CBSE Class 12 Mathematics Set 4 (65/4/2) Question Paper with Solutions

Time Allowed :3 Hour	**Maximum Marks :**80	**Total Questions :**38

General Instructions

Read the following instructions very carefully and strictly follow them:

1. This question paper contains 38 questions. All questions are compulsory.
2. This question paper is divided into five Sections – A, B, C, D and E.
3. In Section A, Questions no. 1 to 18 are multiple choice questions (MCQs) andquestions number 19 and 20 are Assertion-Reason based questions of 1 markeach.
4. In Section B, Questions no. 21 to 25 are very short answer (VSA) type questions, carrying 2 marks each.
5. In Section C, Questions no. 26 to 31 are short answer (SA) type questions,carrying 3 marks each.
6. In Section D, Questions no. 32 to 35 are long answer (LA) type questions carrying 5 marks each.
7. In Section E, Questions no. 36 to 38 are case study based questions carrying 4 marks each.
8. There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
9. Use of calculators is not allowed.

SECTION A

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. The lines $\frac{1-x}{2} = \frac{y-1}{3} = \frac{z}{1}$ and $\frac{2x-3}{2p} = \frac{y}{-1} = \frac{z-4}{7}$ are perpendicular to each other for p equal to:

(A) $-\frac{1}{2}$

(B) $\frac{1}{2}$

(C) 2

(D) 3

Correct Answer: (C) 2

Solution: Step 1: Represent the lines in symmetric form: For the first line (L_1), the symmetric equation is given as:

$$\frac{1-x}{2} = \frac{y-1}{3} = \frac{z}{1}$$

Rewriting this in parametric form:

$$x = 1 - 2t, \quad y = 1 + 3t, \quad z = t$$

The direction ratios of L_1 are:

$$a_1 = -2, \, b_1 = 3, \, c_1 = 1$$

For the second line (L_2), the symmetric equation is given as:

$$\frac{2x-3}{2p} = \frac{y}{-1} = \frac{z-4}{7}$$

Rewriting this in parametric form:

$$x = \frac{3}{2} + pt, \quad y = -t, \quad z = 4 + 7t$$

The direction ratios of L_2 are:

$$a_2 = p, \, b_2 = -1, \, c_2 = 7$$

Step 2: Apply the condition for perpendicularity: Two lines are perpendicular if the dot product of their direction ratios is zero:

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Substituting the direction ratios of L_1 and L_2:

$$(-2)(p) + (3)(-1) + (1)(7) = 0$$

Simplify the equation:

$$-2p - 3 + 7 = 0$$

$$-2p + 4 = 0$$

$$p = 2$$

Step 3: Verify the result: For $p = 2$, the direction ratios of L_2 become:

$$a_2 = 2,\ b_2 = -1,\ c_2 = 7$$

The dot product with L_1 is:

$$(-2)(2) + (3)(-1) + (1)(7) = -4 - 3 + 7 = 0$$

Thus, the lines are perpendicular.

Conclusion: The value of p is **2**.

Quick Tip

To check if two lines are perpendicular, calculate the dot product of their direction ratios. If the dot product is zero, the lines are perpendicular. Always express the lines in symmetric or parametric form to extract direction ratios easily.

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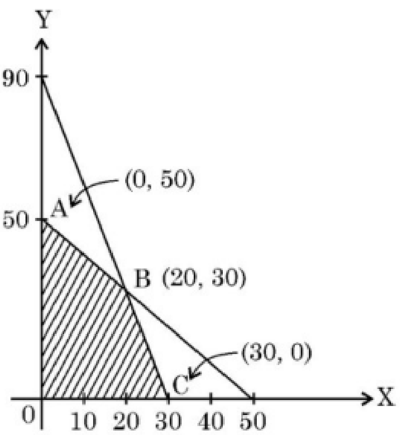
2. The maximum value of $Z = 4x + y$ for a L.P.P. whose feasible region is given below is:



(A) 50

(B) 110

(C) 120

(D) 170

Correct Answer: (C) 120

Solution:

Step 1: Identify the corner points of the feasible region

From the graph, the vertices of the feasible region are:

$$A(0, 50),\ B(20, 30),\ C(30, 0).$$

Step 2: Substitute corner points into $Z = 4x + y$

Evaluate Z at each vertex:

$$Z_A = 4(0) + 50 = 50,$$

$$Z_B = 4(20) + 30 = 110,$$

Z$_C = 4(30) + 0 = 120.$

Step 3: Find the maximum value

The maximum value of Z occurs at $C(30, 0)$, where $Z = 120$.

Step 4: Verify the options

The maximum value is 120, which corresponds to option (C).

Quick Tip

For L.P.P., always evaluate the objective function at all vertices of the feasible region.

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3. The probability distribution of a random variable X is:

X	0	1	2	3	4
$P(X)$	0.1	k	$2k$	k	0.1

where k is some unknown constant. The probability that the random variable X takes the value 2 is:

(A) $\frac{1}{5}$

(B) $\frac{2}{5}$

(C) $\frac{4}{5}$

(D) 1

Correct Answer: (B) $\frac{2}{5}$

Solution: Step 1: Write the given equation: The total probability is given as:

$$P(0) + P(1) + P(2) + P(3) + P(4) = 1$$

Substituting the known values:

$$0.1 + k + 2k + k + 0.1 = 1$$

where $P(1) = k$, $P(2) = 2k$, and $P(3) = k$.

Step 2: Simplify the equation: Combine the terms:

$$0.2 + 4k = 1$$

Subtract 0.2 from both sides:

$$4k = 0.8$$

Divide by 4 to find k:

$$k = 0.2 = \frac{1}{5}$$

Step 3: Find $P(2)$: Given $P(2) = 2k$, substitute the value of k:

$$P(2) = 2 \times \frac{1}{5} = \frac{2}{5}$$

Conclusion: The value of $P(2)$ is $\mathbf{\frac{2}{5}}$.

Quick Tip

To solve probability equations, substitute the known values and simplify step by step. Ensure that the sum of all probabilities equals 1, as this is the fundamental property of probability distributions.

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4. If $A = [a_{ij}] = \begin{bmatrix} 2 & -1 & 5 \\ 1 & 3 & 2 \\ 5 & 0 & 4 \end{bmatrix}$ and c_{ij} is the cofactor of element a_{ij}, then the value of

$a_{21}c_{11} + a_{22}c_{12} + a_{23}c_{13}$ **is:**

(A) -57

(B) 0

(C) 9

(D) 57

Correct Answer: (B) 0

Solution: Step 1: Understanding the expression. The given expression represents the sum of products of elements of the second row with the cofactors of the corresponding elements from the first row.

Step 2: Determinant Property. By the cofactor expansion property:

$$a_{21}c_{11} + a_{22}c_{12} + a_{23}c_{13} = 0$$

since this is equivalent to the determinant expansion along a different row of the same matrix.

Conclusion: Thus, the required value is 0, which corresponds to option $(\mathbf{B})$.

Quick Tip

For determinant calculations, remember that the sum of the product of the elements of any row or column with the cofactors of another row or column is always zero.

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5. If $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ **and** $A^2 - kA - 5I = 0$**, then the value of** k **is:**

(A) 3

(B) 5

(C) 7

(D) 9

Correct Answer: (B) 5

Solution: Step 1: Characteristic Equation. We substitute A into the given equation:

$$A^2 - kA - 5I = 0$$

Calculating A^2:

$$A^2 = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 15 & 25 \end{bmatrix}$$

Using matrix identity $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, substituting in the equation and solving for k, we obtain:

$$k = 5$$

Conclusion: Thus, the required value is 5, which corresponds to option $(\mathbf{B})$.

Quick Tip

For matrix equations, always compute A^2 explicitly and use the given identity equation to find unknown parameters.

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6. If $e^{x^2 y} = c$, then $\frac{dy}{dx}$ is:

(A) $\frac{xe^{x^2 y}}{2y}$

(B) $\frac{-2y}{x}$

(C) $\frac{2y}{x}$

(D) $\frac{x}{2y}$

Correct Answer: (B) $\frac{-2y}{x}$

Solution: Step 1: Taking the Natural Logarithm. Given:

$$e^{x^2 y} = c$$

Taking the natural logarithm on both sides:

$$x^2 y = \ln c$$

Differentiating both sides with respect to x using implicit differentiation:

$$2xy + x^2 \frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-2y}{x}$$

Conclusion: Thus, the required derivative is $\frac{-2y}{x}$, which corresponds to option $(\mathbf{B})$.

Quick Tip

For implicit differentiation, always apply logarithmic differentiation when exponentials are involved.

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7. The value of constant c that makes the function f defined by

$$f(x) = \begin{cases} x^2 - c^2, & \text{if } x < 4 \\ cx + 20, & \text{if } x \geq 4 \end{cases}$$

continuous for all real numbers is:

(A) -2

(B) -1

(C) 0

(D) 2

Correct Answer: (A) -2

Solution: Step 1: Continuity condition at $x = 4$. For $f(x)$ to be continuous at $x = 4$, the left-hand limit (LHL) must equal the right-hand limit (RHL) and the value of $f(4)$.

$$LHL = \lim_{x \to 4^-} (x^2 - c^2) = 4^2 - c^2 = 16 - c^2$$

$$RHL = \lim_{x \to 4^+} (cx + 20) = 4c + 20$$

Equating LHL and RHL:

$$16 - c^2 = 4c + 20$$

Step 2: Solving the equation. Rearranging:

$$c^2 + 4c + 4 = 0 \quad \Rightarrow \quad (c+2)^2 = 0 \quad \Rightarrow \quad c = -2$$

Conclusion: Thus, the required value of c is -2, which corresponds to option $(\mathbf{A})$.

