CUET PG 2024 Statistics Shift 3

Time Allowed : 1 Hours 45 minutes	Maximum Marks : 300	Total Questions : 75

General Instructions

Read the following instructions very carefully and strictly follow them:

1. This question paper comprises 75 questions. All questions are compulsory.
2. Each question carries 04 (four) marks.
3. For each correct response, the candidate will get 04 (four) marks.
4. For each incorrect response, 01 (one) mark will be deducted from the total score.
5. Un-answered/un-attempted response will be given no marks.
6. To answer a question, the candidate needs to choose one option as the correct option.
7. However, after the process of challenges of the Answer Key, in case there are multiple correct options or a change in the key, only those candidates who have attempted it correctly as per the revised Final Answer Key will be awarded marks.
8. In case a question is dropped due to some technical error, full marks shall be given to all the candidates irrespective of the fact who have attempted it or not.

Question:1

Evaluate the following:

$$\lim_{n\to\infty} \frac{1}{n}\left(1 + 2^2 + 3^3 + \ldots + n^n\right)$$

Options:

(A) 0

(B) 1

(C) 3

(D) ∞

Correct Answer: (D) ∞

Solution:

The given expression is:

$$\lim_{n\to\infty} \frac{1}{n}\left(1 + 2^2 + 3^3 + \ldots + n^n\right)$$

Step 1: Analyze the dominant term

In the series, the last term, n^n, grows much faster than any preceding term as $n \to \infty$. Thus, n^n dominates the entire sum.

Step 2: Simplify the limit

The sum can be approximated as:

$$1 + 2^2 + 3^3 + \ldots + n^n \approx n^n$$

So, the limit becomes:

$$\lim_{n\to\infty} \frac{1}{n}(n^n) = \lim_{n\to\infty} \frac{n^n}{n} = \lim_{n\to\infty} n^{n-1}$$

As $n \to \infty$, $n^{n-1} \to \infty$. Hence, the value of the limit is ∞.

> **Quick Tip**
>
> When dealing with series involving terms like n^n, always consider the term that grows the fastest as $n \to \infty$. In most cases, this term will dominate the series, and the limit depends on it.

Question:2

If $\{x_n\}$ is a real sequence such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l \text{ where } |l| < 1, \text{ then } \lim_{n\to\infty} a_n = ?$$

Options:

(A) 0

(B) 1

(C) $\frac{1}{2}$

(D) ∞

Correct Answer: (A) 0

Solution:

The ratio test states that for a sequence $\{a_n\}$, if:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l \quad \text{and} \quad |l| < 1,$$

then $a_n \to 0$ as $n \to \infty$. This is because each term becomes smaller exponentially, approaching zero.

Thus, the given sequence satisfies the conditions of the ratio test, and the limit is:

$$\lim_{n\to\infty} a_n = 0.$$

Quick Tip

For sequences or series, if $|l| < 1$ in the ratio test, the terms converge to 0. Always use this test for geometric or exponential terms.

Question:3

The real series

$$\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1} x^n, \quad x > 0$$

converges under which condition?

Options:

(A) Convergent if $x > 1$

(B) Convergent if $x \geq 1$

(C) Convergent if $x < 1$

(D) Convergent if $x \leq 1$

Correct Answer: (C) Convergent if $x < 1$

Solution:

The given series can be analyzed using the ratio test:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} |x| = |x|,$$

where $a_n = \frac{n^2-1}{n^2+1}x^n$. The series converges when the ratio $|x| < 1$. For $x \geq 1$, the terms do not decay, and the series diverges.

Thus, the series converges for:

$$x < 1.$$

Quick Tip

When analyzing series with terms x^n, always check convergence using the ratio test. The series typically converges for $|x| < 1$.

Question:4

Which of the following statements is/are correct?

(A) A bounded sequence of real numbers which does not converge has at least two limit points.

(B) An unbounded sequence of real numbers from below then $-\infty$ is a limit point of the sequence.

(C) $\lim_{n \to \infty} \sqrt[n]{a} = 1, \text{ if } a > 0.$

(D) A sequence $\{x_n\}$ defined as $x_{n+1} = \sqrt[3]{x_n}$, $x_1 = 1$, $\forall n \geq 1$ converges to zero.

Choose the correct answer from the options given below:

(A) A, C, D

(B) A, B

(C) A, B, C

(D) B, C, D

Correct Answer: (B) A, B

Solution:

1. Statement (A): True.

A bounded sequence of real numbers that does not converge must oscillate between two or more values, creating at least two limit points.

2. Statement (B): True.

An unbounded sequence from below means values can approach $-\infty$, and $-\infty$ can be considered a limit point for such sequences.

3. Statement (C): False.

The root test does not apply here, as the sequence must be re-evaluated for the given boundary conditions.

4. Statement (D): False.

The recursive sequence defined by $x_{n+1} = \sqrt[3]{x_n}$ converges to a constant value greater than zero, not zero.

Quick Tip

For sequence-related problems, analyze boundedness and limit definitions carefully. Unbounded sequences can have limit points like $+\infty$ or $-\infty$ under proper definitions.

Question:5

Let $A\mathrm{x} = \mathrm{b}$ be a non-homogeneous system of linear equations. The augmented matrix $[A : \mathrm{b}]$ is given by:

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & -3 & 0 \\ 0 & 3 & 0 & -1 \end{bmatrix}$$

Which of the following statements is true?

(A) Rank of A is 3.

(B) The system has no solution.

(C) The system has a unique solution.

(D) The system has an infinite number of solutions.

Correct Answer: (B) The system has no solution.

Solution:

1. Calculate the rank of A (Coefficient Matrix):

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

The determinant of A is non-zero, implying that $\text{rank}(A) = 3$.

2. Check consistency of the augmented matrix:

The rank of the augmented matrix $[A : \mathbf{b}]$ exceeds the rank of A, indicating inconsistency.

This inconsistency implies the system is unsolvable.

3. Conclusion:

Since $\text{rank}(A) \neq \text{rank}([A : \mathbf{b}])$, the system has no solution.

Quick Tip

For linear systems, if $\text{rank}(A) \neq \text{rank}([A : \mathbf{b}])$, the system is inconsistent and has no solutions.

Question:6

A function $f(x)$ defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ -x, & \text{if } x \text{ is irrational,} \end{cases}$$

is:

Options:

(A) Discontinuous at every real number.

(B) Discontinuous at $x = 0$.

(C) Continuous at $x = 0$.

(D) Continuous at all non-zero real numbers.

Correct Answer: (C) Continuous at $x = 0$.

Solution:

1. At $x = 0$: For $x = 0$, whether x is rational or irrational, $f(x) = 0$.

Hence, $f(0) = 0$, and the left-hand limit ($\lim_{x\to 0^-}$) and right-hand limit ($\lim_{x\to 0^+}$) are both equal to 0. Thus, $f(x)$ is continuous at $x = 0$.

2. At $x \neq 0$: For non-zero values of x, $f(x)$ takes two distinct values depending on whether x is rational or irrational. Thus, $f(x)$ is not continuous at any $x \neq 0$.

Quick Tip

For piecewise functions involving rational and irrational cases, check continuity separately at specific points like $x = 0$ or $x \neq 0$.

Question 7: Match List I with List II:

List I	Function	List II: Their Series Expansion
A	$\log(1+x)$	(I) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$
B	$\cos x$	(II) $x^2 + \frac{x^4}{12} + \dots$
C	$\log 2$	(III) $\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$ for $-1 < x \le 1$
D	$\log\sec x$	(IV) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \forall x \in \mathbb{R}$

Choose the correct answer from the options given below:

(A) A-IV, B-III, C-I, D-II

(B) A-IV, B-I, C-III, D-II

(C) A-III, B-IV, C-I, D-II

(D) A-II, B-III, C-I, D-IV

Correct Answer: (C) A-III, B-IV, C-I, D-II

Solution:

- **A.** $\log(1+x)$ **(A - III):** The series expansion for $\log(1+x)$ is $\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$ for $-1 < x \le 1$.
- **B.** $\cos x$ **(B - IV):** The expansion for $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \forall x \in \mathbb{R}$.
- **C.** $\log 2$ **(C - I):** The series expansion of $\log 2$ is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$.
- **D.** $\log\sec x$ **(D - II):** The series expansion of $\log\sec x$ is $x^2 + \frac{x^4}{12} + \dots$.

Quick Tip

Tip: Remember common expansions:

- $\log(1 + x)$: Series converges for $-1 < x \leq 1$.
- $\cos x$: Alternating even power terms.
- $\log 2$: Alternating terms starting with 1.
- $\log \sec x$: Involves even power terms only.

Mastering series expansions is essential for calculus and mathematical analysis.

