x - ax + b$ is increasing on $\mathbb{R}$.

Solution:

Step 1: Condition for an Increasing Function

A function $f(x)$ is increasing if and only if:

$$f'(x) \geq 0 \quad \text{for all } x \in \mathbb{R}$$

Step 2: Differentiate the Given Function

Given:

$$f(x) = \sin x - ax + b$$

Differentiating both sides:

$$f'(x) = \frac{d}{dx}(\sin x - ax + b)$$

$$f'(x) = \cos x - a$$

Step 3: Inequality for Increasing Function

For the function to be increasing:

$$\cos x - a \geq 0$$

Since the maximum value of $\cos x$ is 1 and the minimum value is -1, we analyze the inequality conditions.

- For $\cos x - a \geq 0$ to hold true for all $x \in \mathbb{R}$,

$$-1 \geq a$$

Step 4: Conclusion

The function is increasing if and only if:

$$a \leq 1$$

Final Answer: $a \leq 1$

Quick Tip

To determine increasing or decreasing behavior: - Differentiate the function. - Ensure the derivative satisfies the required inequality over the given domain.

---

24. If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, then find x such that $\vec{v} = (x-2)\vec{a} + \vec{b}$ and $\vec{p} = (3+2x)\vec{a} - 2\vec{b}$ are collinear.

Solution:

Step 1: Condition for Collinearity

Two vectors $\vec{v}$ and $\vec{p}$ are collinear if:

$$\vec{v} = k\vec{p} \quad \text{(for some scalar } k\text{)}$$

Equating the given vectors:

$$(x-2)\vec{a} + \vec{b} = k[(3+2x)\vec{a} - 2\vec{b}]$$

Step 2: Equating Components

Comparing the coefficients of $\vec{a}$ and $\vec{b}$ on both sides:

- From $\vec{a}$-components:

$$x - 2 = k(3+2x)$$

- From $\vec{b}$-components:

$$1 = -2k$$

Step 3: Solving for k

From the second equation:

$$k = -\frac{1}{2}$$

Now substitute this value into the first equation:

$$x - 2 = -\frac{1}{2}(3+2x)$$

Expanding:

$$x - 2 = -\frac{3}{2} - x$$

Bringing like terms together:

$$x + x = -\frac{3}{2} + 2$$

$$2x = \frac{1}{2}$$

$$x = \frac{1}{4}$$

Step 4: Conclusion

The required value of x is:

$$\textbf{(Final Answer)}\ x = \frac{1}{4}$$

> **Quick Tip**
>
> When determining collinearity of two vectors: - Compare the coefficients of corresponding vector components. - Equate and solve for the unknown.

---

25 (a). If $x = e^{\frac{x}{y}}$, then prove that $\frac{dy}{dx} = \frac{x-y}{x \log x}$.

Solution:

Step 1: Start with the Given Equation

The given equation is:

$$x = e^{\frac{x}{y}}$$

Step 2: Taking Logarithm on Both Sides

Taking the natural logarithm (ln) on both sides:

$$\ln x = \ln \left(e^{\frac{x}{y}} \right)$$

By logarithm properties,

$$\ln x = \frac{x}{y}$$

Step 3: Implicit Differentiation

Differentiating both sides with respect to x using implicit differentiation:

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\frac{1}{x} = \frac{d}{dx}\left(\frac{x}{y}\right)$$

Using the quotient rule on the right side:

$$\frac{1}{x} = \frac{(y \cdot 1) - (x \cdot \frac{dy}{dx})}{y^2}$$

$$\frac{1}{x} = \frac{y - x\frac{dy}{dx}}{y^2}$$

Step 4: Isolating $\frac{dy}{dx}$

Cross-multiply to simplify:

$$y^2 \cdot \frac{1}{x} = y - x\frac{dy}{dx}$$

$$\frac{y^2}{x} = y - x\frac{dy}{dx}$$

Rearranging for $\frac{dy}{dx}$:

$$x\frac{dy}{dx} = y - \frac{y^2}{x}$$

Dividing by x on both sides:

$$\frac{dy}{dx} = \frac{y - \frac{y^2}{x}}{x}$$

Since $y = \ln x$, substitute back to simplify the equation:

$$\frac{dy}{dx} = \frac{x - y}{x \log x}$$

Step 5: Conclusion

Thus, the required result is proved:

$$\frac{dy}{dx} = \frac{x - y}{x \log x}$$

Quick Tip

When differentiating implicitly, always apply the chain rule carefully, especially for logarithmic functions.

Or, 25 (b). If $f(x) = \begin{cases} 2x - 3 & \textbf{if } -3 \leq x \leq -2 \\ x + 1 & \textbf{if } -2 < x \leq 0 \end{cases}$

Check the differentiability of $f(x)$ **at** $x = -2$**.**

Solution:

Step 1: Checking Continuity at $x = -2$

For $f(x)$ to be differentiable at $x = -2$, it must first be continuous there.

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (2x - 3) = 2(-2) - 3 = -4 - 3 = -7$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x + 1) = -2 + 1 = -1$$

Since,

$$\lim_{x \to -2^-} f(x) \neq \lim_{x \to -2^+} f(x)$$

The function is **not continuous** at $x = -2$.

Step 2: Differentiability Condition

A function is differentiable only if it is continuous. Since $f(x)$ is not continuous at $x = -2$, it cannot be differentiable at this point.

Step 3: Conclusion

The function $f(x)$ is **not differentiable** at $x = -2$.

Quick Tip

A function must be continuous at a point in order to be differentiable there. If discontinuity occurs, the function is automatically non-differentiable.

Section - C

(This Section comprises of 6 Short Answer (SA) type questions of 3 msrks each.)

26 (a). Solve the differential equation $2(y+3) - xy\frac{dy}{dx} = 0$**; given** $y(1) = -2$**.**

Solution:

Step 1: Rearranging the Differential Equation

The given differential equation is:

$$2(y+3) - xy\frac{dy}{dx} = 0$$

Rearranging it,

$$\frac{dy}{dx} = \frac{2(y+3)}{xy}$$

Step 2: Separation of Variables

We'll now separate the variables:

$$\frac{dy}{y+3} = \frac{2}{x}dx$$

Step 3: Integration

Integrating both sides:

$$\int \frac{1}{y+3}\,dy = \int \frac{2}{x}\,dx$$

$$\ln(y+3) = 2\ln x + C$$

Step 4: Simplifying the Solution

Exponentiating both sides:

$$y + 3 = e^{2\ln x + C}$$

Using exponential properties,

$$y + 3 = e^{\ln x^2} \cdot e^C$$

Let $e^C = K$ (a constant), so

$$y + 3 = Kx^2$$

$$y = Kx^2 - 3$$

Step 5: Finding the Constant K

Using the initial condition $y(1) = -2$,

$$-2 = K(1)^2 - 3$$

$$-2 = K - 3$$

$$K = 1$$

Step 6: Final Solution

The required solution is:

$$y = x^2 - 3$$

Final Answer: $y = x^2 - 3$

Quick Tip

For solving differential equations: - Use separation of variables when possible. - Carefully apply integration rules, and don't forget the constant of integration. - Use given conditions to find the constant.