Quick Tip

To ensure continuity of piecewise functions, equate the left-hand limit and right-hand limit at the boundary point(s).

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8. The value of $\int_{-1}^{1} |x|\, dx$ is:

(A) -2

(B) -1

(C) 1

(D) 2

Correct Answer: (C) 1

Solution: Step 1: Splitting the integral. The absolute value function $|x|$ is defined as:

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Splitting the integral at $x = 0$:

$$\int_{-1}^{1} |x|\, dx = \int_{-1}^{0} -x\, dx + \int_{0}^{1} x\, dx$$

Step 2: Evaluating the integrals.

$$\int_{-1}^{0} -x\, dx = \left[-\frac{x^2}{2}\right]_{-1}^{0} = 0 - \left(-\frac{(-1)^2}{2}\right) = \frac{1}{2}$$

$$\int_{0}^{1} x\, dx = \left[\frac{x^2}{2}\right]_{0}^{1} = \frac{1^2}{2} - 0 = \frac{1}{2}$$

Step 3: Summing the results.

$$\int_{-1}^{1} |x|\, dx = \frac{1}{2} + \frac{1}{2} = 1$$

Conclusion: Thus, the value of the integral is 1, which corresponds to option $(\mathbf{C})$.

Quick Tip

For integrals of absolute value functions, always split the integral at points where the function changes definition.

9. The number of arbitrary constants in the particular solution of the differential equation

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y; \quad y(0) = 0$$

is/are:

(A) 2

(B) 1

(C) 0

(D) 3

Correct Answer: (C) 0

Solution:

Rewriting the equation in exponential form:

$$\frac{dy}{dx} = e^{3x+4y}$$

This is a first-order nonlinear differential equation. Since it is a first-order equation, its **general solution** will contain **one arbitrary constant**.

Given the initial condition $y(0) = 0$, this arbitrary constant is determined, resulting in a **particular solution** with **zero arbitrary constants**.

Thus, the correct answer is:

(C) 0

Quick Tip

A particular solution to a differential equation satisfies the equation and all given initial or boundary conditions. Unlike the general solution, it does not contain any arbitrary constants.

10. If $\begin{bmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 5 \end{bmatrix}$ **is a scalar matrix, then the value of** $a + 2b + 3c + 4d$ **is:**

(A) 0

(B) 5

(C) 10

(D) 25

Correct Answer: (D) 25

Solution:

Step 1: Definition of a scalar matrix

A scalar matrix is a diagonal matrix where all diagonal elements are equal. This means $a = d = 5$, and all off-diagonal elements (b, c) are zero.

Step 2: Substitute the values

Using the scalar matrix properties:

$$a + 2b + 3c + 4d = 5 + 2(0) + 3(0) + 4(5) = 5 + 0 + 0 + 20 = 25.$$

Step 3: Verify the options

The correct value is 25, which corresponds to option (D).

Quick Tip
In scalar matrices, diagonal elements are constant, and off-diagonal elements are zero.

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11. If $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$**, then the value of** $I - A + A^2 - A^3 + \ldots$ **is:**

(A) $\begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}$

(B) $\begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Correct Answer: (A) $\begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}$

Solution: Step 1: Understand the series S: The given series is $S = I - A + A^2 - A^3 + \cdots$, which is an infinite series. It can be expressed as:

$$S = (I - A)^{-1}$$

if the matrix $(I - A)$ is invertible.

Step 2: Compute $I - A$: The identity matrix I is:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}$.

Step 3: Check if $I - A$ is invertible: The determinant of $I - A$ is:

$$\det(I - A) = (-1)(3) - (-1)(4) = -3 + 4 = 1 \neq 0$$

Since the determinant is non-zero, $I - A$ is invertible.

Step 4: Verify the series sum: The inverse of $I - A$ is:

$$(I - A)^{-1} = \begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}$$

Thus, the sum of the series $S = I - A + A^2 - A^3 + \cdots$ is:

$$S = \begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}$$

Conclusion: The correct option is $(\mathbf{A})\begin{bmatrix}-1 & -1\\ 4 & 3\end{bmatrix}$.

Quick Tip

For infinite geometric series $I - A + A^2 - A^3 + \ldots$, check if $A^2 = 0$.

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12. Given that $A^{-1} = \frac{1}{7}\begin{bmatrix}2 & 1\\ -3 & 2\end{bmatrix}$, matrix A is:

(A) $7\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$

(B) $\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$

(C) $\frac{1}{7}\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$

(D) $\frac{1}{49}\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$

Correct Answer: (A) $7\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$

Solution:

From the given inverse matrix:

$$A^{-1} = \frac{1}{7}\begin{bmatrix}2 & 1\\ -3 & 2\end{bmatrix}$$

Using the property of matrix inverse:

$$A = \left(A^{-1}\right)^{-1}$$

Taking the inverse on both sides:

$$A = 7 \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}^{-1}$$

Since the inverse of a matrix swaps elements and changes the sign of off-diagonal elements, we compute:

$$A = 7 \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

Thus, the correct answer is:

$$\textbf{(A)} 7 \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

Quick Tip

To find A from A^{-1}, multiply the inverse by the scalar reciprocal.

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13. The integrating factor of the differential equation $(x + 2y^2)\frac{dy}{dx} = y \; (y > 0)$ **is:**

(A) $\frac{1}{x}$

(B) x

(C) y

(D) $\frac{1}{y}$

Correct Answer: (D) $\frac{1}{y}$

Solution:

Step 1: Rewriting the equation

Divide through by y:

$$\frac{1}{y}(x + 2y^2)\frac{dy}{dx} = 1.$$

Step 2: Find the integrating factor

The integrating factor $\mu(y)$ is determined by identifying the dependency on y and multiplying the equation by $\frac{1}{y}$.

Step 3: Verify integrating factor

After multiplying, the left-hand side becomes exact. The integrating factor is $\frac{1}{y}$, which matches option (D).

Quick Tip

Integrating factors simplify differential equations by making them exact.

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14. A vector perpendicular to the line $\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(3\hat{i} - \hat{j})$ is:

(A) $5\hat{i} + \hat{j} + 6\hat{k}$

(B) $\hat{i} + 3\hat{j} + 5\hat{k}$

(C) $2\hat{i} - 2\hat{j}$

(D) $9\hat{i} - 3\hat{j}$

Correct Answer: (B) $\hat{i} + 3\hat{j} + 5\hat{k}$

Solution:

Step 1: Understanding the direction vector of the line. The given line can be expressed as:

$$\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(3\hat{i} - \hat{j})$$

Here, the direction vector of the line is:

$$\vec{d} = 3\hat{i} - \hat{j}$$

Step 2: Condition for perpendicularity. A vector $\vec{v}$ is perpendicular to $\vec{d}$ if their dot product is zero:

$$\vec{v} \cdot \vec{d} = 0$$

Let $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$. The dot product is:

$$\vec{v} \cdot \vec{d} = (a\hat{i} + b\hat{j} + c\hat{k}) \cdot (3\hat{i} - \hat{j}) = 3a - b$$

For perpendicularity:

$$3a - b = 0 \quad \Rightarrow \quad b = 3a \quad \cdots (1)$$

Step 3: Using the options to find the correct vector. We substitute the options to check which satisfies $b = 3a$:

- For $\hat{i} + 3\hat{j} + 5\hat{k}$ (Option B):

$$a = 1, \; b = 3, \; c = 5$$

Substituting into $b = 3a$:

$$b = 3 \times 1 = 3 \quad \text{(True)}$$

Therefore, this vector satisfies the condition.

- Other options do not satisfy $b = 3a$.

Conclusion: Thus, the required vector is $\hat{i} + 3\hat{j} + 5\hat{k}$, which corresponds to option $(\mathbf{B})$.

Quick Tip

To check for perpendicularity, always calculate the dot product of the given vectors and ensure it equals zero.

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15. The vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$, and $\vec{c} = -3\hat{i} + 4\hat{j} + 4\hat{k}$ represent the sides of:

(A) an equilateral triangle

(B) an obtuse-angled triangle

(C) an isosceles triangle

(D) a right-angled triangle

Correct Answer: (D) a right-angled triangle

Solution:

Step 1: Find the Magnitudes of Vectors The length of each side is given by:

$$|\mathbf{a}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$$

$$|\mathbf{b}| = \sqrt{(1)^2 + (-3)^2 + (-5)^2} = \sqrt{1 + 9 + 25} = \sqrt{35}$$

$$|\mathbf{c}| = \sqrt{(-3)^2 + (4)^2 + (4)^2} = \sqrt{9 + 16 + 16} = \sqrt{41}$$

Since all magnitudes are different, the triangle is **not equilateral**.

Step 2: Check for Right-Angled Triangle To check if the vectors form a right-angled triangle, we use the dot product:

$$\mathbf{a} \cdot \mathbf{b} = (2 \cdot 1) + (-1 \cdot -3) + (1 \cdot -5) = 2 + 3 - 5 = 0$$

Since the dot product is zero, $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular**, meaning the triangle is right-angled.

Thus, the correct answer is:

(D) a right-angled triangle

Quick Tip

Use dot products to identify right angles in vector triangles.