Question:8

$$\lim_{x \to 0} \frac{(1 + x)^x - e}{x} \text{ is:}$$

Options:

(A) e

(B) $-e$

(C) $\frac{e}{2}$

(D) $-\frac{e}{2}$

Correct Answer: (D) $-\frac{e}{2}$

Solution:

The given limit is:

$$\lim_{x \to 0} \frac{(1 + x)^x - e}{x}.$$

1. Expand $(1 + x)^x$ using the exponential-logarithmic identity:

$$(1 + x)^x = e^{x \ln(1+x)}.$$

For small x, $\ln(1 + x) \approx x$. Thus:

$$x \ln(1 + x) \approx x \cdot x = x^2.$$

So, $(1 + x)^x \approx e^{x^2} \approx e + e \cdot x^2$.

2. Substitute in the numerator:

$$(1+x)^x - e \approx e \cdot x^2 - e.$$

3. Evaluate the limit:

$$\frac{(1+x)^x - e}{x} \approx \frac{e \cdot x^2 - e}{x} = -\frac{e}{2}.$$

Quick Tip

For limits involving $(1+x)^x$, use the exponential approximation: $(1+x)^x \approx e^{x\ln(1+x)}$ and simplify using Taylor expansion.

Question:9

The function $f(x) = (x-3)^5(x+1)^4$ **has:**

Options:

(A) $x = -1$ is a point of maxima and $x = \frac{7}{9}$ is a point of minima.

(B) $x = \frac{7}{9}$ is a point of maxima and $x = -1$ is a point of minima.

(C) $x = -1$ and $x = 3$ are points of maxima and $x = \frac{7}{9}$ is a point of minima.

(D) Neither a point of maxima nor a point of minima.

Correct Answer: (A) $x = -1$ **is a point of maxima and** $x = \frac{7}{9}$ **is a point of minima.**

Solution:

1. Find the critical points:

The critical points are where $f'(x) = 0$. Differentiate $f(x)$ using the product rule:

$$f'(x) = 5(x-3)^4(x+1)^4 + 4(x-3)^5(x+1)^3.$$

Factoring common terms:

$$f'(x) = (x-3)^4(x+1)^3\left[5(x+1) + 4(x-3)\right].$$

Simplify:

$$f'(x) = (x-3)^4(x+1)^3(9x-7).$$

The critical points are $x = 3$, $x = -1$, and $x = \frac{7}{9}$.

2. Test for maxima or minima using the second derivative test:

- At $x = -1$: The second derivative test indicates $f''(-1) < 0$, so $x = -1$ is a point of maxima. - At $x = \frac{7}{9}$: $f''\left(\frac{7}{9}\right) > 0$, so $x = \frac{7}{9}$ is a point of minima.

3. Conclusion:

$x = -1$ is a point of maxima, and $x = \frac{7}{9}$ is a point of minima.

Quick Tip

Use the first derivative to find critical points and the second derivative to classify them as maxima or minima.

Question:10

The point of local minima of the function $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ **is:**

Options:

(A) $(\sqrt{2}, -\sqrt{2})$

(B) $(-\sqrt{2}, -\sqrt{2})$

(C) $(\sqrt{2}, \sqrt{2})$

(D) $(0, 0)$

Correct Answer: (A) $(\sqrt{2}, -\sqrt{2})$

Solution:

1. Find the critical points:

The critical points are obtained by solving $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

- $\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y = 0$.
- $\frac{\partial f}{\partial y} = 4y^3 - 4y + 4x = 0$.

Solve these equations simultaneously:

$$4x^3 - 4x + 4y = 0, \quad 4y^3 - 4y + 4x = 0.$$

By substituting and solving, we find critical points at $(x, y) = (\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$, $(\sqrt{2}, \sqrt{2})$, and $(0, 0)$.

2. Classify the critical points:

Compute the second partial derivatives:

$$f_{xx} = 12x^2 - 4, \quad f_{yy} = 12y^2 - 4, \quad f_{xy} = 4.$$

Calculate the determinant of the Hessian matrix:

$$H = f_{xx}f_{yy} - (f_{xy})^2.$$

At $(x, y) = (\sqrt{2}, -\sqrt{2})$:

$$f_{xx} = 20, \quad f_{yy} = 20, \quad f_{xy} = 4.$$

$$H = (20)(20) - (4)^2 = 400 - 16 = 384 > 0.$$

Since $f_{xx} > 0$, $(\sqrt{2}, -\sqrt{2})$ is a local minimum.

3. Conclusion:

The point of local minima is $(\sqrt{2}, -\sqrt{2})$.

Quick Tip

For multivariable functions, always calculate the Hessian determinant (H) and use f_{xx} to confirm whether the point is a minimum ($H > 0$, $f_{xx} > 0$).

Question:11

The region R is bounded by $x = 0, x = 2, y = x,$ and $y = x + 2.$ Then

$$\iint_R (x + y)\, dy\, dx$$

is equal to:

Options:

(A) 3 units

(B) 6 units

(C) 9 units

(D) 12 units

Correct Answer: (D) 12 units

Solution:

1. Understand the region R: - The region is bounded by $x = 0$, $x = 2$, $y = x$, and $y = x + 2$. These lines form a quadrilateral in the xy-plane.

2. Set up the double integral: The limits for x are from 0 to 2, and for y, they are from x to $x+2$. The integral is:

$$\iint_R (x+y)\,dy\,dx = \int_0^2 \int_x^{x+2} (x+y)\,dy\,dx.$$

3. Evaluate the inner integral:

$$\int_x^{x+2} (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_x^{x+2}.$$

Substitute the limits:

$$= \left[x(x+2) + \frac{(x+2)^2}{2}\right] - \left[x^2 + \frac{x^2}{2}\right].$$

Simplify:

$$= x(x+2) + \frac{x^2+4x+4}{2} - x^2 - \frac{x^2}{2}.$$

$$= 2x + 2 + 2x + 2.$$

4. Evaluate the outer integral: After simplification, the integral becomes:

$$\int_0^2 (x+x+2)\,dx = \int_0^2 (2x+2)\,dx.$$

Solve:

$$\int_0^2 2x\,dx = [x^2]_0^2 = 4,$$

$$\int_0^2 2\,dx = [2x]_0^2 = 4.$$

Add them:

$$4 + 4 = 12.$$

5. Conclusion: The value of the integral is:

$$\iint_R (x+y)\,dy\,dx = 12 \text{ units}.$$

Quick Tip

Always sketch the region for double integrals. Set up the limits carefully based on the bounding lines or curves and integrate step-by-step.

Question:12

Let the region R be bounded by $x = 0, y = 0, z = 0$ and $x + y + z = a\ (a > 0)$. Then the value of

$$\iiint_R (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

is:

Options:

(A) $\frac{a^5}{20}$

(B) $\frac{a^5}{10}$

(C) $\frac{a^5}{5}$

(D) $\frac{a^5}{2}$

Correct Answer: (A) $\frac{a^5}{20}$

Solution:

1. Understand the region R: The region is the tetrahedron in the first octant bounded by $x = 0$, $y = 0$, $z = 0$, and the plane $x + y + z = a$.

2. Set up the triple integral: The integral can be written as:

$$\iiint_R (x^2 + y^2 + z^2)\, dx\, dy\, dz = \int_0^a \int_0^{a-z} \int_0^{a-y-z} (x^2 + y^2 + z^2)\, dx\, dy\, dz.$$

3. Evaluate the inner integral: The integral over x is:

$$\int_0^{a-y-z} x^2\, dx = \left[\frac{x^3}{3}\right]_0^{a-y-z} = \frac{(a-y-z)^3}{3}.$$

4. Integrate over y and z: Substitute into the double integral for y and z, and simplify the limits:

$$\int_0^a \int_0^{a-z} \left(\frac{(a-y-z)^3}{3} + y^2 + z^2\right) dy\, dz.$$

Solve step-by-step, integrating first with respect to y and then z.

5. Final Result: After evaluating the integral, the result is:

$$\iiint_R (x^2 + y^2 + z^2)\, dx\, dy\, dz = \frac{a^5}{20}.$$

Quick Tip

For triple integrals in tetrahedrons, identify the bounding planes carefully and set up the limits step-by-step. Simplify symmetric expressions where possible.

Question:13

The system of linear equations

$$x + y + z = 6, \quad x + 2y + 3z = 10, \quad x + 2y + \lambda z = \mu$$

has an infinite number of solutions. The values of λ and μ are:

Options:

(A) $\lambda = 3, \mu = -10$

(B) $\lambda = 3, \mu = 10$

(C) $\lambda = 3,$ whatever μ may be

(D) $\lambda = 3, \mu \neq 10$

Correct Answer: **(B)** $\lambda = 3, \mu = 10$

Solution:

1. Condition for infinite solutions:

A system of linear equations has infinite solutions if the rank of the coefficient matrix A equals the rank of the augmented matrix $[A : \mathbf{b}]$, but both are less than the number of variables.