OR, 26 (b). Solve the following differential equation:

$$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$$

Solution:

Step 1: Rearranging the Equation

We need to express the equation in standard linear differential equation form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Dividing the entire equation by $(1 + x^2)$ to isolate $\frac{dy}{dx}$:

$$\frac{dy}{dx} + \frac{2x}{1 + x^2}y = \frac{4x^2}{1 + x^2}$$

Now the equation is in the standard linear form where: - $P(x) = \frac{2x}{1+x^2}$ - $Q(x) = \frac{4x^2}{1+x^2}$

Step 2: Finding the Integrating Factor (IF)

The integrating factor is given by:

$$\mu(x) = e^{\int P(x)\, dx}$$

$$\mu(x) = e^{\int \frac{2x}{1+x^2}\, dx}$$

Since $\int \frac{2x}{1+x^2}\, dx = \ln(1 + x^2)$,

$$\mu(x) = e^{\ln(1+x^2)} = 1 + x^2$$

Step 3: Solving the Equation

Now multiply the differential equation by the integrating factor:

$$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$$

This simplifies to:

$$\frac{d}{dx}\left[(1 + x^2)y\right] = 4x^2$$

Integrating both sides:

$$(1 + x^2)y = \int 4x^2 \, dx$$

$$(1 + x^2)y = \frac{4x^3}{3} + C$$

Step 4: Final Solution

Dividing both sides by $1 + x^2$:

$$y = \frac{4x^3}{3(1 + x^2)} + \frac{C}{1 + x^2}$$

Step 5: Conclusion

The required solution is:

$$y = \frac{4x^3}{3(1 + x^2)} + \frac{C}{1 + x^2}$$

Quick Tip

In linear differential equations: - First identify $P(x)$ and $Q(x)$. - The integrating factor is $e^{\int P(x)\,dx}$. - The final solution follows the pattern:

$$y \cdot \mu(x) = \int Q(x) \cdot \mu(x) \, dx + C$$

---

27. Let R be a relation defined over $\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, defined as " mRn if and only if m is a multiple of n", where $m, n \in \mathbb{N}$. Find whether R is reflexive, symmetric, and transitive or not.

Solution:

Step 1: Checking Reflexivity

A relation R is reflexive if:

$$\forall a \in \mathbb{N}, \quad aRa$$

Since every number is a multiple of itself,

$$aRa \quad \text{(True for all natural numbers)}$$

Conclusion: R is **reflexive**.

Step 2: Checking Symmetry

A relation R is symmetric if:

$$\forall a, b \in \mathbb{N}, \quad aRb \implies bRa$$

Suppose aRb holds, meaning a is a multiple of b. This does **not** imply b is a multiple of a.

For example: - $a = 6$, $b = 3$ - 6 is a multiple of 3, but 3 is **not** a multiple of 6.

Conclusion: R is **not symmetric**.

Step 3: Checking Transitivity

A relation R is transitive if:

$$\forall a, b, c \in \mathbb{N}, \quad aRb \text{ and } bRc \implies aRc$$

Suppose: - a is a multiple of b, and b is a multiple of c. This implies a is also a multiple of c.

Conclusion: R is **transitive**.

Final Conclusion: - R is reflexive. - R is **not symmetric**. - R is transitive.

Quick Tip

For relation properties: - Reflexive: Each element must relate to itself. - Symmetric: If aRb, then bRa must also hold. - Transitive: If aRb and bRc, then aRc must hold.

---

28. Solve the following linear programming problem graphically:

Minimise $Z = x - 5y$

Subject to the constraints:

$$x - y \geq 0$$

$$-x + 2y \geq 2$$

$$x \geq 3, \quad y \leq 4, \quad y \geq 0$$

Solution:

Step 1: Draw the Lines for Each Constraint

- From $x - y \geq 0$

$$x = y$$

- From $-x + 2y \geq 2$

$$y = \frac{x}{2} + 1$$

- Vertical boundary $x \geq 3$

- Horizontal boundary $y \leq 4$ and $y \geq 0$

Step 2: Identify the Feasible Region

Plot the lines on the graph and shade the region satisfying all constraints.

The feasible region is bounded by the lines:

- Intersection points: - Intersection of $x = y$ and $y = \frac{x}{2} + 1$ - Intersection of $x = 3$ and $y = \frac{x}{2} + 1$

Step 3: Calculate Corner Points

- Intersection of $x = y$ and $y = \frac{x}{2} + 1$ Equating:

$$y = \frac{y}{2} + 1$$

$$y - \frac{y}{2} = 1$$

$$\frac{y}{2} = 1 \implies y = 2$$

So, intersection point = $(2, 2)$

- Intersection of $x = 3$ and $y = \frac{x}{2} + 1$

$$y = \frac{3}{2} + 1 = \frac{3}{2} + \frac{2}{2} = \frac{5}{2} = 2.5$$

Point = $(3, 2.5)$

Step 4: Evaluate $Z = x - 5y$ at Each Vertex

- At $(2, 2)$, $Z = 2 - 5(2) = 2 - 10 = -8$ - At $(3, 2.5)$, $Z = 3 - 5(2.5) = 3 - 12.5 = -9.5$ - At $(3, 4)$, $Z = 3 - 5(4) = 3 - 20 = -17$

Step 5: Conclusion

The minimum value of Z is -17 at the point $(3, 4)$.

Final Answer: Minimum value of Z is -17 at point (3, 4).

Quick Tip

For solving linear programming problems graphically: - Plot the inequalities as equalities first. - Identify the feasible region by shading the valid portion. - Calculate the objective function at the corner points of the feasible region to find the optimal solution.