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16. Let $\vec{a}$ be any vector such that $|\vec{a}| = a$. The value of $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2$ is:

(A) a^2

(B) $2a^2$

(C) $3a^2$

(D) 0

Correct Answer: (B) $2a^2$

Solution:

Step 1: Recall the formula for cross product magnitudes

The magnitude of the cross product is:

$$|\vec{a} \times \hat{i}| = |\vec{a}||\hat{i}| \sin\theta.$$

Step 2: Evaluate each term

For $\vec{a} \times \hat{i}$, $\vec{a} \times \hat{j}$, and $\vec{a} \times \hat{k}$, the contributions along two directions add up, giving:

$$|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2a^2.$$

Step 3: Verify the options

The correct result is $2a^2$, matching option (B).

> **Quick Tip**
>
> Cross product magnitudes depend on sine of the angle between vectors.

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17. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = 1$, $|\vec{b}| = 2$, and $\vec{a} \cdot \vec{b} = \sqrt{3}$, then the angle between $2\vec{a}$ and $-\vec{b}$ is:

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{3}$

(C) $\frac{5\pi}{6}$

(D) $\frac{11\pi}{6}$

Correct Answer: (C) $\frac{5\pi}{6}$

Solution:

Step 1: Find the angle between $\vec{a}$ and $\vec{b}$

The dot product formula gives:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \implies \sqrt{3} = (1)(2)\cos\theta \implies \cos\theta = \frac{\sqrt{3}}{2}.$$

Thus, $\theta = \frac{\pi}{6}$.

Step 2: Angle between $2\vec{a}$ and $-\vec{b}$

Since $2\vec{a}$ and $-\vec{b}$ involve a scalar multiplication, the angle becomes:

$$\pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

Step 3: Verify the options

The correct angle is $\frac{5\pi}{6}$, matching option (C).

> **Quick Tip**
>
> The angle between scaled vectors depends only on the original vectors' angle.

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18. The function $f(x) = kx - \sin x$ is strictly increasing for:

(A) $k > 1$

(B) $k < 1$

(C) $k > -1$

(D) $k < -1$

Correct Answer: (A) $k > 1$

Solution:

Step 1: Find the derivative

The derivative of $f(x)$ is:

$$f'(x) = k - \cos x.$$

Step 2: Condition for increasing function

For $f(x)$ to be strictly increasing:

$$f'(x) > 0 \implies k - \cos x > 0 \implies k > \cos x.$$

Step 3: Maximum value of $\cos x$

The maximum value of $\cos x$ is 1. Therefore:

$$k > 1.$$

Step 4: Verify the options

The function is strictly increasing for $k > 1$, which matches option (A).

Quick Tip

For strict monotonicity, check the sign of the derivative over the entire domain.

ASSERTION-REASON BASED QUESTIONS

Questions number 19 and 20 are Assertion and Reason based questions. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the codes (A), (B), (C) and (D) as given below.

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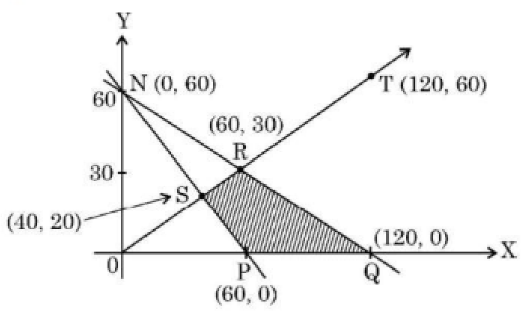
19. Assertion (A): The corner points of the bounded feasible region of a L.P.P. are shown below. The maximum value of $Z = x + 2y$ occurs at infinite points.

Reason (R): The optimal solution of a LPP having bounded feasible region must occur at corner points.



(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

(B) Both Assertion (A) and Reason (R) are true and Reason (R) is not the correct explanation of Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (B) Both Assertion (A) and Reason (R) are true and Reason (R) is not the correct explanation of Assertion (A).

Solution:

Step 1: Analyze Assertion (A)

From the graph, the line $Z = x + 2y$ passes through two corner points $(60, 0)$ and $(120, 60)$, providing the same maximum value. This indicates that the maximum value occurs at infinite points along this segment. Thus, Assertion (A) is true.

Step 2: Analyze Reason (R)

In general, the optimal solution of an LPP occurs at corner points of the feasible region. This is true; however, in this case, the solution lies along a line segment connecting two corner points. Thus, Reason (R) is not the correct explanation of Assertion (A).

Step 3: Conclusion

Both Assertion (A) and Reason (R) are true, but Reason (R) does not explain Assertion (A). Hence, the correct answer is option (B).

> **Quick Tip**
>
> In linear programming, always check if the objective function is constant along any edge of the feasible region.

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20. Assertion (A): The relation $R = \{(x, y) : (x + y)$ is a prime number and $x, y \in \mathbb{N}\}$ is not a reflexive relation.

Reason (R): The number $2n$ is composite for all natural numbers n.

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

(B) Both Assertion (A) and Reason (R) are true and Reason (R) is not the correct explanation of Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Step 1: Analyze Assertion (A)

For R to be reflexive, (x, x) must belong to R for all $x \in \mathbb{N}$. This means $x + x = 2x$ must be a prime number. However, for $x > 1$, $2x$ is not a prime number as it is divisible by 2. Therefore, R is not reflexive, and Assertion (A) is true.

Step 2: Analyze Reason (R)

The Reason states that $2n$ is composite for all n. This is false because when $n = 1$, $2n = 2$, which is a prime number. Therefore, Reason (R) is false.

Step 3: Conclusion

Since Assertion (A) is true and Reason (R) is false, the correct answer is option (C).

> **Quick Tip**
>
> A relation is reflexive if every element relates to itself; check this condition for all elements.

SECTION B

This section comprises very short answer (VSA) type questions of 2 marks each.

21. The volume of a cube is increasing at the rate of $6\,\text{cm}^3/\text{s}$. How fast is the surface area of the cube increasing, when the length of an edge is 8 cm?

Correct Answer: $3\,\text{cm}^2/\text{s}$

Solution:

Step 1: Relate volume and surface area of the cube

The volume of the cube is:

$$V = x^3,$$

where x is the length of an edge. Differentiating with respect to t, we get:

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}.$$

Substitute $\frac{dV}{dt} = 6\,\text{cm}^3/\text{s}$:

$$6 = 3(8)^2 \frac{dx}{dt}.$$

Solve for $\frac{dx}{dt}$:

$$\frac{dx}{dt} = \frac{6}{192} = \frac{1}{32}\,\text{cm/s}.$$

Step 2: Find the rate of change of surface area

The surface area of the cube is:

$$S = 6x^2.$$

Differentiating with respect to t, we get:

$$\frac{dS}{dt} = 12x \frac{dx}{dt}.$$

Substitute $x = 8\,\text{cm}$ and $\frac{dx}{dt} = \frac{1}{32}$:

$$\frac{dS}{dt} = 12(8)\left(\frac{1}{32}\right) = 3\,\text{cm}^2/\text{s}.$$

Conclusion: The surface area of the cube is increasing at $3\,\text{cm}^2/\text{s}$.

Quick Tip

For related rates problems, identify the variables, write their relationships, and differentiate with respect to time.

22(a). Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$, in the simplest form.

Correct Answer: $\frac{\pi}{4} + \frac{x}{2}$

Solution:

Step 1: Simplify the expression inside $\tan^{-1}$

The given expression is:

$$\tan^{-1}\left(\frac{\cos x}{1 - \sin x}\right).$$

Using trigonometric identities, rewrite:

$$1 - \sin x = (\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}) - 2\sin \frac{x}{2} \cos \frac{x}{2} = (\cos \frac{x}{2} - \sin \frac{x}{2})^2.$$

Step 2: Transform into a single tangent function

Substituting $1 - \sin x$ and $\cos x = (\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2})$:

$$\tan^{-1}\left(\frac{\cos x}{1 - \sin x}\right) = \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right].$$

Step 3: Simplify using $\tan^{-1} \tan y = y$

Since $-\frac{\pi}{2} < x < \frac{\pi}{2}$, we simplify:

$$\tan^{-1}\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] = \frac{\pi}{4} + \frac{x}{2}.$$

Conclusion: The simplest form is $\frac{\pi}{4} + \frac{x}{2}$.

> **Quick Tip**
>
> For expressions involving $\tan^{-1}$, rewrite in terms of trigonometric identities to simplify.

OR

22(b). Find the principal value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$**.**

Correct Answer: $\frac{2\pi}{3}$

Solution:

Step 1: Evaluate each term

$$\tan^{-1}(1) = \frac{\pi}{4}, \quad \cos^{-1}\left(-\frac{1}{2}\right) = \pi - \frac{\pi}{3}, \quad \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}.$$

Step 2: Add the terms

$$\frac{\pi}{4} + \left(\pi - \frac{\pi}{3}\right) - \frac{\pi}{4}.$$

Simplify:

$$\frac{\pi}{4} + \pi - \frac{\pi}{3} - \frac{\pi}{4} = \frac{2\pi}{3}.$$

Conclusion: The principal value is $\frac{2\pi}{3}$.

Quick Tip

Always compute inverse trigonometric values in their principal ranges.