2. Coefficient matrix and augmented matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}, \quad [A : \mathbf{b}] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}.$$

3. Rank condition:

To have infinite solutions, the determinant of A must be zero. Compute $\det(A)$:

$$\det(A) = 1 \begin{vmatrix} 2 & 3 \\ 2 & \lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 1 & \lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}.$$

Simplify:

$$\det(A) = 1[(2\lambda - 6)] - 1[(\lambda - 3)] + 1[0] = 2\lambda - 6 - \lambda + 3 = \lambda - 3.$$

For $\det(A) = 0$, $\lambda = 3$.

4. Augmented matrix:

Substitute $\lambda = 3$ into $[A : \mathbf{b}]$. For consistency, $\mu = 10$.

Quick Tip

For linear systems with infinite solutions, set $\det(A) = 0$ to find parameter conditions, then check consistency with the augmented matrix.

Question:14

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then:

Options:

(A) $f'(x)$ is continuous at $x = 0$.

(B) $f''(x)$ is continuous at $x = 0$.

(C) $f'(0)$ exists.

(D) $f''(0)$ exists.

Correct Answer: (C) $f'(0)$ exists.

Solution:

1. Compute $f'(x)$:

For $x \neq 0$, using the product rule:

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

At $x = 0$, by definition:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x}.$$

Simplify:

$$f'(0) = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right).$$

Since $\sin\left(\frac{1}{x}\right)$ is bounded between -1 and 1, $f'(0) = 0$.

2. Check for $f''(0)$:

Compute the second derivative. It involves terms like $x\cos\left(\frac{1}{x}\right)$, which do not have a well-defined limit as $x \to 0$. Thus, $f''(0)$ does not exist.

Quick Tip

When evaluating derivatives at points where the function is piecewise-defined, use the limit definition carefully to confirm existence.

Question:15

Consider the two series:

$$S_1 = \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+3)}, \quad S_2 = \sum_{k=1}^{\infty} \frac{1}{\sqrt{(k+1)(k+3)}}.$$

Then:

Options:

(A) S_1 and S_2 converge.

(B) S_1 diverges, S_2 converges.

(C) S_1 converges, S_2 diverges.

(D) S_1 and S_2 diverge.

Correct Answer: (C) S_1 converges, S_2 diverges.

Solution:

1. Analyze S_1:

The terms of S_1 can be simplified using partial fractions:

$$\frac{1}{(k+1)(k+3)} = \frac{1}{2(k+1)} - \frac{1}{2(k+3)}.$$

This is a telescoping series. The terms cancel out, leaving a finite value. Hence, S_1 converges.

2. Analyze S_2:

For S_2, note that:

$$\frac{1}{\sqrt{(k+1)(k+3)}} > \frac{1}{k+1}.$$

Since the harmonic series $\sum \frac{1}{k+1}$ diverges, S_2 also diverges.

Quick Tip

For series, simplify terms using partial fractions or compare with standard divergent series like the harmonic series to determine convergence.

Question:16

The solution of

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = e^{2x}\cos 3x$$

where c_1 and c_2 are arbitrary constants, is:

Options:

(A) $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) - \frac{1}{6}xe^{2x}\sin 3x$

(B) $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - \frac{1}{6}xe^{3x}\sin 3x$

(C) $y = e^{3x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{6}xe^{3x}\sin 3x$

(D) $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - \frac{1}{6}xe^{3x}\sin 3x$

Correct Answer: (C)

Solution:

1. Solve the complementary equation: The auxiliary equation is:

$$r^2 - 4r + 13 = 0.$$

Solve for r:

$$r = 2 \pm 3i.$$

Hence, the complementary solution is:

$$y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x).$$

2. Find the particular integral: Since the RHS is $e^{2x}\cos 3x$, and this matches the complementary solution form, assume a particular solution of:

$$y_p = xe^{2x}(A\cos 3x + B\sin 3x).$$

Substitute into the differential equation to solve for A and B. After simplification:

$$A = 0, \quad B = \frac{1}{6}.$$

3. General solution: Combine y_c and y_p:

$$y = e^{3x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{6}xe^{3x} \sin 3x.$$

Quick Tip

For non-homogeneous equations where the RHS matches the complementary solution, multiply the assumed particular solution by x to avoid duplication.

Question:17

The solution of the differential equation:

$$\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$$

is (where c is an arbitrary constant):

Options:

(A) $x^2 - y^2 - 4x - 8y - 14 = c$

(B) $x^2 + 2xy - 4x - 8y - 14 = c$

(C) $x^2 + 2xy - y^2 - 4x + 8y - 14 = c$

(D) $x^2 - 2xy - y^2 - 4x - 8y - 14 = c$

Correct Answer: (C)

Solution:

1. Rewrite the equation: Rewrite the given equation as:

$$(y - x - 4)\, dy = (y + x - 2)\, dx.$$

2. Simplify and integrate: Expand and rearrange terms:

$$(y\, dy - x\, dy - 4\, dy) = (y\, dx + x\, dx - 2\, dx).$$

Integrate both sides:

$$\int y\, dy - \int x\, dy - \int 4\, dy = \int y\, dx + \int x\, dx - \int 2\, dx.$$

3. Solve the integrals: After integrating:

$$\frac{y^2}{2} - \frac{x^2}{2} - 4y = xy - x^2 - 2x + c.$$

Simplify into standard form:

$$x^2 + 2xy - y^2 - 4x + 8y - 14 = c.$$

Quick Tip

For exact differential equations, separate variables carefully, and integrate each term systematically.

Question:18

If A is a skew-Hermitian matrix, then for a matrix B of appropriate order, B^*AB is a:

Options:

(A) Hermitian matrix

(B) Skew-Hermitian matrix

(C) Either Hermitian or skew-Hermitian matrix

(D) Neither Hermitian nor skew-Hermitian matrix

Correct Answer: (B)

Solution:

1. Property of skew-Hermitian matrices: A matrix A is skew-Hermitian if $A^* = -A$, where A^* is the conjugate transpose.

2. Transformation with B: Consider B^*AB. The conjugate transpose of B^*AB is:

$$(B^*AB)^* = B^*A^*B.$$

3. Substitute the skew-Hermitian property: Since $A^* = -A$:

$$(B^*AB)^* = B^*(-A)B = -B^*AB.$$

This satisfies the property of a skew-Hermitian matrix.

Quick Tip

For skew-Hermitian matrices, apply the conjugate transpose property directly to verify transformations.

Question:19

The eigenvalues of the matrix:

$$\begin{bmatrix} 2 & 1 & 0 \\ 9 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

are:

Options:

(A) $-2, 5, -1$

(B) $2, 5, -1$

(C) $2, 2, 5$

(D) $1, 2, 5$

Correct Answer: (B)

Solution:

1. Eigenvalue equation: Compute the determinant of $(A - \lambda I)$, where A is the given matrix:

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 9 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) = 0.$$

2. Expand the determinant:

$$(2-\lambda)\left[(2-\lambda)(2-\lambda) - 9\right] = 0.$$

Solve:

$$(2-\lambda)[(2-\lambda)^2 - 9] = 0.$$

Factorize:

$$(2-\lambda)^2(5-\lambda) = 0.$$

3. Eigenvalues: The eigenvalues are $\lambda = 2, 2, 5$.

Quick Tip

For eigenvalues of triangular matrices, use the determinant condition directly to simplify calculations.

Question 20: Match List I with List II:

List I	Matrix	List II: Characteristics Root
A	Hermitian matrix	(I) Unit modules
B	Skew-Hermitian matrix	(II) Diagonal elements of matrix
C	Unitary matrix	(III) Real
D	Diagonal matrix	(IV) Either zero or pure imaginary

Choose the correct answer from the options given below:

(A) A-IV, B-III, C-I, D-II

(B) A-IV, B-III, C-II, D-I

(C) A-III, B-IV, C-II, D-I

(D) A-III, B-IV, C-I, D-II

Correct Answer: (D) A-III, B-IV, C-I, D-II

Solution:

- **A. Hermitian matrix (A - III):** The eigenvalues of a Hermitian matrix are always real.
- **B. Skew-Hermitian matrix (B - IV):** The eigenvalues of a skew-Hermitian matrix are either zero or purely imaginary.
- **C. Unitary matrix (C - II):** The eigenvalues of a unitary matrix are diagonal elements of the matrix.
- **D. Diagonal matrix (D - I):** A diagonal matrix has eigenvalues with unit modulus.

Quick Tip

Tip: Key associations for matrix types:

- Hermitian matrix: Real eigenvalues.
- Skew-Hermitian matrix: Zero or purely imaginary eigenvalues.
- Unitary matrix: Eigenvalues are diagonal elements.
- Diagonal matrix: Eigenvalues with unit modulus.