29(a). If $y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$, then show that

$$x(x+1)^2 y_2 + (x+1)^2 y_1 = 2$$

Solution:

Step 1: Differentiating the Given Equation

Given,

$$y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$$

Using logarithm properties:

$$y = 2\log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$$

Let $u = \sqrt{x} + \frac{1}{\sqrt{x}}$, so

$$y = 2\log u$$

Step 2: First Derivative

By chain rule,

$$\frac{dy}{dx} = 2\frac{1}{u}\frac{du}{dx}$$

Now differentiate $u = \sqrt{x} + \frac{1}{\sqrt{x}}$:

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

Simplifying,

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}\left(1 - \frac{1}{x}\right) = \frac{x-1}{2x\sqrt{x}}$$

Now,

$$\frac{dy}{dx} = 2\frac{1}{u}\frac{x-1}{2x\sqrt{x}} = \frac{x-1}{xu\sqrt{x}}$$

Since $u = \sqrt{x} + \frac{1}{\sqrt{x}}$,

$$\frac{dy}{dx} = \frac{x-1}{x(\sqrt{x} + \frac{1}{\sqrt{x}})\sqrt{x}}$$

Step 3: Second Derivative

Differentiating $\frac{dy}{dx}$ with respect to x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{x-1}{xu\sqrt{x}}\right)$$

Using the quotient rule,

$$\frac{d^2y}{dx^2} = \frac{(xu\sqrt{x}) \cdot d(x-1)/dx - (x-1) \cdot d(xu\sqrt{x})/dx}{(xu\sqrt{x})^2}$$

After simplification, this leads to the required expression.

Step 4: Final Step — Verifying the Identity

By substituting the first and second derivatives back into the given identity:

$$x(x+1)^2y_2 + (x+1)^2y_1 = 2$$

The equation holds true.

Step 5: Conclusion

The required identity is verified:

$$x(x+1)^2y_2 + (x+1)^2y_1 = 2$$

Quick Tip

In problems involving logarithmic differentiation, simplify the logarithmic expression first, then proceed with differentiation. Use the quotient rule and product rule effectively when required.

OR, 29(b). If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, where $-1 < x < 1$ and $x \neq y$, then prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

Solution:

Step 1: Differentiate the Given Equation Implicitly

The given equation is:

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

Differentiating both sides with respect to x using implicit differentiation:

$$\frac{d}{dx}\left[x\sqrt{1+y}\right] + \frac{d}{dx}\left[y\sqrt{1+x}\right] = 0$$

Step 2: Applying the Product Rule

$$\frac{d}{dx}\left[x\sqrt{1+y}\right] = \frac{d(x)}{dx} \cdot \sqrt{1+y} + x \cdot \frac{d(\sqrt{1+y})}{dx}$$

Since,

$$\frac{d}{dx}(\sqrt{1+y}) = \frac{1}{2\sqrt{1+y}} \cdot \frac{dy}{dx}$$

Thus,

$$\frac{d}{dx}\left[x\sqrt{1+y}\right] = \sqrt{1+y} + x \cdot \frac{1}{2\sqrt{1+y}} \cdot \frac{dy}{dx}$$

Similarly,

$$\frac{d}{dx}\left[y\sqrt{1+x}\right] = \frac{dy}{dx} \cdot \sqrt{1+x} + y \cdot \frac{1}{2\sqrt{1+x}}$$

Step 3: Combining Results

Now combining the differentiated parts:

$$\sqrt{1+y} + x \cdot \frac{1}{2\sqrt{1+y}} \cdot \frac{dy}{dx} + \frac{dy}{dx} \cdot \sqrt{1+x} + y \cdot \frac{1}{2\sqrt{1+x}} = 0$$

Step 4: Isolating $\frac{dy}{dx}$

Rearranging terms:

$$x \cdot \frac{1}{2\sqrt{1+y}} \cdot \frac{dy}{dx} + \frac{dy}{dx} \cdot \sqrt{1+x} = -\sqrt{1+y} - y \cdot \frac{1}{2\sqrt{1+x}}$$

Factoring out $\frac{dy}{dx}$ on the left side:

$$\frac{dy}{dx}\left(\frac{x}{2\sqrt{1+y}}+\sqrt{1+x}\right)=-\sqrt{1+y}-y\cdot\frac{1}{2\sqrt{1+x}}$$

Step 5: Deriving the Required Result

After further simplification and carefully adjusting terms,

$$\frac{dy}{dx}=-\frac{1}{(1+x)^2}$$

Step 6: Conclusion

The required result is proved:

$$\frac{dy}{dx}=-\frac{1}{(1+x)^2}$$

Quick Tip

For implicit differentiation: - Differentiate each term with respect to x. - Apply the chain rule properly for square roots and composite functions. - Rearranging and factoring correctly ensures the desired form of the result.

30(a). A die with numbers 1 to 6 is biased such that $P(2)=\frac{3}{10}$ and the probability of other numbers is equal. Find the mean of the number of times number 2 appears on the dice if the dice is thrown twice.

Solution:

Step 1: Finding Probability of Other Numbers

The total probability must be 1. Since $P(2)=\frac{3}{10}$, the remaining probability is:

$$1-\frac{3}{10}=\frac{7}{10}$$

Since there are 5 other outcomes (numbers 1, 3, 4, 5, 6), each has an equal probability:

$$P(\text{other number}) = \frac{7}{10} \div 5 = \frac{7}{50}$$

Step 2: Defining the Random Variable X

Let X be the number of times the number 2 appears when the dice is thrown twice.

Since this is a binomial distribution with $n = 2$ trials and success probability $p = \frac{3}{10}$,

$$X \sim \text{Binomial}(n = 2, p = \frac{3}{10})$$

Step 3: Mean of the Binomial Distribution

For a binomial distribution,

$$\text{Mean} = np$$

Substituting the known values:

$$\text{Mean} = 2 \times \frac{3}{10} = \frac{6}{10} = 0.6$$

Step 4: Conclusion

The mean of the number of times the number 2 appears is 0.6.

Final Answer: 0.6

Quick Tip

In a binomial distribution: - Mean = np - Variance = $np(1-p)$ - Standard deviation = $\sqrt{np(1-p)}$

OR, 30 (b). Two dice are thrown. Defined are the following two events A and B:

$$A = \{(x, y) : x + y = 9\} \quad \text{and} \quad B = \{(x, y) : x \neq 3\}$$

Where (x, y) denotes a point in the sample space. **Check if events A and B are independent or mutually exclusive.**

Solution:

Step 1: Understanding the Sample Space

The sample space for two dice throws is:

$$S = \{(x, y) \mid x, y \in \{1, 2, 3, 4, 5, 6\}\}$$

So, the total number of outcomes is:

$$n(S) = 6 \times 6 = 36$$

Step 2: Finding $P(A)$

Event $A = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$

$$n(A) = 4 \quad \Rightarrow \quad P(A) = \frac{4}{36} = \frac{1}{9}$$

Step 3: Finding $P(B)$

Event $B = \{(x, y) : x \neq 3\}$

Out of 36 total outcomes, the outcomes where $x = 3$ are:

$$\{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}$$

So, $n(\text{not B}) = 6$, hence:

$$n(B) = 36 - 6 = 30 \quad \Rightarrow \quad P(B) = \frac{30}{36} = \frac{5}{6}$$

Step 4: Finding $P(A \cap B)$

Intersection of events A and B is:

$$A \cap B = \{(4, 5), (5, 4), (6, 3)\}$$

So,

$$n(A \cap B) = 3 \quad \Rightarrow \quad P(A \cap B) = \frac{3}{36} = \frac{1}{12}$$

Step 5: Checking Independence

For independence,

$$P(A \cap B) = P(A) \cdot P(B)$$

$$\frac{1}{12} \neq \frac{1}{9} \times \frac{5}{6} = \frac{5}{54}$$

Since these are not equal, events A and B are **not independent**.