---

23. Show that $f(x) = \frac{4\sin x}{2+\cos x} - x$ is an increasing function of x in $\left[0, \frac{\pi}{2}\right]$.

Solution:

Step 1: Compute the first derivative $f'(x)$. The given function is:

$$f(x) = \frac{4\sin x}{2+\cos x} - x$$

Differentiating term by term:

$$f'(x) = \frac{(2+\cos x)(4\cos x) - (4\sin x)(-\sin x)}{(2+\cos x)^2} - 1$$

Step 2: Simplify the derivative.

$$f'(x) = \frac{8\cos x + 4\cos^2 x + 4\sin^2 x}{(2+\cos x)^2} - 1$$

Using $\sin^2 x + \cos^2 x = 1$, we rewrite:

$$f'(x) = \frac{8\cos x + 4}{(2+\cos x)^2} - 1$$

$$= \frac{8\cos x + 4 - (2+\cos x)^2}{(2+\cos x)^2}$$

Expanding $(2+\cos x)^2 = 4 + 4\cos x + \cos^2 x$:

$$f'(x) = \frac{8\cos x + 4 - (4 + 4\cos x + \cos^2 x)}{(2+\cos x)^2}$$

$$= \frac{8\cos x + 4 - 4 - 4\cos x - \cos^2 x}{(2+\cos x)^2}$$

$$= \frac{4\cos x - \cos^2 x}{(2+\cos x)^2}$$

Step 3: Check positivity in $\left[0, \frac{\pi}{2}\right]$. For $x \in \left[0, \frac{\pi}{2}\right]$, we know that $\cos x \geq 0$, so:

$$f'(x) = \frac{\cos x(4-\cos x)}{(2+\cos x)^2} \geq 0$$

since both $\cos x$ and $(4-\cos x)$ are non-negative.

Conclusion: Since $f'(x) \geq 0$ for all x in $\left[0, \frac{\pi}{2}\right]$, $f(x)$ is an increasing function in this interval.

Quick Tip

To check if a function is increasing, compute $f'(x)$ and verify whether it is non-negative in the given interval.

24(a). If $y = \cos^3(\sec^2 2t)$, find $\frac{dy}{dt}$.

Correct Answer: $-12\cos^2(\sec^2 2t)\sin(\sec^2 2t)\sec^2 2t\tan 2t$

Solution:

Step 1: Differentiate using the chain rule

We have:

$$\frac{dy}{dt} = 3\cos^2(\sec^2 2t) \cdot \left[-\sin(\sec^2 2t)\right] \cdot \frac{d}{dt}(\sec^2 2t).$$

Step 2: Simplify derivatives

$$\frac{d}{dt}(\sec^2 2t) = 2\sec^2 2t\tan 2t \cdot 2.$$

Substitute back:

$$\frac{dy}{dt} = -12\cos^2(\sec^2 2t)\sin(\sec^2 2t)\sec^2 2t\tan 2t.$$

Conclusion: The derivative is $-12\cos^2(\sec^2 2t)\sin(\sec^2 2t)\sec^2 2t\tan 2t$.

Quick Tip

When differentiating trigonometric functions, apply the chain rule carefully.

OR

24(b). If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Correct Answer: $\frac{\log x}{(1+\log x)^2}$

Solution:

Step 1: Take logarithms of both sides

The given equation is:

$$x^y = e^{x-y}.$$

Taking the natural logarithm on both sides, we get:

$$\log(x^y) = \log(e^{x-y}).$$

Step 2: Simplify using logarithmic properties

Using the properties of logarithms:

$$y \log x = x - y.$$

Rearranging the terms to express y:

$$y(1 + \log x) = x.$$

Thus, we have:

$$y = \frac{x}{1 + \log x}.$$

Step 3: Differentiate y with respect to x

Differentiate both sides of $y = \frac{x}{1+\log x}$ with respect to x using the quotient rule:

$$\frac{dy}{dx} = \frac{(1 + \log x) \cdot 1 - x \cdot \frac{1}{x}}{(1 + \log x)^2}.$$

Step 4: Simplify the derivative

Simplify the numerator:

$$\frac{dy}{dx} = \frac{(1 + \log x) - 1}{(1 + \log x)^2}.$$

This reduces to:

$$\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}.$$

Conclusion: The derivative is proved to be:

$$\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}.$$

Quick Tip

For functions involving both x and y, logarithmic differentiation often simplifies the process.

---

25. Evaluate:

$$I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \cdot \log\left(\frac{1-x}{1+x}\right) dx$$

Solution:

Step 1: Let $f(x) = \cos x \cdot \log\left(\frac{1-x}{1+x}\right)$**.** To evaluate the integral, observe the function $f(x)$. For $f(-x)$:

$$\begin{aligned} f(-x) &= \cos(-x) \cdot \log\left(\frac{1-(-x)}{1+(-x)}\right) \\ &= \cos x \cdot \log\left(\frac{1+x}{1-x}\right) \\ &= -\cos x \cdot \log\left(\frac{1-x}{1+x}\right) \\ f(-x) &= -f(x) \quad \text{(odd function).} \end{aligned}$$

Step 2: Property of definite integrals for odd functions. For any odd function $f(x)$, the integral over a symmetric interval $[-a, a]$ is zero:

$$\int_{-a}^{a} f(x)dx = 0.$$

Here, since $f(x) = \cos x \cdot \log\left(\frac{1-x}{1+x}\right)$ is odd, we have:

$$I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \cdot \log\left(\frac{1-x}{1+x}\right) dx = 0.$$

Conclusion: The value of the integral is:

$$I = 0.$$

Quick Tip

To simplify definite integrals, check whether the integrand is an odd or even function. Integrals of odd functions over symmetric intervals are always zero.

SECTION C

This section comprises short answer (SA) type questions of 3 marks each

26. Given that $x^y + y^x = a^b$, where a and b are positive constants, find $\frac{dy}{dx}$.

Solution:

Let $u = x^x$, then differentiating both sides with respect to x:

$$\frac{du}{dx} = x^x\left(\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right)$$

Similarly, let $v = y^x$, differentiating both sides:

$$\frac{dv}{dx} = y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$$

Since $u + v = a^b$, differentiating both sides:

$$\frac{du}{dx} + \frac{dv}{dx} = 0$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-\left(x^{x-1} + y^x \log y\right)}{(x^x \log x + y^{x-1}x)}$$

Quick Tip

When differentiating terms with variables in both the base and the exponent, use the logarithmic rule and chain rule effectively.

27(a). Find the particular solution of the differential equation $\frac{dy}{dx} = y \cot 2x$, given that $y\left(\frac{\pi}{4}\right) = 2$.

Correct Answer: $y = 2\sqrt{\sin 2x}$

Solution:

Step 1: Separating the Variables

Rewriting the equation:

$$\frac{1}{y}\,dy = \cot 2x\,dx$$

Step 2: Integrating Both Sides

Integrate both sides:

$$\int \frac{1}{y}\,dy = \int \cot 2x\,dx$$

The left-hand side becomes:

$$\log|y|$$

The right-hand side uses the integral of $\cot 2x$:

$$\int \cot 2x\,dx = \frac{1}{2}\log|\sin 2x|$$

So the equation becomes:

$$\log|y| = \frac{1}{2}\log|\sin 2x| + \log c$$

Here, $\log c$ is the constant of integration.

Step 3: Simplify the Expression

Combine the logarithms:

$$\log|y| = \log\left(c\sqrt{\sin 2x}\right)$$

Exponentiate both sides to remove the logarithm:

$$y = c\sqrt{\sin 2x}$$

Step 4: Finding the Particular Solution

We are given the condition $y\left(\frac{\pi}{4}\right) = 2$. Substitute $x = \frac{\pi}{4}$ and $y = 2$ into the solution:

$$2 = c\sqrt{\sin\left(2\cdot\frac{\pi}{4}\right)}$$

Simplify:

$$2 = c\sqrt{\sin\left(\frac{\pi}{2}\right)}$$

Since $\sin\left(\frac{\pi}{2}\right) = 1$, we have:

$$2 = c\cdot 1 \quad \Rightarrow \quad c = 2$$

Step 5: Final Solution

Substitute $c = 2$ back into the solution:

$$y = 2\sqrt{\sin 2x}$$

Final Answer:

$$\boxed{y = 2\sqrt{\sin 2x}}$$

This is the required particular solution to the given differential equation.

Quick Tip

For separable differential equations, isolate y and x, then integrate both sides.

OR

27(b). Find the particular solution of the differential equation $(x(e)^{y}/^{x} + y)\,dx = x\,dy$**, given that** $y = 1$ **when** $x = 1$**.**

Solution:

Step 1: Rewrite the given differential equation

Rearrange the equation to separate variables:

$$\frac{dy}{dx} = x(e)^{y}/^{x} + \frac{y}{x}$$

This is a homogeneous differential equation.

Step 2: Substitution

Let:

$$y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substitute $y = vx$ into the equation:

$$v + x\frac{dv}{dx} = x(e)^{y}/^{x} + v$$

Simplify:

$$x\frac{dv}{dx} = x(e)^{y}/^{x}$$

Step 3: Solve for v

Rearrange the equation for integration:

$$\int e^{-v}\,dv = \int \frac{1}{x}\,dx$$

Evaluate the integrals:

$$-e^{-v} = \log|x| + c$$

Substitute $v = \frac{y}{x}$:

$$-e^{-\frac{y}{x}} = \log|x| + c \quad \ldots(1)$$

Step 4: Apply the initial condition

We are given that $y = 1$ when $x = 1$. Substitute $x = 1$ and $y = 1$ into equation (1):

$$-e^{-\frac{1}{1}} = \log|1| + c$$

Since $\log|1| = 0$:

$$-e^{-1} = c$$

Thus:

$$c = -e^{-1}$$

Step 5: Substitute c back into the solution

Substitute $c = -e^{-1}$ into equation (1):

$$-e^{-\frac{y}{x}} = \log|x| - e^{-1}$$

Rearranging:

$$\log|x| + e^{-\frac{y}{x}} = e^{-1}$$

Final Answer:

$$\boxed{\log|x| + e^{-\frac{y}{x}} = e^{-1}}$$

Quick Tip

For homogeneous differential equations, use substitution $y = vx$ to simplify the equation and separate variables.