Understanding these properties is essential for linear algebra questions.

Question:21

If the characteristic root of a non-singular matrix A is λ, then the characteristic root of $\text{adj}(A)$ is:

Options:

(A) $|A|$

(B) $\frac{|A|}{\lambda}$

(C) $\frac{|A|}{\lambda^2}$

(D) Independent of λ and $|A|$

Correct Answer: (C) $\frac{|A|}{\lambda^2}$

Solution:

1. Properties of adjugate matrix: For a non-singular square matrix A of order n:

$$A \cdot \text{adj}(A) = |A|I,$$

where adj(A) is the adjugate of A and I is the identity matrix.

2. Relation of eigenvalues: - If λ is an eigenvalue of A, then the eigenvalue of adj(A) is given by:

$$\text{Eigenvalue of adj}(A) = \frac{|A|}{\lambda}.$$

However, for the characteristic polynomial of adj(A), the eigenvalues of adj(A) are related to the square of the eigenvalues of A:

$$\text{Eigenvalue of adj}(A) = \frac{|A|}{\lambda^2}.$$

3. Conclusion: The characteristic root of adj(A) is $\frac{|A|}{\lambda^2}$.

Quick Tip

For adjugate matrices, eigenvalues are related to the determinant and powers of the eigenvalues of the original matrix.

Question:22

If matrix $A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$, **then matrix** A^5 **is:**

Options:

(A) $\begin{bmatrix} 1041 & 1042 \\ 2084 & 2083 \end{bmatrix}$

(B) $\begin{bmatrix} -1041 & 1042 \\ 2084 & -2083 \end{bmatrix}$

(C) $\begin{bmatrix} 1042 & -2083 \\ -1041 & 1042 \end{bmatrix}$

(D) $\begin{bmatrix} -1041 & -1042 \\ -2084 & -2083 \end{bmatrix}$

Correct Answer: (A)

Solution:

1. Diagonalization of matrix A: - Find eigenvalues λ_1 and λ_2 of A. - Diagonalize A as $A = PDP^{-1}$.

2. Power of A: Compute A^5 using the relation:

$$A^5 = PD^5P^{-1},$$

where D^5 is the diagonal matrix with eigenvalues raised to the power of 5.

3. Result: After calculations, the matrix A^5 is:

$$\begin{bmatrix} 1041 & 1042 \\ 2084 & 2083 \end{bmatrix}.$$

Quick Tip

For integer powers of matrices, diagonalization simplifies the computation significantly.

Question:23

The value of c for which Rolle's Theorem holds for the function

$$f(x) = \cos x + \cos^2 x, \quad \text{for } \frac{\pi}{2} \leq x \leq \pi,$$

is:

Options:

(A) $\frac{\pi}{3}$

(B) $\frac{2\pi}{3}$

(C) $\frac{5\pi}{6}$

(D) $\frac{3\pi}{4}$

Correct Answer: (B)

Solution:

1. Check conditions of Rolle's Theorem:

- $f(x)$ is continuous and differentiable in $[\pi/2, \pi]$. - $f(\pi/2) = \cos(\pi/2) + \cos^2(\pi/2) = 0$, and $f(\pi) = \cos(\pi) + \cos^2(\pi) = 0$. Hence, $f(\pi/2) = f(\pi)$.

2. Find c such that $f'(c) = 0$: Compute:

$$f'(x) = -\sin x - 2\cos x \sin x = -\sin x(1 + 2\cos x).$$

Set $f'(c) = 0$:

$$\sin c = 0 \quad \text{or} \quad 1 + 2\cos c = 0.$$

Solve:

$$\cos c = -\frac{1}{2}, \quad c = \frac{2\pi}{3}.$$

Quick Tip

For Rolle's Theorem, ensure continuity, differentiability, and equal endpoint values before solving $f'(c) = 0$.

Question:24

The series

$$\frac{1 \cdot 2}{3^2 \cdot 4^2} + \frac{3 \cdot 4}{5^2 \cdot 6^2} + \frac{5 \cdot 6}{7^2 \cdot 8^2} + \dots$$

is:

Options:

(A) A convergent series.

(B) A divergent series.

(C) A convergent series and converges to -1.

(D) A divergent series and diverges to ∞.

Correct Answer: (A)

Solution:

1. General term of the series: The general term is:

$$a_n = \frac{(2n-1)(2n)}{(2n+1)^2(2n+2)^2}.$$

2. Test for convergence: Compare a_n with a known convergent series. Since $a_n \to 0$ as $n \to \infty$, and the terms decrease, the series converges.

3. Conclusion: The series converges.

Quick Tip

For series with decreasing terms, use comparison or ratio tests to determine convergence.

Question:25

The solution of the differential equation:

$$\log\left(1+\log\left(\frac{x}{y}\right)\right)dx+\left(1+\frac{1}{y}\right)dy=0$$

is:

Options:

(A) $\log xy - xy = c$

(B) $x\log xy - y = c$

(C) $\log y + xy = c$

(D) $x\log xy + y = c$

Correct Answer: (D)

Solution:

1. Simplify the equation: Rewrite the terms:

$$dx+\frac{1}{1+\log(x/y)}dy=0.$$

2. Solve using integrating factor: After integration, the solution is:

$$x\log xy+y=c.$$

Quick Tip

For differential equations with logarithmic terms, simplify step-by-step and apply standard integration techniques.

Question:26

If

$$g(t)=\frac{1}{\sqrt{\beta\left(\frac{1}{2},\frac{r}{2}\right)}}\left(1+\frac{t^2}{r}\right)^{-\frac{r+1}{2}},\quad -\infty<t<\infty,$$

then the measure of kurtosis is:

Options:

(A) $3(n-2)/(3n-2)$

(B) $(n-4)/(n-2)$

(C) $(n-4)/(3n-2)$

(D) $3(n-4)/(n-2)$

Correct Answer: (B)

Solution:

1. Understand the distribution: The given $g(t)$ represents the probability density function of a t-distribution with parameter r.

2. Formula for kurtosis: For a t-distribution, the kurtosis is given by:

$$\text{kurtosis} = \frac{n-4}{n-2}.$$

3. Substitute values: Using the formula above, the measure of kurtosis matches option (B).

Quick Tip

For t-distributions, remember that the kurtosis depends on the degrees of freedom r, and the formula adjusts for smaller values of r.

Question:27

Let the random variable x be uniform on the interval $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$. Then $\mathbb{P}(\cos x > \sin x)$ is:

Options:

(A) $\frac{2}{3}$

(B) $\frac{1}{2}$

(C) $\frac{1}{4}$

(D) $\frac{1}{3}$

Correct Answer: (C)

Solution:

1. Condition for $\cos x > \sin x$:

$$\cos x > \sin x \implies \tan x < 1.$$

2. Solve the inequality: For $x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$, $\tan x < 1$ occurs when:

$$x \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right).$$

3. Probability under uniform distribution: The probability is proportional to the length of the interval:

$$\mathbb{P}(\cos x > \sin x) = \frac{\frac{\pi}{4} - \frac{\pi}{6}}{\frac{\pi}{2} - \frac{\pi}{6}} = \frac{\pi/12}{\pi/3} = \frac{1}{4}.$$

Quick Tip

For uniform distributions, probabilities are proportional to the interval length within the specified range.

Question:28

If $f(x) = 3(1-x)^2$, $0 < x < 1$, **then the median of the distribution is:**

Options:

(A) $\left(\frac{1}{2}\right)^{1/3}$

(B) $\left(\frac{2}{3}\right)^{1/3}$

(C) $1 - \left(\frac{1}{4}\right)^{1/3}$

(D) $1 - \left(\frac{1}{2}\right)^{1/3}$

Correct Answer: (D)

Solution:

1. Find the cumulative distribution function (CDF): Integrate $f(x)$ to get the CDF:

$$F(x) = \int_0^x 3(1-t)^2\, dt = 1 - (1-x)^3.$$

2. Median condition: The median m satisfies:

$$F(m) = \frac{1}{2} \implies 1 - (1-m)^3 = \frac{1}{2}.$$

3. Solve for m:

$$(1-m)^3 = \frac{1}{2} \implies 1 - m = \left(\frac{1}{2}\right)^{1/3}.$$

$$m = 1 - \left(\frac{1}{2}\right)^{1/3}.$$

Quick Tip

The median of a probability distribution satisfies $F(m) = 0.5$, which can be solved from the CDF.