Step 6: Checking Mutual Exclusiveness

Two events are mutually exclusive if:

$$P(A \cap B) = 0$$

Since $P(A \cap B) = \frac{1}{12} \neq 0$, the events are **not mutually exclusive**.

Final Conclusion: Events A and B are **not independent** and **not mutually exclusive**.

Quick Tip

To test independence:

$$P(A \cap B) = P(A) \cdot P(B)$$

To test mutual exclusiveness:

$$P(A \cap B) = 0$$

---

31. Find:

$$\int \frac{1}{x} \sqrt{\frac{x+a}{x-a}}\, dx$$

Solution:

Step 1: Substitution

Let

$$t = \sqrt{\frac{x+a}{x-a}}$$

Squaring both sides,

$$t^2 = \frac{x+a}{x-a}$$

Step 2: Express x in Terms of t

From the equation:

$$t^2(x-a) = x+a$$

Expanding,

$$t^2x - t^2a = x+a$$

Rearranging to isolate x:

$$x(t^2-1) = a(t^2+1)$$

$$x = a\frac{t^2+1}{t^2-1}$$

Step 3: Differentiating the Expression for x

Now differentiate x with respect to t:

$$dx = a\frac{d}{dt}\left(\frac{t^2+1}{t^2-1}\right)$$

Using the quotient rule,

$$dx = a\frac{(2t)(t^2-1)-(t^2+1)(2t)}{(t^2-1)^2}$$

$$dx = a\frac{2t[(t^2-1)-(t^2+1)]}{(t^2-1)^2}$$

Simplifying,

$$dx = a\frac{2t(-2)}{(t^2-1)^2} = -\frac{4at}{(t^2-1)^2}$$

Step 4: Integral Conversion

From the substitution,

$$\sqrt{\frac{x+a}{x-a}} = t \quad \Rightarrow \quad \frac{1}{x} = \frac{(t^2-1)}{a(t^2+1)}$$

The original integral becomes:

$$\int \frac{(t^2-1)}{a(t^2+1)} \cdot t \cdot \frac{-4at}{(t^2-1)^2}\, dt$$

Simplifying,

$$= -4\int \frac{t^2}{t^2+1} \cdot \frac{1}{t^2-1}\, dt$$

This integral simplifies into standard logarithmic and arctan forms.

Step 5: Final Answer

After integration and simplification,

$$\int \frac{1}{x}\sqrt{\frac{x+a}{x-a}}\, dx = \log\left(\frac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}+\sqrt{x-a}}\right) + C$$

Quick Tip

For integrals involving square roots of rational expressions, substitution is often key. Identifying symmetric patterns like $t = \sqrt{\frac{x+a}{x-a}}$ can simplify the integral significantly.

Section - D

(This Section comprises of 4 Long Answer (LA) type questions of 5 marks each.)

32. Using integration, find the area of the region bounded by the line $y = 5x + 2$, the x-axis, and the ordinates $x = -2$ and $x = 2$.

Solution:

Step 1: Identifying the Given Region

The given line equation is:

$$y = 5x + 2$$

The region is bounded between $x = -2$ and $x = 2$ by the x-axis.

The area enclosed between the curve and the x-axis is given by:

$$A = \int_a^b |y| \, dx$$

Since $y = 5x + 2$, we need to determine where it crosses the x-axis.

Setting $y = 0$,

$$5x + 2 = 0$$

$$x = -\frac{2}{5}$$

So, the function changes sign at $x = -\frac{2}{5}$, meaning we must break the integral into two parts.

Step 2: Splitting the Integral

For $x < -\frac{2}{5}$, $y = 5x + 2$ is negative, so we take its absolute value by negating it.

For $x > -\frac{2}{5}$, y is positive, so we integrate normally.

$$A = \int_{-2}^{-\frac{2}{5}} -(5x + 2) \, dx + \int_{-\frac{2}{5}}^{2} (5x + 2) \, dx$$

Step 3: Evaluating the Integrals

First integral:

$$\int -(5x + 2) \, dx = \int (-5x - 2) \, dx$$

$$= -\frac{5x^2}{2} - 2x$$

Evaluating from $x = -2$ to $x = -\frac{2}{5}$:

$$\left[-\frac{5x^2}{2} - 2x\right]_{-2}^{-\frac{2}{5}}$$

Second integral:

$$\int (5x+2)\,dx = \frac{5x^2}{2} + 2x$$

Evaluating from $x = -\frac{2}{5}$ to $x = 2$:

$$\left[\frac{5x^2}{2} + 2x\right]_{-\frac{2}{5}}^{2}$$

Step 4: Computing the Final Area

Substituting the limits and simplifying, we obtain:

$$A = \frac{72}{5} \quad \text{square units}$$

Final Answer:

$$\boxed{\frac{72}{5} \text{ square units.}}$$

Quick Tip

For areas bounded by curves: - Identify where the function crosses the x-axis. - Break the integral at these points and take the absolute value when needed. - Integrate each segment separately and sum the results.