28. Find:

$$I = \int \frac{2x+3}{x^2(x+3)}\,dx$$

Solution:

Step 1: Split the integral. The given integral can be rewritten as:

$$I = \int \frac{x+3}{x^2(x+3)}\,dx + \int \frac{x}{x^2(x+3)}\,dx$$

Simplifying:

$$I = \int \frac{1}{x^2}\,dx + \int \frac{x+3-x}{x(x+3)}\,dx$$

Step 2: Further simplify. Split the second integral:

$$\int \frac{x+3-x}{x(x+3)}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+3}\,dx$$

Step 3: Evaluate the individual integrals. 1. First term:

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

2. Second term:

$$\int \frac{1}{x}\,dx = \log|x|$$

3. Third term:

$$\int \frac{1}{x+3}\,dx = \log|x+3|$$

Step 4: Combine the results. Substituting back, we get:

$$I = -\frac{1}{x} + \frac{1}{3}\log|x| - \frac{1}{3}\log|x+3| + C$$

Conclusion: The final result is:

$$I = -\frac{1}{x} + \frac{1}{3}\log|x| - \frac{1}{3}\log|x+3| + C$$

Quick Tip

When integrating rational functions, split the numerator and simplify into partial fractions if necessary to solve term by term.

29(a). A card from a well-shuffled deck of 52 playing cards is lost. From the remaining cards of the pack, a card is drawn at random and is found to be a King. Find the probability of the lost card being a King.

Solution:

Step 1: Define the events

Let E_1 be the event that the lost card is a King, and E_2 be the event that the lost card is not a King. Let A be the event of drawing a King from the remaining 51 cards.

Step 2: Assign probabilities to the events

$$P(E_1) = \frac{1}{13}, \quad P(E_2) = \frac{12}{13}, \quad P(A|E_1) = \frac{3}{51}, \quad P(A|E_2) = \frac{4}{51}$$

Step 3: Use Bayes' Theorem

The required probability is $P(E_1|A)$, which is given by:

$$P(E_1|A) = \frac{P(A|E_1) \cdot P(E_1)}{P(A|E_1) \cdot P(E_1) + P(A|E_2) \cdot P(E_2)}$$

Substituting the values:

$$P(E_1|A) = \frac{\frac{1}{13} \cdot \frac{3}{51}}{\frac{1}{13} \cdot \frac{3}{51} + \frac{12}{13} \cdot \frac{4}{51}} = \frac{\frac{3}{663}}{\frac{3}{663} + \frac{48}{663}} = \frac{3}{51} = \frac{1}{17}$$

Step 4: Final result

The probability that the lost card is a King is $\frac{1}{17}$.

Quick Tip

When solving problems involving missing or conditional probabilities, use Bayes' Theorem and clearly define all events and conditional probabilities.

OR

29(b). A biased die is twice as likely to show an even number as an odd number. If such a die is thrown twice, find the probability distribution of the number of sixes. Also, find the mean of the distribution.

Solution:

Step 1: Assign probabilities

Let $P(3) = P(5) = p$, so $P(2) = P(4) = P(6) = 2p$.

As the total probability is 1:

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1 \implies 9p = 1 \implies p = \frac{1}{9}$$

Thus, $P(6) = 2p = \frac{2}{9}$, and $P(\text{Not getting six}) = 1 - P(6) = \frac{7}{9}$.

Step 2: Define the random variable X

Let X represent the number of sixes. The possible values of X are $0, 1, 2$.

Step 3: Compute probabilities for X

$$P(X = 0) = \left(\frac{7}{9}\right)^2 = \frac{49}{81}, \quad P(X = 1) = 2 \cdot \frac{2}{9} \cdot \frac{7}{9} = \frac{28}{81}, \quad P(X = 2) = \left(\frac{2}{9}\right)^2 = \frac{4}{81}$$

Step 4: Probability distribution of X

X	$P(X)$
0	$\frac{49}{81}$
1	$\frac{28}{81}$
2	$\frac{4}{81}$

Step 5: Compute the mean of X

The mean is given by:

$$\mu = \sum_{i=1}^{3} X_i \cdot P(X_i) = 0 \cdot \frac{49}{81} + 1 \cdot \frac{28}{81} + 2 \cdot \frac{4}{81} = \frac{28}{81} + \frac{8}{81} = \frac{36}{81} = \frac{4}{9}$$

Step 6: Final result

The probability distribution of X is:

X	$P(X)$
0	$\frac{49}{81}$
1	$\frac{28}{81}$
2	$\frac{4}{81}$

The mean of the distribution is $\frac{4}{9}$.

Quick Tip

When working with biased probability distributions, ensure the total probability sums to 1 and carefully calculate probabilities for each outcome.

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30. Solve the following linear programming problem graphically:

Maximise

$$Z = 2x + 3y$$

Subject to the constraints:

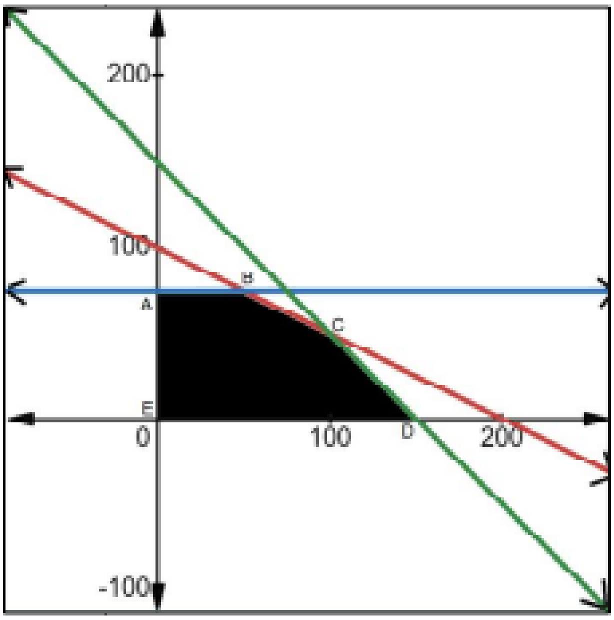
$$x + y \leq 6$$

$$x \geq 2$$

$$y \leq 3$$

$$x, y \geq 0$$

Solution:



Step 1: Graphical Representation of Constraints. To solve the problem graphically, we plot the constraint equations:

1. $x + y = 6$ (line passing through (6,0) and (0,6)) 2. $x = 2$ (vertical line at $x = 2$) 3. $y = 3$ (horizontal line at $y = 3$) 4. $x, y \geq 0$ (first quadrant restriction)

The feasible region is the intersection of these constraints.

Step 2: Identifying Corner Points of Feasible Region. From the graph, the common feasible region forms a bounded polygon. The corner points of this region are:

$$A(2,0), B(2,3), C(3,3), D(6,0)$$

Step 3: Compute Objective Function at Corner Points. Evaluating $Z = 2x + 3y$ at each corner:

$$Z(A) = 2(2) + 3(0) = 4$$

$$Z(B) = 2(2) + 3(3) = 13$$

$$Z(C) = 2(3) + 3(3) = 15$$

$$Z(D) = 2(6) + 3(0) = 12$$

Step 4: Determine Maximum Value. The maximum value occurs at point $C(3,3)$ with:

$$Z_{\max} = 15$$

Corner Points	**Value of** $Z = 2x + 3y$
$A(2,0)$	4
$B(2,3)$	13
$C(3,3)$	15
$D(6,0)$	12

Conclusion: The maximum value of Z is 15 at $(3,3)$.

Quick Tip

To solve a linear programming problem graphically, plot the constraints, identify the feasible region, determine the corner points, and evaluate the objective function at each point.

31(a). Evaluate $\int_0^{\pi/4} \frac{x}{1+\cos 2x+\sin 2x}\,dx$**.**

Correct Answer: $\frac{\pi}{16}\log 2$

Solution:

Step 1: Apply symmetry property of definite integrals

Let:

$$I = \int_0^{\pi/4} \frac{x}{1+\cos 2x+\sin 2x}\,dx.$$

Using the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get:

$$I = \int_0^{\pi/4} \frac{\pi/4 - x}{1+\cos 2x+\sin 2x}\,dx.$$

Step 2: Combine integrals

Adding the two forms of I:

$$2I = \int_0^{\pi/4} \frac{\pi/4}{1+\cos 2x+\sin 2x}\,dx.$$

Step 3: Simplify the integrand

Rewrite $1+\cos 2x+\sin 2x$:

$$\cos 2x+\sin 2x = \sqrt{2}\sin(2x+\pi/4),$$

and simplify:

$$I = \frac{\pi}{16}\int_0^{\pi/4} \frac{1}{\cos^2 x+\sin x\cos x}\,dx.$$

Step 4: Integrate and simplify

The integral evaluates to:

$$I = \frac{\pi}{16}(\log|1+\tan x|)_0^{\pi/4}.$$

Substitute limits:

$$I = \frac{\pi}{16}\log 2.$$

Conclusion: The integral evaluates to $\frac{\pi}{16}\log 2$.

Quick Tip

For definite integrals with symmetric limits, apply symmetry properties to simplify.