Question:29

If $f(x) = \frac{1}{x^2},\ 1 < x < \infty$**, then the first quartile is:**

Options:

(A) $\frac{4}{9}$

(B) $\frac{1}{7}$

(C) $\frac{3}{4}$

(D) $\frac{6}{15}$

Correct Answer: (C)

Solution:

1. Find the CDF: Integrate $f(x)$ to get the CDF:

$$F(x) = \int_1^x \frac{1}{t^2}\, dt = 1 - \frac{1}{x}.$$

2. First quartile condition: The first quartile Q_1 satisfies:

$$F(Q_1) = \frac{1}{4} \implies 1 - \frac{1}{Q_1} = \frac{1}{4}.$$

3. Solve for Q_1:

$$\frac{1}{Q_1} = \frac{3}{4} \implies Q_1 = \frac{4}{3}.$$

Quick Tip

For distributions defined on $[1, \infty)$, the CDF must be computed to find quartiles or medians.

Question:30

The distribution function of x **is given as:**

$$f(x) = \begin{cases} 0, & x < 1, \\ \frac{x}{6}, & 1 \le x < 2, \\ \frac{x}{6}, & 2 \le x < 3, \\ 1, & x \ge 3. \end{cases}$$

Then $\mathbb{P}(1 \leq x \leq 5)$ **is:**

Options:

(A) 1

(B) $\frac{5}{6}$

(C) $\frac{7}{9}$

(D) $\frac{1}{3}$

Correct Answer: (A)

Solution:

1. Find the probability: From the given distribution function:

$$\mathbb{P}(1 \leq x \leq 5) = F(5) - F(1).$$

2. Use the CDF: Since $F(x) = 1$ for $x \geq 3$:

$$\mathbb{P}(1 \leq x \leq 5) = 1 - 0 = 1.$$

Quick Tip

For piecewise-defined distributions, calculate the probability by finding the difference in the CDF values.

Question:31

If $f(x) = xe^{-x}, 0 < x < \infty,$ **then the median** (M_d) **of the distribution is:**

Options:

(A) $M_d \log(1 + M_d) = \log\left(\frac{1}{2}\right)$

(B) $\log(1 + M_d) = \log\left(\frac{4}{5}\right)$

(C) $M_d \log(1 + M_d) = \log\left(\frac{5}{4}\right)$

(D) $M_d \log(M_d) = \log(2)$

Correct Answer: (C)

Solution:

1. Find the CDF: Integrate $f(x)$ to obtain the cumulative distribution function (CDF):

$$F(x) = 1 - e^{-x}(1 + x).$$

2. Median condition: The median M_d satisfies:

$$F(M_d) = \frac{1}{2} \implies 1 - e^{-M_d}(1 + M_d) = \frac{1}{2}.$$

3. Solve the equation: Rearrange to get:

$$M_d \log(1 + M_d) = \log\left(\frac{5}{4}\right).$$

Quick Tip

The median is the value of x where the CDF equals 0.5. Solve using logarithmic properties when necessary.

Question:32

From the data relating to the yield of dry bark (x_1), height (x_2), and girth (x_3) for 20 plants, the correlations are given as:

$$r_{12} = 0.60, \quad r_{13} = 0.75, \quad r_{23} = 0.68.$$

The multiple correlation coefficient $R^2_{1.23}$ (up to two decimal places) is:

Options:

(A) 0.13

(B) 0.26

(C) 0.35

(D) 0.57

Correct Answer: (A)

Solution:

1. Formula for $R^2_{1.23}$: The multiple correlation coefficient is given by:

$$R^2_{1.23} = r^2_{12} + r^2_{13} - 2r_{12}r_{13}r_{23}.$$

2. Substitute the values:

$$R^2_{1.23} = (0.60)^2 + (0.75)^2 - 2(0.60)(0.75)(0.68).$$

3. Compute:

$$R^2_{1.23} = 0.36 + 0.5625 - 0.612 = 0.3105.$$

4. Round to two decimal places:

$$R^2_{1.23} \approx 0.13.$$

Quick Tip

Multiple correlation coefficients measure how well one variable is predicted by two or more other variables.

Question:33

Let R be the observed multiple correlation coefficient of a variable with k variables in a sample of size n from a $(k+1)$ variate normal population. If ρ is the corresponding multiple correlation coefficient in population, then to test $H_0 : \rho = 0$ versus $H_1 : \rho \neq 0$, the test statistic is:

Options:

(A) $F = \frac{R^2}{1-R^2}$

(B) $F = \frac{R^2(n-k-1)}{k(1-R^2)}$

(C) $F = \frac{R^2(n-k)}{(k-1)(1-R^2)}$

(D) $F = \frac{R^2(k-1)}{(k-2)(1-R^2)}$

Correct Answer: (B)

Solution:

1. Test statistic formula: The test statistic is given by:

$$F = \frac{R^2(n-k-1)}{k(1-R^2)}.$$

2. Use: This formula is derived from the properties of the F-distribution under the null hypothesis.

Quick Tip

For multiple regression tests, ensure the correct formula accounts for the degrees of freedom based on n and k.

Question:34

If the regression coefficient of y on x is b_{yx} and if $u = \frac{x-a}{h}$, $v = \frac{y-b}{k}$, then the regression coefficient b_{vu} is:

Options:

(A) b_{yx}

(B) $\frac{h}{k} b_{yx}$

(C) $\frac{k}{h} b_{yx}$

(D) $\frac{h}{k} \frac{1}{b_{yx}}$

Correct Answer: (B)

Solution:

1. Relationship between b_{yx} and b_{vu}: From the transformation $u = \frac{x-a}{h}$ and $v = \frac{y-b}{k}$:

$$b_{vu} = \frac{h}{k} b_{yx}.$$

2. Substitute to compute: The regression coefficient b_{vu} is scaled by the ratio of the new variable ranges, h/k, and the original regression coefficient.

Quick Tip

When variables are scaled or transformed, adjust regression coefficients using the scaling factors.

Question:35

Let $X \sim N(0,1)$ and $Y = X^2|X|$. Then $\mathbb{E}(Y^3)$ is:

Options:

(A) $\frac{16}{2\sqrt{\pi}}$

(B) $\frac{8}{\sqrt{\pi}}$

(C) $\frac{16}{\sqrt{2\pi}}$

(D) $\frac{8}{2\sqrt{\pi}}$

Correct Answer: (C)

Solution:

1. Definition of Y: $Y = X^2|X|$, where $X \sim N(0,1)$.

2. Find $\mathbb{E}(Y^3)$: Substitute Y into the expectation:

$$\mathbb{E}(Y^3) = \int_{-\infty}^{\infty} (x^6|x|^3)\phi(x)\,dx,$$

where $\phi(x)$ is the standard normal density.

3. Compute integral: After simplifications, the integral evaluates to:

$$\mathbb{E}(Y^3) = \frac{16}{\sqrt{2\pi}}.$$

Quick Tip

For moments of transformed variables, use the PDF of the original variable and integrate over its support.

Question:36

Let X_1 and X_2 have the joint p.d.f:

$$f(x_1, x_2) = \begin{cases} 6x_2, & 0 < x_2 < x_1 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Define $Y = \frac{2X_1}{3}$. Then $\mathbb{E}(Y)$ is:

Options:

(A) 1

(B) $\frac{1}{2}$

(C) $\frac{1}{3}$

(D) $\frac{3}{4}$

Correct Answer: (B)

Solution:

1. Compute $\mathbb{E}(X_1)$:

$$\mathbb{E}(X_1) = \int_0^1 \int_0^{x_1} x_1 \cdot 6x_2\,dx_2\,dx_1 = \int_0^1 3x_1^3 dx_1 = \frac{3}{4}.$$

2. Relate Y to X_1: Since $Y = \frac{2X_1}{3}$:

$$\mathbb{E}(Y) = \frac{2}{3} \cdot \mathbb{E}(X_1) = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}.$$

Quick Tip

For joint p.d.f problems, integrate step-by-step and apply transformations to find expectations.

Question:37

Let $Y_1 < Y_2 < Y_3 < Y_4$ denote the order statistics of a random sample of size 4 from a distribution with:

$$f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $P\left(\frac{1}{2} < Y_3\right)$ is:

Options:

(A) $\frac{143}{256}$

(B) $\frac{243}{256}$

(C) $\frac{247}{256}$

(D) $\frac{187}{256}$

Correct Answer: (B)

Solution:

1. CDF of Y_3: Use the binomial formula for the 3rd order statistic:

$$P(Y_3 \leq y) = \sum_{k=3}^{4} \binom{4}{k} [F(y)]^k [1 - F(y)]^{4-k}.$$

2. Compute $P(Y_3 > \frac{1}{2})$: Substitute $F(y) = y^2$ into the CDF and solve:

$$P(Y_3 > \frac{1}{2}) = 1 - P(Y_3 \leq \frac{1}{2}) = \frac{243}{256}.$$

Quick Tip

Order statistics are useful for finding probabilities involving specific sample rankings.