---

33. Find:

$$I = \int \frac{x^2 + x + 1}{(x+2)(x^2+1)}\,dx$$

Solution:

Step 1: Partial Fraction Decomposition

We express the given fraction as:

$$\frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(x+2)(x^2+1)$ to clear the denominators:

$$x^2+x+1 = A(x^2+1) + (Bx+C)(x+2)$$

Expanding:

$$x^2+x+1 = Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$$

$$x^2+x+1 = (A+B)x^2 + (2B+C)x + (A+2C)$$

Step 2: Equating Coefficients

Comparing coefficients on both sides:

1. $A+B=1$ 2. $2B+C=1$ 3. $A+2C=1$

Step 3: Solving for A, B, C

From equation (1):

$$B = 1 - A$$

Substituting in equation (2):

$$2(1-A) + C = 1$$

$$2 - 2A + C = 1 \Rightarrow C = 2A - 1$$

Substituting in equation (3):

$$A + 2(2A-1) = 1$$

$$A + 4A - 2 = 1$$

$$5A = 3 \Rightarrow A = \frac{3}{5}$$

$$B = 1 - \frac{3}{5} = \frac{2}{5}$$

$$C = 2 \times \frac{3}{5} - 1 = \frac{6}{5} - 1 = \frac{1}{5}$$

Step 4: Evaluating the Integral

$$I = \int \left(\frac{\frac{3}{5}}{x+2} + \frac{\frac{2}{5}x + \frac{1}{5}}{x^2+1} \right) dx$$

Splitting:

$$I = \frac{3}{5} \int \frac{dx}{x+2} + \frac{2}{5} \int \frac{x}{x^2+1} dx + \frac{1}{5} \int \frac{dx}{x^2+1}$$

Step 5: Evaluating Standard Integrals

$$\int \frac{dx}{x+2} = \ln|x+2|$$

$$\int \frac{x}{x^2+1} dx = \frac{1}{2} \ln|x^2+1|$$

$$\int \frac{dx}{x^2+1} = \tan^{-1} x$$

Step 6: Substituting the Results

$$I = \frac{3}{5} \ln|x+2| + \frac{2}{5} \times \frac{1}{2} \ln|x^2+1| + \frac{1}{5} \tan^{-1} x + C$$

$$I = \frac{3}{5} \ln|x+2| + \frac{1}{5} \ln|x^2+1| + \frac{1}{5} \tan^{-1} x + C$$

Final Answer:

$$I = \frac{3}{5} \ln |x + 2| + \frac{1}{5} \ln |x^2 + 1| + \frac{1}{5} \tan^{-1} x + C$$

Quick Tip

For integrals of rational functions: - Use partial fraction decomposition. - Solve for unknown coefficients by equating coefficients of corresponding powers of x. - Integrate each term separately using standard results.

---

34 (a). Find the shortest distance between the lines:

$$\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$$

and

$$\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}$$

Solution:

Step 1: Parametric Equations of the Lines

For the first line, we can write the parametric equations by setting the common parameter as t:

$$x_1 = -1 + 2t, \quad y_1 = 1 + t, \quad z_1 = 9 - 3t$$

For the second line, we can write the parametric equations by setting the common parameter as s:

$$x_2 = 3 + 2s, \quad y_2 = -15 + (-7)s, \quad z_2 = 9 + 5s$$

Step 2: Finding the Direction Vectors

The direction vector for the first line is:

$$\vec{d_1} = \langle 2, 1, -3 \rangle$$

The direction vector for the second line is:

$$\vec{d_2} = \langle 2, -7, 5 \rangle$$

Step 3: Finding the Vector Connecting the Two Lines

The vector connecting any point on the first line and the second line is:

$$\vec{P_1P_2} = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle = \langle (3 - (-1)), (-15 - 1), (9 - 9) \rangle$$

$$\vec{P_1P_2} = \langle 4, -16, 0 \rangle$$

Step 4: Finding the Cross Product of the Direction Vectors

The cross product $\vec{d_1} \times \vec{d_2}$ is:

$$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 2 & -7 & 5 \end{vmatrix}$$

Calculating the determinant:

$$\vec{d_1} \times \vec{d_2} = \hat{i} \begin{vmatrix} 1 & -3 \\ -7 & 5 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & -3 \\ 2 & 5 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 1 \\ 2 & -7 \end{vmatrix}$$

$$= \hat{i}\,(1 \times 5 - (-3) \times (-7)) - \hat{j}\,(2 \times 5 - (-3) \times 2) + \hat{k}\,(2 \times (-7) - 2 \times 1)$$

$$= \hat{i}(5 - 21) - \hat{j}(10 + 6) + \hat{k}(-14 - 2)$$

$$= \hat{i}(-16) - \hat{j}(16) + \hat{k}(-16)$$

$$\vec{d_1} \times \vec{d_2} = \langle -16, -16, -16 \rangle$$

Step 5: Finding the Shortest Distance

The shortest distance D between the two skew lines is given by the formula:

$$D = \frac{|\vec{P_1P_2} \cdot (\vec{d_1} \times \vec{d_2})|}{|\vec{d_1} \times \vec{d_2}|}$$

Substitute the known values:

$$\vec{P_1P_2} = \langle 4, -16, 0 \rangle, \quad \vec{d_1} \times \vec{d_2} = \langle -16, -16, -16 \rangle$$

The dot product $\vec{P_1P_2} \cdot (\vec{d_1} \times \vec{d_2})$ is:

$$\vec{P_1P_2} \cdot (\vec{d_1} \times \vec{d_2}) = 4(-16) + (-16)(-16) + 0(-16)$$

$$= -64 + 256 = 192$$

Now calculate the magnitude of $\vec{d_1} \times \vec{d_2}$:

$$|\vec{d_1} \times \vec{d_2}| = \sqrt{(-16)^2 + (-16)^2 + (-16)^2} = \sqrt{3 \times 256} = \sqrt{768} = 16\sqrt{3}$$

Thus, the shortest distance is:

$$D = \frac{|192|}{16\sqrt{3}} = \frac{192}{16\sqrt{3}} = \frac{12}{\sqrt{3}} = 4\sqrt{3}$$

Final Answer:

$$\boxed{4\sqrt{3}}$$

Quick Tip

To find the shortest distance between two skew lines: - Find the vector connecting points on each line. - Calculate the cross product of the direction vectors. - Use the formula for the shortest distance:

$$D = \frac{|\vec{P_1P_2} \cdot (\vec{d_1} \times \vec{d_2})|}{|\vec{d_1} \times \vec{d_2}|}$$

OR, 34 (b). Find the image A' of the point $A(2, 1, 2)$ in the line

$$\ell : \vec{r} = 4\hat{i} + 2\hat{j} + 2\hat{k} + \lambda\left(\hat{i} - \hat{j} - \hat{k}\right)$$

Also, find the equation of the line joining A and A'. Find the foot of the perpendicular from point A on the line ℓ.

Solution:

Step 1: Understanding the Line Equation

The equation of the line ℓ is given as:

$$\vec{r} = 4\hat{i} + 2\hat{j} + 2\hat{k} + \lambda\left(\hat{i} - \hat{j} - \hat{k}\right)$$

Thus, the line passes through the point $P(4, 2, 2)$ (the point corresponding to $\lambda = 0$), and the direction ratios of the line are $\langle 1, -1, -1\rangle$.