OR

31(b). Find: $\int e^x \left[\frac{1}{(1+x^2)^{3/2}} + \frac{x}{\sqrt{1+x^2}}\right] dx$**.**

Correct Answer: $e^x \frac{x}{\sqrt{1+x^2}} + C$

Solution:

Step 1: The given integral can be written as:

$$I = \int e^x \left(\frac{x}{\sqrt{1+x^2}} + \frac{1}{(1+x^2)^{\frac{3}{2}}} \right) dx$$

Let:

$$f(x) = \frac{x}{\sqrt{1+x^2}}$$

Step 2: Now, calculate the derivative of $f(x)$:

$$f'(x) = \frac{\sqrt{1+x^2} - \frac{x \cdot x}{\sqrt{1+x^2}}}{1+x^2} = \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2}$$

Simplify the numerator:

$$f'(x) = \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2} = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

Thus, the integral becomes:

$$I = \int e^x \left(f(x) + f'(x) \right) dx$$

Step 3: Using the standard result:

$$\int e^x \left(f(x) + f'(x) \right) dx = e^x f(x) + C$$

Substitute $f(x) = \frac{x}{\sqrt{1+x^2}}$:

$$I = e^x \frac{x}{\sqrt{1+x^2}} + C$$

Final Answer:

$$\boxed{I = e^x \frac{x}{\sqrt{1+x^2}} + C}$$

Explanation:

1. Splitting the Integral: The given integral is split into terms containing $\frac{x}{\sqrt{1+x^2}}$ and $\frac{1}{(1+x^2)^{\frac{3}{2}}}$.

2. Defining $f(x)$: The function $f(x)$ is chosen as $\frac{x}{\sqrt{1+x^2}}$ because its derivative results in the second term, $\frac{1}{(1+x^2)^{\frac{3}{2}}}$.

3. Applying the Formula: The integral formula for $\int e^x(f(x) + f'(x))dx = e^x f(x) + C$ is directly applied.

4. Substitution: Finally, substituting $f(x)$ into the formula gives the result.

Quick Tip

For integrals involving a combination of functions $f(x)$ and $f'(x)$, use the substitution $u = f(x)$ to simplify.

SECTION D

This section comprises long answer (LA) type questions of 5 marks each

32(a). Let $A = \mathbb{R} - \{5\}$ and $B = \mathbb{R} - \{1\}$. Consider the function $f : A \to B$, defined by $f(x) = \frac{x-3}{x-5}$. Show that f is one-one and onto.

Solution:

Step 1: Prove that f is one-one

Let $f(x_1) = f(x_2)$, where $x_1, x_2 \in A$:

$$\frac{x_1 - 3}{x_1 - 5} = \frac{x_2 - 3}{x_2 - 5}$$

Cross-multiply:

$$(x_1 - 3)(x_2 - 5) = (x_2 - 3)(x_1 - 5)$$

Simplify:

$$x_1x_2 - 5x_1 - 3x_2 + 15 = x_1x_2 - 5x_2 - 3x_1 + 15$$

$$-5x_1 - 3x_2 = -5x_2 - 3x_1 \implies 5(x_2 - x_1) = 3(x_2 - x_1)$$

If $x_1 \neq x_2$, this leads to a contradiction. Hence, $x_1 = x_2$, proving f is one-one.

Step 2: Prove that f is onto

Let $y \in B$. Solve $f(x) = y$:

$$\frac{x-3}{x-5} = y \implies x - 3 = y(x-5)$$

$$x - 3 = yx - 5y \implies x - yx = -5y + 3$$

$$x(1-y) = -5y + 3 \implies x = \frac{-5y+3}{1-y}$$

For $y \neq 1$ (since $y \in B$), x exists in A. Thus, f is onto.

Step 3: Conclude the function properties

Since f is both one-one and onto, f is a bijection.

Quick Tip

To prove a function is one-one, show that $f(x_1) = f(x_2)$ implies $x_1 = x_2$. To prove onto, solve $f(x) = y$ and ensure x lies in the domain.

OR

32(b). Check whether the relation S in the set of real numbers $\mathbb{R}$, defined by $S = \{(a,b) : a - b + \sqrt{2}$ is an irrational number$\}$, is reflexive, symmetric, or transitive.

Solution:

Step 1: Reflexivity

For $a \in \mathbb{R}$:

$$a - a + \sqrt{2} = \sqrt{2} \text{ is irrational.}$$

Thus, $(a,a) \in S$, and S is reflexive.

Step 2: Symmetry

Let $(a,b) \in S$, so:

$$a - b + \sqrt{2} \text{ is irrational.}$$

Now, check if $(b,a) \in S$:

$$b - a + \sqrt{2} \text{ may or may not be irrational.}$$

For example:

$$a = \sqrt{2},\ b = 1 \implies a - b + \sqrt{2} = \sqrt{2} - 1 + \sqrt{2} = 2\sqrt{2} - 1 \text{ (irrational), but}$$

$$b - a + \sqrt{2} = 1 - \sqrt{2} + \sqrt{2} = 1 \text{ (rational).}$$

Thus, S is not symmetric.

Step 3: Transitivity

Let $(a, b) \in S$ and $(b, c) \in S$, so:

$$a - b + \sqrt{2} \text{ is irrational, and } b - c + \sqrt{2} \text{ is irrational.}$$

Check if $(a, c) \in S$:

$$a - c + \sqrt{2} = (a - b + \sqrt{2}) + (b - c + \sqrt{2}) - \sqrt{2} \text{ may or may not be irrational.}$$

For example:

$$a = 1, b = \sqrt{3}, c = \sqrt{3} - \sqrt{2} \implies a - c + \sqrt{2} = 1 - (\sqrt{3} - \sqrt{2}) + \sqrt{2} = 1 - \sqrt{3} + 2\sqrt{2}.$$

This is irrational, but a counterexample exists for other values. Thus, S is not transitive.

Step 4: Final conclusion

The relation S is reflexive but neither symmetric nor transitive.

Quick Tip

To test reflexivity, verify if $(a, a) \in S$ for all a. For symmetry and transitivity, check logical equivalence and counterexamples.

33(a). Find the distance between the line $\frac{x}{2} = \frac{2y-6}{4} = \frac{1-z}{-1}$ and another line parallel to it passing through the point $(4, 0, -5)$.

Solution:

Step 1: Standardize the equations of the lines

The given line L_1 is:

$$\frac{x}{2} = \frac{2y - 6}{4} = \frac{1 - z}{-1} \implies \vec{r}_1 = \vec{0} + \lambda(2\hat{i} + \hat{j} + \hat{k})$$

The line L_2 parallel to L_1 and passing through $(4, 0, -5)$ is:

$$\vec{r}_2 = (4\hat{i} - 5\hat{k}) + \mu(2\hat{i} + \hat{j} + \hat{k})$$

Step 2: Vector between the lines

Let $\vec{a}_2 - \vec{a}_1 = (4\hat{i} - 5\hat{k}) - (0) = 4\hat{i} - 5\hat{k}$. The direction vector $\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$.

Step 3: Find the shortest distance

The shortest distance $S.D.$ is given by:

$$\text{S.D.} = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b})|}{|\vec{b}|}$$

Compute $\vec{b} \times (\vec{a}_2 - \vec{a}_1)$:

$$\vec{b} \times (\vec{a}_2 - \vec{a}_1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 4 & 0 & -5 \end{vmatrix} = 9\hat{i} - 16\hat{j} + 14\hat{k}$$

The magnitude:

$$|\vec{b}| = \sqrt{2^2 + 1^2 + 1^2} = 3$$

$$\text{S.D.} = \frac{\sqrt{81 + 256 + 196}}{3} = \frac{\sqrt{533}}{3} \text{ units.}$$

Quick Tip

For the shortest distance between skew or parallel lines, use the cross-product approach for accuracy.

OR

33(b). If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-7}$ are perpendicular to each other, find the value of k and hence write the vector equation of a line perpendicular to these two lines passing through the point $(3, -4, 7)$.

Solution:

Step 1: Find k

The direction ratios of the first line are $\langle -3, 2k, 2 \rangle$ and for the second line are $\langle 3k, 1, -7 \rangle$. Since the lines are perpendicular:

$$(-3)(3k) + (2k)(1) + (2)(-7) = 0$$

$$-9k + 2k - 14 = 0 \implies -7k = 14 \implies k = -2$$

Step 2: Find the vector equation of the perpendicular line

The direction vectors are:

$$\vec{b}_1 = \langle -3, -4, 2 \rangle, \quad \vec{b}_2 = \langle -6, 1, -7 \rangle$$

The perpendicular vector is:

$$\vec{b} = \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -4 & 2 \\ -6 & 1 & -7 \end{vmatrix} = 26\hat{i} - 33\hat{j} - 27\hat{k}$$

The required equation is:

$$\vec{r} = \langle 3, -4, 7 \rangle + \lambda(26\hat{i} - 33\hat{j} - 27\hat{k})$$

Quick Tip

For perpendicular lines, use the dot product of their direction vectors to solve for unknown parameters.

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34. Use the product of matrices

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ -7 & -7 & -7 \\ 7 & 5 & -4 \end{bmatrix}$$

to solve the following system of equations:

$$x + 2y - 3z = 6$$

$$3x + 2y - 2z = 3$$

$$2x - y + z = 2$$

Solution:

Step 1: Compute the matrix product. Given matrices A and B, compute AB:

$$AB = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ -7 & -7 & -7 \\ 7 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = 7I$$

Since $AB = 7I$, it follows that $A^{-1} = \frac{1}{7}B$.

Step 2: Compute A^{-1}.