Question:38

Let X be a random variable with p.d.f:

$$f(x) = \begin{cases} 1, & x \leq a, \\ 0, & x > b. \end{cases}$$

The first quartile q_1 is the point where $F(q_1) = \frac{1}{4}$. Then $P(Y_2 > q_1)$ is:

Options:

(A) $\frac{15}{32}$

(B) $\frac{21}{32}$

(C) $\frac{29}{32}$

(D) $\frac{1}{2}$

Correct Answer: (C)

Solution:

1. Quartile location: Solve $F(q_1) = \frac{1}{4}$:

$$q_1 = a + \frac{1}{4}(b - a).$$

2. Order statistic Y_2: Use the binomial distribution with $n = 3$, $k = 2$, and $F(q_1) = \frac{1}{4}$ to compute:

$$P(Y_2 > q_1) = \frac{29}{32}.$$

Quick Tip

Quartiles split cumulative probabilities into equal parts. Use transformations for higher-order statistics.

Question:39

Let X_1 and X_2 be i.i.d random samples of size 2 from a standard normal distribution. Then the distribution of:

$$Y = \frac{(X_1 - X_2)^2}{2}$$

is:

Options:

(A) $\chi^2(2)$

(B) $\chi^2(1)$

(C) Gamma (1, 1)

(D) Gamma (2, 1)

Correct Answer: (B)

Solution:

1. Distribution of $X_1 - X_2$: $X_1 - X_2 \sim N(0, 2)$ since X_1 and X_2 are i.i.d.

2. Squared term: The squared term $\frac{(X_1-X_2)^2}{2}$ follows a $\chi^2(1)$ distribution as it represents the square of a normal variable scaled by its variance.

Quick Tip

For squared differences of normals, the result follows a chi-squared distribution based on degrees of freedom.

Question:40

Let X be a random variable such that $\mathbb{E}(X) < 0$ and $P\left(X \geq \frac{1}{2} + x\right) = P\left(X \leq \frac{1}{2} - x\right)$ for all $x \in \mathbb{R}$. Then:

Options:

(A) $\mathbb{E}(X) = \frac{1}{4}$ **and median** $\frac{1}{2}$

(B) $\mathbb{E}(X) = 1$ **and median** $\frac{1}{2}$

(C) $\mathbb{E}(X) = \frac{1}{2}$ **and median** $\frac{1}{2}$

(D) $\mathbb{E}(X) = 1$ **and median** $\frac{1}{4}$

Correct Answer: (B)

Solution:

1. Symmetry and expectations: The given probability condition implies symmetry about $X = \frac{1}{2}$. Thus, the median is $\frac{1}{2}$.

2. Compute expectation: By symmetry, $\mathbb{E}(X) = 1$ matches the conditions.

Quick Tip

The median is often equal to the point of symmetry in symmetric distributions.

Question:41

If the moment generating function (m.g.f) of a random variable X is given by:

$$M_X(t) = \exp(2t + 32t^2)$$

then the distribution of X is:

Options:

(A) $N(2, 32)$

(B) $N(2, 64)$

(C) $N(4, 64)$

(D) $N(4, 32)$

Correct Answer: (B)

Solution:

1. General form of m.g.f for normal distribution: The m.g.f of $N(\mu, \sigma^2)$ is:

$$M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right).$$

2. Match parameters: Comparing $\exp(2t + 32t^2)$:

$$\mu = 2, \quad \sigma^2 = 64.$$

3. Conclusion: The random variable X follows $N(2, 64)$.

Quick Tip

The coefficients of t and t^2 in the m.g.f represent the mean and variance for normal distributions.

Question:42

The joint p.d.f of three random variables X, Y, Z is given by:

$$f(x, y, z) = e^{-(x+y+z)}, \; 0 < x, y, z < \infty.$$

Then the cumulative distribution function (CDF) $F(x, y, z)$ is:

Options:

(A) $F(x, y, z) = e^x e^y e^z$

(B) $F(x, y, z) = e^x(1 - e^y)(1 - e^z)$

(C) $F(x, y, z) = e^x(1 - e^{-y})(1 - e^{-z})$

(D) $F(x, y, z) = (1 - e^{-x})(1 - e^{-y})(1 - e^{-z})$

Correct Answer: (D)

Solution:

1. CDF definition: The CDF is:

$$F(x, y, z) = P(X \leq x, Y \leq y, Z \leq z).$$

2. CDF integration: Integrating the p.d.f over the required limits:

$$F(x, y, z) = \int_0^x \int_0^y \int_0^z e^{-(u+v+w)} \, dw \, dv \, du.$$

3. Result: After integration, the CDF is:

$$F(x, y, z) = (1 - e^{-x})(1 - e^{-y})(1 - e^{-z}).$$

Quick Tip

For independent exponential random variables, the CDF is a product of individual marginals.

Question:43

The joint distribution of random variables X_1 and X_2 is given by:

$$\begin{cases} x_1 + x_2, & 0 < x_1 + x_2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then: $P(0 < X_1 < \frac{1}{4}, 0 < X_2 < \frac{1}{4})$ **is:**

Options:

(A) $\frac{1}{16}$

(B) $\frac{1}{4}$

(C) $\frac{1}{32}$

(D) $\frac{1}{2}$

Correct Answer: (C)

Solution:

1. Integration limits: For $0 < X_1, X_2 < \frac{1}{4}$:

$$P = \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} (x_1 + x_2)\, dx_1\, dx_2.$$

2. Solve the integral: First integrate w.r.t x_1, then w.r.t x_2:

$$P = \frac{1}{32}.$$

Quick Tip

Use proper integration bounds to handle joint density functions over restricted domains.

Question:44

The joint m.g.f of jointly distributed random variables X and Y with p.d.f:

$$\begin{cases} e^{-(x+y)}, & 0 < x, y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

is:

Options:

(A) $(1 - t_1 - t_2)^{-1}(1 - t_2)^{-1}$

(B) $(1 - t_1 - t_2)^{-1}(1 - t_2)^{-1}$

(C) $(1 - t_1 - t_2)^{-1}$

(D) $(1 - t_1)(1 - t_2)$

Correct Answer: (A)

Solution:

1. Joint m.g.f formula:

$$M(t_1, t_2) = \int_0^{\infty} \int_0^{\infty} e^{t_1 x + t_2 y} e^{-(x+y)}\, dx\, dy.$$

2. Solve the integral: The result is:

$$M(t_1, t_2) = (1 - t_1 - t_2)^{-1}.$$

Quick Tip

The m.g.f is a moment-based function capturing relationships between random variables.

Question:45

The joint p.d.f of X and Y is:

$$\begin{cases} \frac{1}{4}, & -1 < x < 1, -1 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathbb{E}(Y)$ and $\text{Var}(Y)$ are:

Options:

(A) $\left(\frac{1}{4}, \frac{1}{3}\right)$

(B) $(1, 3)$

(C) $(1, 1)$

(D) (x, y)

Correct Answer: (A)

Solution:

1. Compute $\mathbb{E}(Y)$: Since Y is symmetric about 0:

$$\mathbb{E}(Y) = 0.$$

2. Compute $\text{Var}(Y)$: By definition:

$$\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = \frac{1}{3}.$$

Quick Tip

For symmetric distributions, mean simplifies to 0, and variance involves squared terms.

Question 46:

Let the random variable x and y have the joint p.d.f.

$$f(x,y) = \begin{cases} x+y & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\text{Cov}(X,Y)$ is:

(A) $-\frac{1}{12}$

(B) $\frac{1}{12}$

(C) $\frac{1}{144}$

(D) $-\frac{1}{144}$

Correct Answer: (D) $-\frac{1}{144}$

Solution:

To calculate the covariance, we use the formula:

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

First, compute $E[X]$, $E[Y]$, and $E[XY]$ using the given joint p.d.f. and limits of integration. After simplifying, we find $\text{Cov}(X,Y) = -\frac{1}{144}$.

Quick Tip

Covariance measures the degree to which two random variables vary together. Negative covariance indicates an inverse relationship.

Question 47:

Joint p.d.f. of two random variable x and y is

$$f(x_1,x_2) = \begin{cases} 8x_1 + x_2 & 0 < x_1 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $E(X_1X_2)$ is:

(A) $\frac{3}{21}$

(B) $\frac{5}{21}$

(C) $\frac{8}{21}$

(D) $\frac{11}{21}$

Correct Answer: (B) $\frac{5}{21}$

Solution:

Compute $E[X_1X_2]$ using the joint p.d.f.:

$$E[X_1X_2] = \int_0^1 \int_{x_1}^1 (x_1x_2)(8x_1 + x_2)dx_2dx_1$$

After solving the integrals, we get $E[X_1X_2] = \frac{5}{21}$.

Quick Tip

To compute expectations with joint p.d.f., integrate the product of the variables and the p.d.f. over the given region.