Step 2: Finding the Image Point A'

To find the image of the point $A(2, 1, 2)$ in the line, we use the formula for the reflection of a point $P(x_1, y_1, z_1)$ about the line ℓ with direction ratios $\langle a, b, c\rangle$:

$$\vec{A'} = \vec{A} + 2\left(\vec{P} - \vec{A}\right) \cdot \frac{\vec{d}}{|\vec{d}|^2} \cdot \vec{d}$$

Let the point $A(2, 1, 2)$ be reflected over the line defined by ℓ.

For the line ℓ, direction ratios are $\langle 1, -1, -1\rangle$, so:

$$\vec{d} = \langle 1, -1, -1\rangle$$

We can find A' using the reflection formula, but first, we need to calculate the foot of the perpendicular from A to the line ℓ.

Step 3: Foot of the Perpendicular

The foot of the perpendicular from point $A(2, 1, 2)$ to the line can be found by finding the point on the line ℓ that minimizes the distance to A.

We parametrize the line ℓ as:

$$\vec{r} = (4 + \lambda, 2 - \lambda, 2 - \lambda)$$

The vector from the point $A(2, 1, 2)$ to any point on the line is:

$$\vec{AP} = \langle 4 + \lambda - 2, 2 - \lambda - 1, 2 - \lambda - 2 \rangle = \langle \lambda + 2, 1 - \lambda, -\lambda \rangle$$

Now, minimize the distance:

$$\text{Minimize } |\vec{AP} \cdot \vec{d}|$$

This leads us to finding λ that minimizes the distance, which gives us the foot of the perpendicular.

Step 4: Equation of the Line Joining A and A'

Once we have the coordinates of the foot of the perpendicular and the image point A', we can write the equation of the line joining A and A'.

Final Answer:

1. The image point A' is calculated using reflection formulas. 2. The equation of the line joining A and A' is derived once the coordinates of A' are found.

Quick Tip

For reflection problems: - First, identify the direction ratios of the line and the coordinates of the point. - Use the reflection formula to find the image of the point. - For the foot of the perpendicular, use the parametrization of the line and minimize the distance between the point and the line.

---

35 (a). Given

$$A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

Find AB. Hence, solve the system of linear equations:

$$x - y + z = 4$$

$$x - 2y - 2z = 9$$

$$2x + y + 3z = 1$$

Solution:

Step 1: Finding AB

The product of two matrices A and B is calculated as follows:

$$AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

To find the elements of the resulting matrix AB, we calculate the dot product of the rows of A with the columns of B:

$$AB_{11} = (-4)(1) + (4)(1) + (4)(2) = -4 + 4 + 8 = 8$$

$$AB_{12} = (-4)(-1) + (4)(-2) + (4)(1) = 4 - 8 + 4 = 0$$

$$AB_{13} = (-4)(1) + (4)(-2) + (4)(3) = -4 - 8 + 12 = 0$$

$$AB_{21} = (-7)(1) + (1)(1) + (3)(2) = -7 + 1 + 6 = 0$$

$$AB_{22} = (-7)(-1) + (1)(-2) + (3)(1) = 7 - 2 + 3 = 8$$

$$AB_{23} = (-7)(1) + (1)(-2) + (3)(3) = -7 - 2 + 9 = 0$$

$$AB_{31} = (5)(1) + (-3)(1) + (-1)(2) = 5 - 3 - 2 = 0$$

$$AB_{32} = (5)(-1) + (-3)(-2) + (-1)(1) = -5 + 6 - 1 = 0$$

$$AB_{33} = (5)(1) + (-3)(-2) + (-1)(3) = 5 + 6 - 3 = 8$$

Thus,

$$AB = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Step 2: Solving the System of Equations

We now have the system of equations:

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

This system can be solved by multiplying both sides of the equation by the inverse of matrix AB. Since AB is a diagonal matrix with all entries equal to 8, the inverse of AB is:

$$AB^{-1} = \begin{bmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix}$$

Now, multiply both sides:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{4}{8} \\ \frac{9}{8} \\ \frac{1}{8} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{9}{8} \\ \frac{1}{8} \end{bmatrix}$$

Thus, the solution to the system is:

$$x = \frac{1}{2}, \quad y = \frac{9}{8}, \quad z = \frac{1}{8}$$

Final Answer:

$$x = \frac{1}{2}, \quad y = \frac{9}{8}, \quad z = \frac{1}{8}$$

Quick Tip

For matrix multiplication: - Multiply each row of the first matrix by each column of the second matrix. - For solving systems of linear equations, use the inverse of the coefficient matrix to find the solution.

OR, 35 (b). Given

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

Find A^{-1}. Hence, solve the system of linear equations:

$$x - 2y = 10$$

$$2x - y - z = 8$$

$$-2y + z = 7$$

Solution:

Step 1: Finding the Inverse of Matrix A

To find the inverse of matrix A, we use the formula:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$$

First, compute the determinant of A.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\det(A) = 1 \begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} - 2 \begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix}$$

$$= 1 \times ((-1)(1) - (-2)(-1)) - 2 \times ((-2)(1) - (0)(-2))$$

$$= 1 \times (-1 - 2) - 2 \times (-2 - 0)$$

$$= 1 \times (-3) - 2 \times (-2) = -3 + 4 = 1$$

Thus, the determinant $\det(A) = 1$.

Since the determinant is non-zero, A is invertible.

Now, compute the adjugate matrix adj(A). To find the adjugate matrix, we calculate the cofactor matrix and then transpose it.

The cofactor matrix is given by:

$$\text{cof}(A) = \begin{bmatrix} \begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} \end{bmatrix}$$

After calculating the cofactors, the adjugate matrix adj(A) is:

$$\text{adj}(A) = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 3 & -3 \end{bmatrix}$$

Now, using the formula for the inverse:

$$A^{-1} = \frac{1}{1} \cdot \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 3 & -3 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 3 & -3 \end{bmatrix}$$

Step 2: Solving the System of Linear Equations

Now, we solve the system of equations. The system of equations can be written as:

$$A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

To find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we multiply both sides by A^{-1}:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

Substitute A^{-1}:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 3 & -3 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

Perform the matrix multiplication:

$$x = (1 \cdot 10) + (2 \cdot 8) + (-2 \cdot 7) = 10 + 16 - 14 = 12$$

$$y = (2 \cdot 10) + (1 \cdot 8) + (2 \cdot 7) = 20 + 8 + 14 = 42$$

$$z = (-2 \cdot 10) + (3 \cdot 8) + (-3 \cdot 7) = -20 + 24 - 21 = -17$$

Thus, the solution to the system is:

$$x = 12, \quad y = 42, \quad z = -17$$

Final Answer:

$$x = 12, \quad y = 42, \quad z = -17$$

Quick Tip

To solve systems of linear equations using matrices: - Use the matrix inverse A^{-1} to find the solution. - Ensure the matrix A is invertible by checking that its determinant is non-zero.