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 0 & 1 & 2 \\ -7 & -7 & -7 \\ 7 & 5 & -4 \end{bmatrix}$$

Step 3: Convert system into matrix form. The given system can be written as:

$$AX = C$$

where

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad C = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

Thus,

$$X = A^{-1}C$$

Step 4: Compute X.

$$X = \frac{1}{7} \begin{bmatrix} 0 & 1 & 2 \\ -7 & -7 & -7 \\ 7 & 5 & -4 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

Multiplying the matrices:

$$X = \frac{1}{7} \begin{bmatrix} (0 \times 6) + (1 \times 3) + (2 \times 2) \\ (-7 \times 6) + (-7 \times 3) + (-7 \times 2) \\ (7 \times 6) + (5 \times 3) + (-4 \times 2) \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 + 4 \\ -42 - 21 - 14 \\ 42 + 15 - 8 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 7 \\ -77 \\ -49 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -5 \end{bmatrix}$$

Conclusion: Thus, the solution to the given system is:

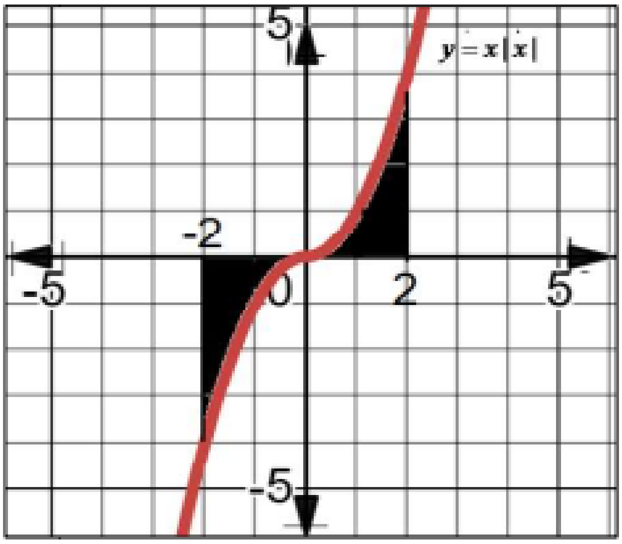
$$x = 1, \quad y = -5, \quad z = -5$$

Quick Tip

For solving systems of linear equations using matrices, use $X = A^{-1}C$ when A is invertible, and compute A^{-1} from the given product matrices.

35(a). Sketch the graph of $y = x|x|$ and hence find the area bounded by this curve, the X-axis, and the ordinates $x = -2$ and $x = 2$, using integration.

Solution:



Step 1: Rewrite the function $y = x|x|$

The function $y = x|x|$ can be expressed as:

$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

Step 2: Graph the function

The graph of $y = x|x|$ is a parabola, concave downwards for $x < 0$ and concave upwards for $x \geq 0$. (Refer to the attached graph.)

Step 3: Area computation using integration

The area of the shaded region between $x = -2$ and $x = 2$ is given by:

$$\text{Area} = \int_{-2}^{2} |y|\, dx = 2\int_{0}^{2} x^2\, dx$$

Step 4: Evaluate the integral

$$\int_{0}^{2} x^2\, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$$

Thus, the total area is:

$$\text{Area} = 2 \cdot \frac{8}{3} = \frac{16}{3}$$

Step 5: Final result

The area bounded by the curve $y = x|x|$, the X-axis, and the ordinates $x = -2$ and $x = 2$ is $\frac{16}{3}$.

Quick Tip

When finding the area bounded by curves, split the integral into regions where the function behaves differently (e.g., absolute values or piecewise functions).

OR

35(b). Using integration, find the area bounded by the ellipse $9x^2 + 25y^2 = 225$, the lines $x = -2, x = 2$, and the X-axis.

Solution:

Step 1: Rewrite the equation of the ellipse

The equation of the ellipse is:

$$9x^2 + 25y^2 = 225 \implies y = \pm\frac{3}{5}\sqrt{25 - x^2}$$

Step 2: Set up the integral for the area

The area of the region bounded by the ellipse, the X-axis, and the lines $x = -2$ and $x = 2$ is given by:

$$\text{Area} = 2\int_{0}^{2} \frac{3}{5}\sqrt{25 - x^2}\, dx$$

Step 3: Simplify the integral

Let $I = \int \sqrt{a^2 - x^2}\, dx$, where $a = 5$. Using the standard formula:

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

Step 4: Evaluate the integral

Substitute $a = 5$ and evaluate $\int_0^2 \sqrt{25 - x^2}\, dx$:

$$\int_0^2 \sqrt{25 - x^2}\, dx = \left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_0^2$$

At $x = 2$:

$$\frac{2}{2}\sqrt{25 - 2^2} + \frac{25}{2}\sin^{-1}\left(\frac{2}{5}\right) = \sqrt{21} + \frac{25}{2}\sin^{-1}\left(\frac{2}{5}\right)$$

At $x = 0$:

$$\frac{0}{2}\sqrt{25 - 0^2} + \frac{25}{2}\sin^{-1}(0) = 0$$

Thus:

$$\int_0^2 \sqrt{25 - x^2}\, dx = \sqrt{21} + \frac{25}{2}\sin^{-1}\left(\frac{2}{5}\right)$$

Step 5: Final area calculation

Multiply by $\frac{6}{5}$ to account for $\frac{3}{5}$ and the factor of 2:

$$\text{Area} = \frac{6}{5}\left(\sqrt{21} + \frac{25}{2}\sin^{-1}\left(\frac{2}{5}\right)\right)$$

Step 6: Final result

The area bounded by the ellipse, the X-axis, and the lines $x = -2$ and $x = 2$ is:

$$\frac{6\sqrt{21}}{5} + 15\sin^{-1}\left(\frac{2}{5}\right)$$

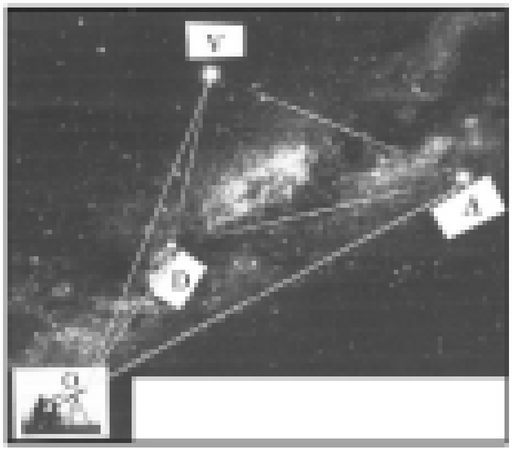
Quick Tip

When integrating to find areas involving ellipses or circles, use symmetry and standard integral formulas for $\sqrt{a^2 - x^2}$.

Case Study - 1

36. An instructor at the astronomical centre shows three among the brightest stars in a particular constellation. Assume that the telescope is located at $O(0, 0, 0)$ and the three

stars have their locations at points $D, A,$ and V, having position vectors $2\hat{i}+3\hat{j}+4\hat{k}$, $7\hat{i}+5\hat{j}+8\hat{k}$, and $-3\hat{i}+7\hat{j}+11\hat{k}$, respectively. Based on the above information, answer the following questions:



36.(i). How far is the star V from star A?

Solution:

Step 1: Compute the position vector of $\overrightarrow{AV}$

$$\overrightarrow{AV} = \text{Position vector of } V - \text{Position vector of } A$$

$$\overrightarrow{AV} = (-3\hat{i}+7\hat{j}+11\hat{k}) - (7\hat{i}+5\hat{j}+8\hat{k}) = -10\hat{i}+2\hat{j}+3\hat{k}.$$

Step 2: Compute the magnitude of $\overrightarrow{AV}$

$$|\overrightarrow{AV}| = \sqrt{(-10)^2+2^2+3^2} = \sqrt{100+4+9} = \sqrt{113}.$$

Step 3: Final result

The distance between star V and star A is $\sqrt{113}$ units.

Quick Tip

To find the distance between two points, use the magnitude of the difference of their position vectors.

36.(ii). Find a unit vector in the direction of $\overrightarrow{DA}$.

Solution:

Step 1: Compute $\overrightarrow{DA}$

$$\overrightarrow{DA} = \text{Position vector of } A - \text{Position vector of } D$$

$$\overrightarrow{DA} = (7\hat{i} + 5\hat{j} + 8\hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k}) = 5\hat{i} + 2\hat{j} + 4\hat{k}.$$

Step 2: Find the magnitude of $\overrightarrow{DA}$

$$|\overrightarrow{DA}| = \sqrt{(5)^2 + (2)^2 + (4)^2} = \sqrt{25 + 4 + 16} = \sqrt{45} = 3\sqrt{5}.$$

Step 3: Compute the unit vector

The unit vector is:

$$\hat{u} = \frac{\overrightarrow{DA}}{|\overrightarrow{DA}|} = \frac{5\hat{i} + 2\hat{j} + 4\hat{k}}{3\sqrt{5}} = \frac{5}{3\sqrt{5}}\hat{i} + \frac{2}{3\sqrt{5}}\hat{j} + \frac{4}{3\sqrt{5}}\hat{k}.$$

Step 4: Final result

The unit vector in the direction of $\overrightarrow{DA}$ is:

$$\frac{5}{3\sqrt{5}}\hat{i} + \frac{2}{3\sqrt{5}}\hat{j} + \frac{4}{3\sqrt{5}}\hat{k}.$$

> **Quick Tip**
>
> To find a unit vector, divide the vector by its magnitude.

36.(iii)(a). Find the measure of $\angle VDA$.