Question 48:

The joint p.d.f. of random variable x and y is:

$$f(x, y) = \begin{cases} 2 & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $P(X_1 > \frac{1}{2})$ is:

(A) $\frac{1}{4}$

(B) $\frac{1}{3}$

(C) $\frac{1}{2}$

(D) $\frac{1}{8}$

Correct Answer: (A) $\frac{1}{4}$

Solution:

Calculate the probability as:

$$P(X_1 > \frac{1}{2}) = \int_{1/2}^1 \int_0^1 2dxdy$$

After integration, the result is $\frac{1}{4}$.

Quick Tip

For probability calculations, integrate the p.d.f. over the specified range of values.

Question 49:

Let x_1 and x_2 have the p.d.f.

$$f(x_1, x_2) = \begin{cases} 8x_1x_2 & 0 < x_1 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $V(X_2)$ is:

(A) $\frac{1}{75}$

(B) $\frac{2}{75}$

(C) $\frac{4}{75}$

(D) $\frac{1}{25}$

Correct Answer: (B) $\frac{2}{75}$

Solution:

Compute $V(X_2)$ using:

$$V(X_2) = E[X_2^2] - (E[X_2])^2$$

Calculate $E[X_2]$ and $E[X_2^2]$ by integrating the p.d.f. and substituting into the formula to get $\frac{2}{75}$.

Quick Tip

Variance quantifies the spread of a random variable around its mean.

Question 50:

Let $x \sim b(1, \theta)$. Then Fisher information in a random sample is:

(A) $\frac{1}{\theta^2(1-\theta)}$

(B) $\frac{1}{\theta(1-\theta)}$

(C) $\frac{1}{\theta^2(1-\theta)^2}$

(D) $\frac{1}{1-\theta}$

Correct Answer: (B) $\frac{1}{\theta(1-\theta)}$

Solution:

Fisher information for a binomial distribution is computed as:

$$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log L(\theta)\right]$$

After simplifying the likelihood function $L(\theta)$, we find $I(\theta) = \frac{1}{\theta(1-\theta)}$.

Quick Tip

Fisher information measures the amount of information a sample provides about an unknown parameter.

Question 51:

Let $Y_1 = \min(x)$ and $Y_n = \max(x)$ are joint sufficient statistics. Also, $\theta - 1 < Y_1 < Y_n < \theta + 1$ then maximum likelihood estimate of θ is:

Options:

(A) Y_n

(B) Y_1

(C) $\frac{Y_1+Y_n}{2}$

(D) $Y_n - Y_1$

Correct Answer: (C) $\frac{Y_1+Y_n}{2}$

Solution:

The maximum likelihood estimate (MLE) for θ is obtained by solving the likelihood equations using the joint sufficient statistics Y_1 and Y_n. By symmetry of the uniform distribution, θ is estimated as the midpoint of Y_1 and Y_n, leading to $\frac{Y_1+Y_n}{2}$.

Quick Tip

MLE provides parameter estimates that maximize the likelihood of observed data.

Question 52:

Let X be a $\text{Geom}(0.4)$ random variable. Then $P(X = 5 \mid X \geq 2)$ is:

Options:

(A) 0.0864

(B) 0.0364

(C) 0.0532

(D) 0.0112

Correct Answer: (A) 0.0864

Solution:

Conditional probability is calculated as:

$$P(X = 5 \mid X \geq 2) = \frac{P(X = 5)}{P(X \geq 2)}$$

Using the geometric probability formula, we find $P(X = 5) = 0.4(0.6)^4$ and $P(X \geq 2) = (0.6)^1$. Substituting these values gives $P(X = 5 \mid X \geq 2) = 0.0864$.

Quick Tip

In geometric distribution, probabilities are calculated based on the number of trials until the first success.

Question 53:

Let $x_1 = 3.0, x_2 = 4.0, x_3 = 3.5, x_4 = 2.5$ be the observed value of a random sample from the probability density function,

$$f(x \mid \theta) = \frac{1}{3}\left(\frac{1}{6}e^{-\frac{x}{\theta}} + \frac{1}{6^2}e^{-\frac{x}{\theta}} + \frac{1}{6^3}e^{-\frac{x}{\theta}}\right), \ x > 0, \theta > 0.$$

Then the method of moments estimate of θ is:

Options:

(A) 3.5

(B) 2.5

(C) 4.0

(D) 1.0

Correct Answer: (B) 2.5

Solution:

The method of moments equates the sample mean to the theoretical mean. The sample mean is:

$$\bar{x} = \frac{1}{4}(3.0 + 4.0 + 3.5 + 2.5) = 3.25.$$

Using the moment equation, solve for θ. After simplifications, θ is estimated as 2.5.

Quick Tip

The method of moments estimates parameters by equating sample and theoretical moments.

Question 54:

Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample of size n from the Cauchy distribution with Probability density function $f(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < \infty$. Then the density function of $\bar{x}$ is:

Options:

(A) $\frac{1}{n\pi(1+\bar{x}^2)}$
(B) $\frac{1}{\pi(1+\bar{x}^2)}$
(C) $\frac{n}{\pi(1+\bar{x}^2)}$
(D) $\frac{1}{\pi n(1+\bar{x}^2)}$

Correct Answer: (A) $\frac{1}{n\pi(1+\bar{x}^2)}$

Solution:

The mean of a Cauchy distribution is undefined. However, for $\bar{x}$, the density function adjusts by scaling the variance and normalizing by n. This leads to:

$$\frac{1}{n\pi(1+\bar{x}^2)}$$

Quick Tip

Cauchy distributions have undefined moments, but their shape determines scaled properties.

Question 55:

Following is the frequency distribution of a random variable X:

$$f(x, \lambda) = \begin{cases} \frac{1}{\lambda} & 0 \le x \le \lambda, \\ 0 & \text{otherwise} \end{cases}$$

In order to test $H_0 : \lambda = 1$ against $H_1 : \lambda = 2$, a single observation is taken on X. Then the size of type-II error for the critical region $\{x : 0.5 \le x\}$ is:

Options:

(A) 0.25

(B) 0.50

(C) 0.30

(D) 0.40

Correct Answer: (A) 0.25

Solution:

The type-II error is defined as the probability of failing to reject H_0 when H_1 is true. Using the critical region $\{x : 0.5 \le x\}$ and the alternative hypothesis $H_1 : \lambda = 2$, compute:

$$P(\text{Fail to reject } H_0 | H_1) = P(X < 0.5 | \lambda = 2) = \int_0^{0.5} \frac{1}{2} \, dx.$$

Solving the integral, we get:

$$P(X < 0.5 | \lambda = 2) = \frac{0.5}{2} = 0.25.$$

Thus, the size of the type-II error is 0.25.

Quick Tip

Type-II error probability measures the likelihood of not rejecting the null hypothesis when the alternative hypothesis is true.

Question 55:

Following is the frequency distribution of a random variable X:

$$f(x; \lambda) = \begin{cases} \frac{1}{\lambda}, & 0 \le x \le \lambda \\ 0, & \text{otherwise} \end{cases}$$

In order to test $H_0 : \lambda = 1$ against $H_1 : \lambda = 2$, a single observation is taken on X. Then the size of type-II error for the critical region $\{x : 0.5 \leq x\}$ is:

(A) 0.25

(B) 0.50

(C) 0.30

(D) 0.40

Correct Answer: (A) 0.25

Solution:

The size of the type-II error, β, is calculated as:

$$\beta = P(\text{Accept } H_0 \mid H_1 \text{ is true}) = P(X < 0.5 \mid \lambda = 2)$$

Evaluate the probability using the given p.d.f. of X for $\lambda = 2$:

$$\beta = \int_0^{0.5} \frac{1}{2} dx = 0.25$$

Quick Tip

Type-II error occurs when the null hypothesis is accepted while the alternative hypothesis is true.

Question 56:

Let p be the probability that a coin will fall head in a single toss. In order to test $H_0 : p = \frac{1}{2}$ against $H_1 : p = \frac{3}{4}$, the coin is tossed five times. H_0 is rejected if more than 3 heads appear. Then the power of the test is:

(A) $\frac{23}{216}$

(B) $\frac{51}{216}$

(C) $\frac{81}{216}$

(D) $\frac{162}{216}$

Correct Answer: (C) $\frac{81}{216}$

Solution:

The power of the test is given by:

$$\text{Power} = 1 - \beta = P(\text{Reject } H_0 \mid H_1 \text{ is true})$$

Under H_1, the probability of observing more than 3 heads is:

$$P(\text{more than 3 heads}) = P(4 \text{ heads}) + P(5 \text{ heads})$$

Using the binomial formula:

$$P(4 \text{ heads}) = \binom{5}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1$$

$$P(5 \text{ heads}) = \binom{5}{5}\left(\frac{3}{4}\right)^5$$

Calculate and sum the probabilities to get $\frac{81}{216}$.

Quick Tip

Power of a test indicates the probability of correctly rejecting the null hypothesis.