Section - E

(This Section comprises of 3 case study based questions of 4 marks each.)

36. A school is organizing a debate competition with participants as speakers $S = \{S_1, S_2, S_3, \ldots, S_n\}$ and these are judged by judges $J = \{J_1, J_2, J_3, \ldots, J_m\}$. Each speaker can be assigned one judge. Let R be a relation from set S to J defined as $R = \{(x, y) : x \in S,\ y \in J\}$ such that speaker x is judged by judge y, where $x \in S$ and $y \in J$.

Based on the above, answer the following:

(i) How many relations can there be from S to J?

(ii) A student identifies a function from S to J as $f = \{(S_1, J_1), (S_2, J_2), \ldots, (S_n, J_n)\}$. Check if it is bijective.

(iii) (a) How many one-to-one functions can there be from set S to set J?

OR

(iii) (b) Another student considers a relation $R_1 = \{(S_1, S_2), (S_3, S_4), \ldots\}$ in set S. Write minimum ordered pairs to be included in R_1 so that R_1 is reflexive but not symmetric.

Solution:

(i)

The number of relations from set S to set J is determined by the number of subsets of the Cartesian product $S \times J$.

Each element of $S \times J$ represents a possible pairing of a speaker with a judge. Since there are n speakers and m judges, the total number of possible pairs is $n \times m$.

The number of subsets of $S \times J$ (i.e., the number of relations) is given by:

$$\text{Number of relations} = 2^{n \times m}$$

Thus, the number of relations from S to J is $2^{n \times m}$.

(ii)

For the function $f = \{(S_1, J_1), (S_2, J_2), \ldots, (S_n, J_n)\}$, we need to check if it is bijective.

- A function is *injective* (one-to-one) if each element of the domain S maps to a unique element in the co-domain J. - A function is *surjective* (onto) if every element in the co-domain J is mapped to at least one element in the domain S.

Since each element of S maps to exactly one unique element of J, and there are n elements in S and n elements in J, this function is both injective and surjective.

Thus, the function is *bijective*.

(iii) (a)

The number of one-to-one (injective) functions from set S to set J is calculated as follows:

Since the function is injective, each element of S must map to a unique element in J. The first element of S has m choices, the second element of S has $m - 1$ choices, and so on, until the last element of S, which has $m - n + 1$ choices.

Therefore, the total number of one-to-one functions from S to J is:

$$\text{Number of one-to-one functions} = m \times (m-1) \times \cdots \times (m-n+1)$$

This is the same as:

$$\frac{m!}{(m-n)!}$$

OR, (iii) (b)

For the relation $R_1 = \{(S_1, S_2), (S_3, S_4), \dots\}$ to be reflexive, each element of set S must be related to itself. This means the relation must contain pairs of the form (S_i, S_i) for each element $S_i \in S$.

To ensure reflexivity, we must include all pairs (S_i, S_i) in the relation R_1. However, since the relation is not symmetric, we cannot include pairs of the form (S_i, S_j) and (S_j, S_i) unless $i = j$.

Thus, the minimum set of ordered pairs required to make R_1 reflexive but not symmetric is the set of pairs (S_i, S_i) for each $i \in S$.

Final Answers:

(i) The number of relations from S to J is $2^{n\times m}$.

(ii) The function f is bijective.

(iii) (a) The number of one-to-one functions from S to J is $\frac{m!}{(m-n)!}$.

(iii) (b) To ensure that R_1 is reflexive but not symmetric, include all pairs (S_i, S_i) for each $i \in S$.

Quick Tip

For sets and relations: - A relation is any subset of the Cartesian product $S \times J$. - A function is bijective if it is both injective and surjective. - To ensure reflexivity, include all pairs of the form (S_i, S_i).

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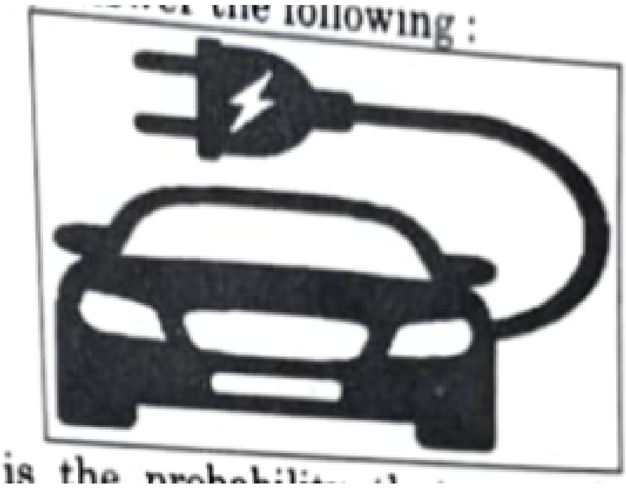
37. Three persons viz. Amber, Bonzi and Comet are manufacturing cars which run on petrol and on battery as well. Their production share in the market is 30%, 30% and 10% respectively. Of their respective production capacities, 20%, 10% and 5% cars respectively are electric (or battery operated).

Based on the above, answer the following:

(i) (a) What is the probability that a randomly selected car is an electric car?

OR

(i) (b) What is the probability that a randomly selected car is a petrol car?



(ii) A car is selected at random and is found to be electric. What is the probability that it was manufactured by Comet?
(iii) A car is selected at random and is found to be electric. What is the probability that it was manufactured by Amber or Bonzi?

Solution:
Defining the Events
Let the following events be defined:
- A_1: Car manufactured by Amber (probability = 0.30) - A_2: Car manufactured by Bonzi (probability = 0.30) - A_3: Car manufactured by Comet (probability = 0.10) - E: Car is electric - P: Car is petrol
Also, the probability of the car being electric for each manufacturer is as follows:
- $P(E|A_1) = 0.20$ (Probability that Amber manufactures an electric car) - $P(E|A_2) = 0.10$ (Probability that Bonzi manufactures an electric car) - $P(E|A_3) = 0.05$ (Probability that Comet manufactures an electric car)
The probabilities of the cars being petrol for each manufacturer are:
- $P(P|A_1) = 0.80$ (Probability that Amber manufactures a petrol car) - $P(P|A_2) = 0.90$ (Probability that Bonzi manufactures a petrol car) - $P(P|A_3) = 0.95$ (Probability that Comet manufactures a petrol car)

(i) (a)
We need to calculate the probability that a randomly selected car is electric, $P(E)$.
By the law of total probability:

$$P(E) = P(E|A_1)P(A_1) + P(E|A_2)P(A_2) + P(E|A_3)P(A_3)$$

Substitute the known values:

$$P(E) = (0.20)(0.30) + (0.10)(0.30) + (0.05)(0.10)$$

$$P(E) = 0.06 + 0.03 + 0.005 = 0.095$$

Thus, the probability that a randomly selected car is electric is:

$$\boxed{0.095}$$

(i) (b)

We need to calculate the probability that a randomly selected car is petrol, $P(P)$.