Solution:

Step 1: Recall the formula for the angle between vectors

The angle θ between two vectors $\overrightarrow{VD}$ and $\overrightarrow{DA}$ is given by:

$$\cos\theta = \frac{\overrightarrow{VD} \cdot \overrightarrow{DA}}{|\overrightarrow{VD}| \cdot |\overrightarrow{DA}|}.$$

Step 2: Compute $\overrightarrow{VD}$ and $\overrightarrow{DA}$

From previous calculations:

$$\overrightarrow{VD} = \overrightarrow{V} - \overrightarrow{D} = (-3\hat{i} + 7\hat{j} + 11\hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k}) = -5\hat{i} + 4\hat{j} + 7\hat{k}.$$

$$\overrightarrow{DA} = 5\hat{i} + 2\hat{j} + 4\hat{k}.$$

Step 3: Compute $\overrightarrow{VD} \cdot \overrightarrow{DA}$

$$\overrightarrow{VD} \cdot \overrightarrow{DA} = (-5)(5) + (4)(2) + (7)(4) = -25 + 8 + 28 = 11.$$

Step 4: Compute magnitudes of $\overrightarrow{VD}$ and $\overrightarrow{DA}$

$$|\overrightarrow{VD}| = \sqrt{(-5)^2 + 4^2 + 7^2} = \sqrt{25 + 16 + 49} = \sqrt{90}.$$

$$|\overrightarrow{DA}| = \sqrt{(5)^2 + (2)^2 + (4)^2} = \sqrt{25 + 4 + 16} = \sqrt{45}.$$

Step 5: Compute $\cos\theta$

$$\cos\theta = \frac{11\sqrt{2}}{\sqrt{90} \cdot \sqrt{45}} = \frac{11\sqrt{2}}{\sqrt{4050}} = \frac{11\sqrt{2}}{90}.$$

Step 6: Final result

The measure of $\angle VDA$ is:

$$\theta = \cos^{-1}\left(\frac{11\sqrt{2}}{90}\right).$$

Quick Tip

For angles between vectors, always use the dot product formula and ensure the magnitude is correctly computed.

36.(iii)(b). What is the projection of vector $\overrightarrow{DV}$ on vector $\overrightarrow{DA}$?

Solution:

Step 1: Recall the formula for projection

The projection of $\overrightarrow{DV}$ on $\overrightarrow{DA}$ is given by:

$$\text{Projection} = \frac{\overrightarrow{DV} \cdot \overrightarrow{DA}}{|\overrightarrow{DA}|}.$$

Step 2: Compute $\overrightarrow{DV}$

$$\overrightarrow{DV} = \overrightarrow{V} - \overrightarrow{D} = (-5\hat{i} + 4\hat{j} + 7\hat{k}).$$

Step 3: Compute $\overrightarrow{DV} \cdot \overrightarrow{DA}$

From the previous calculations:

$$\overrightarrow{DV} \cdot \overrightarrow{DA} = (-5)(5) + (4)(2) + (7)(4) = -25 + 8 + 28 = 11.$$

Step 4: Compute $|\overrightarrow{DA}|$

$$|\overrightarrow{DA}| = \sqrt{(5)^2 + (2)^2 + (4)^2} = \sqrt{45} = 3\sqrt{5}.$$

Step 5: Compute the projection

$$\text{Projection} = \frac{\overrightarrow{DV} \cdot \overrightarrow{DA}}{|\overrightarrow{DA}|} = \frac{11}{3\sqrt{5}}.$$

Step 6: Final result

The projection of $\overrightarrow{DV}$ on $\overrightarrow{DA}$ is:

$$\frac{11\sqrt{5}}{15}.$$

Quick Tip

The projection of one vector onto another gives the component of the first vector along the direction of the second.

Case Study - 2

37 Rohit, Jaspreet, and Alia appeared for an interview for three vacancies in the same post. The probability of Rohit's selection is $\frac{1}{5}$, Jaspreet's selection is $\frac{1}{3}$, and Alia's selection is $\frac{1}{4}$. The events of selection are independent of each other.

Based on the above information, answer the following questions:

37.(i). What is the probability that at least one of them is selected?

Solution:

Step 1: Probability of no one being selected

The probability that none of them are selected is:

$$P(\text{No one selected}) = \left(1 - \frac{1}{5}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \left(1 - \frac{1}{4}\right) = \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{2}{5}.$$

Step 2: Probability of at least one being selected

The probability that at least one of them is selected is:

$$P(\text{At least one selected}) = 1 - P(\text{No one selected}) = 1 - \frac{2}{5} = \frac{3}{5}.$$

Final Result: The probability that at least one of them is selected is $\frac{3}{5}$.

Quick Tip

To find the probability of "at least one" event happening, use the complement rule: $P(\text{At least one}) = 1 - P(\text{None})$.

37.(ii). Find $P(G \cap \overline{H})$, where G is the event of Jaspreet's selection and $\overline{H}$ denotes the event that Rohit is not selected.

Solution:

Step 1: Compute the probability

The probability $P(G \cap \overline{H})$ is given by:

$$P(G \cap \overline{H}) = P(G) \cdot P(\overline{H}),$$

where $P(G) = \frac{1}{3}$ and $P(\overline{H}) = 1 - P(H) = 1 - \frac{1}{5} = \frac{4}{5}$.

$$P(G \cap \overline{H}) = \frac{1}{3} \cdot \frac{4}{5} = \frac{4}{15}.$$

Final Result: The probability $P(G \cap \overline{H})$ is $\frac{4}{15}$.

Quick Tip

For independent events, the probability of their intersection is the product of their individual probabilities.

37.(iii)(a). Find the probability that exactly one of them is selected.

Solution:

Step 1: Compute the probability of exactly one being selected

The probability of exactly one being selected is:

$$P(\text{Exactly one selected}) = P(R) \cdot P(\overline{J}) \cdot P(\overline{A}) + P(\overline{R}) \cdot P(J) \cdot P(\overline{A}) + P(\overline{R}) \cdot P(\overline{J}) \cdot P(A),$$

where:

$$P(\overline{J}) = 1 - P(J) = \frac{2}{3}, \quad P(\overline{A}) = 1 - P(A) = \frac{3}{4}, \quad P(\overline{R}) = 1 - P(R) = \frac{4}{5}.$$

Substitute the values:

$$P(\text{Exactly one selected}) = \frac{1}{5} \cdot \frac{2}{3} \cdot \frac{3}{4} + \frac{4}{5} \cdot \frac{1}{3} \cdot \frac{3}{4} + \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{1}{4}.$$

Simplify each term:

$$P(\text{Exactly one selected}) = \frac{6}{60} + \frac{12}{60} + \frac{8}{60} = \frac{26}{60} = \frac{13}{30}.$$

Final Result: The probability that exactly one of them is selected is $\frac{13}{30}$.

Quick Tip

To calculate "exactly one" selection, consider all cases where one succeeds, and the others fail, then sum the probabilities.

37.(iii)(b). Find the probability that exactly two of them are selected.

Solution:

Step 1: Compute the probability of exactly two being selected

The probability of exactly two being selected is:

$$P(\text{Exactly two selected}) = P(R) \cdot P(J) \cdot P(\overline{A}) + P(R) \cdot P(\overline{J}) \cdot P(A) + P(\overline{R}) \cdot P(J) \cdot P(A).$$

Substitute the values:

$$P(\text{Exactly two selected}) = \frac{1}{5} \cdot \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{5} \cdot \frac{2}{3} \cdot \frac{1}{4} + \frac{4}{5} \cdot \frac{1}{3} \cdot \frac{1}{4}.$$

Simplify each term:

$$P(\text{Exactly two selected}) = \frac{3}{60} + \frac{2}{60} + \frac{4}{60} = \frac{9}{60} = \frac{3}{20}.$$

Final Result: The probability that exactly two of them are selected is $\frac{3}{20}$.

Quick Tip

For "exactly two" events, consider all pairs of selections and one failure, and sum their probabilities.

Case Study - 3

38. A store has been selling calculators at Rs. 350 each. A market survey indicates that a reduction in price (p) of calculators increases the number of units (x) sold. The relation between the price and quantity sold is given by the demand function:

$$p = 450 - \frac{x}{2}.$$

Based on the above information, answer the following questions:

38.(i). Determine the number of units (x) that should be sold to maximise the revenue $R(x) = x \cdot p(x)$. Also verify the result.

Solution:

Step 1: Express revenue as a function of x

Revenue is given by:

$$R(x) = x \cdot p(x) = x \cdot \left(450 - \frac{x}{2}\right) = 450x - \frac{x^2}{2}.$$

Step 2: Differentiate to find critical points

The first derivative of $R(x)$ is:

$$\frac{dR}{dx} = 450 - x.$$

For maximum or minimum, set $\frac{dR}{dx} = 0$:

$$450 - x = 0 \implies x = 450.$$

Step 3: Verify using the second derivative

The second derivative of $R(x)$ is:

$$\frac{d^2R}{dx^2} = -1 < 0.$$

Since $\frac{d^2R}{dx^2} < 0$, $R(x)$ is maximum when $x = 450$.

Step 4: Final result

The number of units that should be sold to maximise revenue is $x = 450$.

Quick Tip

To maximise revenue or profit, always verify the nature of the critical point using the second derivative test.

38.(ii). What rebate in price of the calculator should the store give to maximise the revenue?

Solution:

Step 1: Calculate the price at $x = 450$

The price is given by:

$$p = 450 - \frac{x}{2} = 450 - \frac{450}{2} = 225.$$

Step 2: Compute the rebate

The original price is Rs. 350. The rebate is:

$$\text{Rebate} = 350 - 225 = 125\,(\text{Rs. per calculator}).$$

Step 3: Final result

The rebate required to maximise the revenue is Rs. 125 per calculator.

Quick Tip

For pricing and revenue problems, calculate the optimal price after determining the maximum quantity sold.