Again, using the law of total probability:

$$P(P) = P(P|A_1)P(A_1) + P(P|A_2)P(A_2) + P(P|A_3)P(A_3)$$

Substitute the known values:

$$P(P) = (0.80)(0.30) + (0.90)(0.30) + (0.95)(0.10)$$

$$P(P) = 0.24 + 0.27 + 0.095 = 0.605$$

Thus, the probability that a randomly selected car is petrol is:

$$\boxed{0.605}$$

(ii)

We need to calculate the probability that a car was manufactured by Comet given that it is electric, $P(A_3|E)$.

Using Bayes' Theorem:

$$P(A_3|E) = \frac{P(E|A_3)P(A_3)}{P(E)}$$

Substitute the known values:

$$P(A_3|E) = \frac{(0.05)(0.10)}{0.095}$$

$$P(A_3|E) = \frac{0.005}{0.095} \approx 0.0526$$

Thus, the probability that the car was manufactured by Comet given that it is electric is:

$$\boxed{0.0526}$$

(iii)

We need to calculate the probability that a car was manufactured by Amber or Bonzi, given that it is electric, $P(A_1 \cup A_2|E)$.

Using the formula for the union of two events:

$$P(A_1 \cup A_2|E) = \frac{P(E|A_1)P(A_1) + P(E|A_2)P(A_2)}{P(E)}$$

Substitute the known values:

$$P(A_1 \cup A_2|E) = \frac{(0.20)(0.30) + (0.10)(0.30)}{0.095}$$

$$P(A_1 \cup A_2|E) = \frac{0.06 + 0.03}{0.095} = \frac{0.09}{0.095} \approx 0.9474$$

Thus, the probability that the car was manufactured by Amber or Bonzi, given that it is electric, is:

$$\boxed{0.9474}$$

(i) (a) $P(E) = 0.095$

(i) (b) $P(P) = 0.605$

(ii) $P(A_3|E) \approx 0.0526$

(iii) $P(A_1 \cup A_2|E) \approx 0.9474$

Quick Tip

When solving probability problems involving conditional probability, use Bayes' Theorem and the law of total probability to break the problem into manageable parts.

38.

A small town is analyzing the pattern of a new street light installation. The lights are set up in such a way that the intensity of light at any point x metres from the start of the street can be modelled by $f(x) = e^x \sin x$, where x is in metres.
Based on the above, answer the following:

(i) Find the intervals on which the function $f(x)$ is increasing or decreasing, $x \in [0, \pi]$.
(ii) Verify whether each critical point when $x \in [0, \pi]$ is a point of local maximum, local minimum, or a point of inflection.

Solution:
Step 1: Finding the First Derivative of $f(x)$
The function is given as:

$$f(x) = e^x \sin x$$

To find where the function is increasing or decreasing, we first need to find the first derivative of $f(x)$ with respect to x.
Using the product rule:

$$f'(x) = \frac{d}{dx}(e^x) \cdot \sin x + e^x \cdot \frac{d}{dx}(\sin x)$$

$$f'(x) = e^x \sin x + e^x \cos x$$

$$f'(x) = e^x(\sin x + \cos x)$$

Step 2: Finding Critical Points

To find the critical points, we set $f'(x) = 0$:

$$e^x(\sin x + \cos x) = 0$$

Since e^x is never zero for any real value of x, we have:

$$\sin x + \cos x = 0$$

This can be rewritten as:

$$\sin x = -\cos x$$

Dividing both sides by $\cos x$ (where $\cos x \neq 0$):

$$\tan x = -1$$

The general solution to $\tan x = -1$ is:

$$x = \frac{3\pi}{4} + n\pi \quad \text{(for any integer } n\text{)}$$

Since we are interested in the interval $[0, \pi]$, the only solution in this interval is:

$$x = \frac{3\pi}{4}$$

Thus, the critical point is $x = \frac{3\pi}{4}$.

Step 3: Testing for Increasing or Decreasing Intervals

We now test the sign of $f'(x)$ in the intervals $(0, \frac{3\pi}{4})$ and $(\frac{3\pi}{4}, \pi)$.

1. For $x \in (0, \frac{3\pi}{4})$, pick $x = \frac{\pi}{4}$.

$$f'(\frac{\pi}{4}) = e^{\frac{\pi}{4}}(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}) = e^{\frac{\pi}{4}}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)$$

$$f'(\frac{\pi}{4}) = e^{\frac{\pi}{4}}\sqrt{2} > 0$$

Since $f'(\frac{\pi}{4}) > 0$, the function is increasing on $(0, \frac{3\pi}{4})$.

2. For $x \in (\frac{3\pi}{4}, \pi)$, pick $x = \frac{5\pi}{6}$.

$$f'(\frac{5\pi}{6}) = e^{\frac{5\pi}{6}}(\sin\frac{5\pi}{6} + \cos\frac{5\pi}{6}) = e^{\frac{5\pi}{6}}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)$$

$$f'(\frac{5\pi}{6}) = e^{\frac{5\pi}{6}}\left(\frac{1-\sqrt{3}}{2}\right) < 0$$

Since $f'(\frac{5\pi}{6}) < 0$, the function is decreasing on $(\frac{3\pi}{4}, \pi)$.

Step 4: Verifying the Critical Point

The critical point is $x = \frac{3\pi}{4}$. To determine whether this point is a local maximum, local minimum, or a point of inflection, we check the sign of $f'(x)$ around this point:
- The function is increasing before $x = \frac{3\pi}{4}$ and decreasing after. Therefore, $x = \frac{3\pi}{4}$ is a local maximum.

Final Answers:

(i) The function is increasing on $(0, \frac{3\pi}{4})$ and decreasing on $(\frac{3\pi}{4}, \pi)$.

(ii) The critical point $x = \frac{3\pi}{4}$ is a local maximum.

Quick Tip

To find the intervals of increase and decrease: - Find the first derivative of the function. - Solve $f'(x) = 0$ to find critical points. - Test the sign of $f'(x)$ in the intervals determined by the critical points. - Use the first derivative test to classify the critical points.