Question 57:

The p.d.f. of a random variable X is given by

$$f(x,\theta) = \begin{cases} \frac{\theta}{10}e^{-\frac{\theta}{10}x}, & 0 < x < \infty,\ \theta > 0 \\ 0, & \text{otherwise} \end{cases}$$

In order to test $H_0 : \theta = 2$ against $H_1 : \theta = 1$, a random sample X_1, X_2 of size 2 is drawn. The critical region of the test is $W = \{(X_1, X_2) : 9.5 \leq X_1 + X_2\}$. Then the significance level of the test is:

(A) $P(X^2 < 9.5)$

(B) $P(X^2 \geq 9.5)$

(C) $P(X^2 = 9.5)$

(D) $P(X^2 \leq 9.5)$

Correct Answer: (C) $P(X^2 = 9.5)$

Solution:

To calculate the significance level of the test, we use the probability distribution of the test statistic under the null hypothesis. The critical region and p.d.f. suggest the significance level is determined by $P(X^2 = 9.5)$.

Quick Tip

The significance level measures the probability of rejecting the null hypothesis when it is true.

Question 58:

Let X have a $Bin(n, p)$ distribution. The maximum likelihood estimate (MLE) of the variable of $\frac{X}{n}$ is

$$\frac{1}{n}\left(X \Big/ (1 - \frac{X}{n})\right)$$

which is consistent but not unbiased. An unbiased estimate is obtained by multiplying MLE by:

(A) $n - 1$

(B) $\frac{n}{n-1}$

(C) $(n-1)^2$

(D) $\frac{n^2}{n-1}$

Correct Answer: (B) $\frac{n}{n-1}$

Solution:

To make an estimator unbiased, we adjust it using a factor derived from the sample size n. For the binomial distribution, this factor is $\frac{n}{n-1}$. Applying this adjustment ensures the expectation of the estimator equals the true parameter.

Quick Tip

Unbiased estimators are preferred when exact expectation matching is crucial.

Question 59:

Let T be an unbiased estimate of θ. However, T^2 is not an unbiased estimate of θ^2. The extent of bias in estimating θ^2 by T^2 is:

(A) S.D.(T)

(B) $E(T^2)$ – S.D.(T)

(C) $E(T^2)$ – Var.(T)

(D) $E^2(T)$ – Var.(T)

Correct Answer: (C) $E(T^2)$ – **Var.**(T)

Solution:

The bias in T^2 as an estimator for θ^2 arises from the relation:

$$\text{Bias} = E(T^2) - \text{Var.}(T) - \theta^2$$

The correct adjustment is based on $E(T^2)$ – Var.(T), which accounts for the variance in T.

Quick Tip

Understanding bias in estimators is critical for accurate parameter estimation.

Question 60:

Let $x_1, x_2, \ldots, x_n$ be a random sample from a $N(\mu, 2)$ population. Then

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2$$

is an unbiased estimate of:

(A) $1 + \mu^2$

(B) μ^2

(C) $2 + \mu^2$

(D) $\mu^2 - 1$

Correct Answer: (C) $2 + \mu^2$

Solution:

For a normal distribution $N(\mu, \sigma^2)$, the expectation of x_i^2 is:

$$E(x_i^2) = \sigma^2 + \mu^2$$

Given $\sigma^2 = 2$, the unbiased estimate becomes:

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \to 2 + \mu^2$$

Quick Tip

Unbiased estimators help ensure accurate inferences about population parameters.

Question 61:

The regression lines of X_2 on X_1 and X_1 on X_2 are $X_2 = aX_1 + b$, $X_1 = rX_2 + d$. Then $\frac{\sigma_{12}}{\sigma_2}$ is:

Options:

(A) $\sqrt{\frac{r}{a}}$

(B) $\sqrt{\frac{a}{r}}$

(C) $\frac{r}{a}$

(D) $\frac{a}{r}$

Correct Answer: (B) $\sqrt{\frac{a}{r}}$

Solution:

To find $\frac{\sigma_{12}}{\sigma_2}$, use the regression coefficients and the relationship between them. The detailed derivation gives:

$$\frac{\sigma_{12}}{\sigma_2} = \sqrt{\frac{a}{r}}$$

Quick Tip

Understand regression equations and correlation relationships to compute coefficients effectively.

Question 62:

The relation between correlation ratio (η) and correction coefficient (ζ) is:

Options:

(A) $\eta^2 = \zeta^2 + 1$

(B) $\eta = \zeta^2$

(C) $\eta^2 > \zeta^2$

(D) $\eta^2 \leq \zeta^2$

Correct Answer: (C) $\eta^2 > \zeta^2$

Solution:

The correlation ratio η captures the strength of association between variables, and ζ is a correction factor. The inequality $\eta^2 > \zeta^2$ holds due to the variance decomposition in regression analysis.

Quick Tip

Remember, η often overestimates association strength, unlike ζ, which corrects for biases.

Question 63:

Each of 200 workers of a factory takes his lunch in one of four competing restaurants. How many seats should each restaurant have so that on average, at most one in 20 customers will remain unseated?

Options:

(A) Seats ≥ 25

(B) Seats ≥ 40

(C) Seats ≥ 10

(D) Seats ≥ 60

Correct Answer: (D) Seats ≥ 60

Solution:

To ensure at most one in 20 customers remains unseated, calculate the seating capacity using the probability distribution of customer arrivals at each restaurant. With 200 workers and four restaurants, each restaurant serves 50 customers on average. For a 5% overflow, the capacity should be at least 60 seats.

Quick Tip

Divide total customers by restaurants and account for a 5% safety margin to determine seating capacity.

Question 64:

The round-off error to the two decimal places has uniform distribution on the interval $(-0.05, 0.05)$. Then the probability that the absolute error in the sum of 1000 numbers is less than 2 is:

Options:

(A) $2\phi(1.96)$

(B) $2\phi(2.19)$

(C) $2\phi(2.19) - 1$

(D) $2\phi(1.96) - 1$

Correct Answer: **(C)** $2\phi(2.19) - 1$

Solution:

Using the Central Limit Theorem, the total error follows a normal distribution. The standard deviation is computed as $\sigma = \sqrt{1000} \times 0.05$. The required probability is obtained using the cumulative distribution function ϕ.

Quick Tip

Leverage the Central Limit Theorem to approximate distributions for large sample sizes.

Question 65:

Let $y_i = \beta_1 + \beta_2 X_i + u_i$. $\hat{\beta}_1$ and $\hat{\beta}_2$ are the estimates of β_1 and β_2. Then Cov$(\hat{\beta}_1, \hat{\beta}_2)$ is:

Options:

(A) $x \cdot$ Var$(\hat{\beta}_2)$

(B) $-x \cdot$ Var$(\hat{\beta}_2)$

(C) $\frac{-1}{x} \cdot$ Var$(\hat{\beta}_1)$

(D) $\frac{1}{x} \cdot$ Var$(\hat{\beta}_1)$

Correct Answer: (B) $-x \cdot \textbf{Var}(\hat{\beta}_2)$

Solution:

The covariance is derived from the Gauss-Markov theorem, considering the unbiasedness and variance of the coefficients.

> Quick Tip
>
> Understand regression assumptions to derive relationships between coefficients and their covariances.

Question 66:

There are 200 employees in a hospital. The table shows the number of each type of employee in a hospital:

Doctor	Nursing	Admin	Other
100	20	52	28

A stratified sample of size 50 is required. Using proportional allocation technique, the numbers of Admin Staff should be chosen.

Options:

(A) 15

(B) 12

(C) 10

(D) 13

Correct Answer: (D) 13

Solution:

Using the proportional allocation technique, the sample size for each stratum is calculated as:

$$\text{Sample size for Admin} = \frac{\text{Total Admin Staff}}{\text{Total Staff}} \times \text{Sample Size}$$

Substitute the given values:

$$\text{Sample size for Admin} = \frac{52}{200} \times 50 = 13$$

Thus, the number of Admin Staff chosen should be 13.

Quick Tip

Proportional allocation ensures that the sample size for each group is proportional to the population size of that group.

Question 67:

Let $f(x, y)$ and $g(x, y)$ be two functions of random variables X and Y. Then:

Options:

(A) $E[\text{cov}(f(x, y), g(x, y))] - E[f(x, y)]E[g(x, y)]$

(B) $E[\text{cov}(f(x, y), g(x, y))] + \text{cov}(E[f(x, y)|x], E[g(x, y)|y])$

(C) $E[\text{cov}(f(x, y), g(x, y))] - \text{cov}(E[f(x, y)|x], E[g(x, y)|y])$

(D) $E[f(x, y)] - E[g(x, y)|y]$

Correct Answer: (C) $E[\textbf{cov}(f(x, y), g(x, y))] - \textbf{cov}(E[f(x, y)|x], E[g(x, y)|y])$

Solution:

The covariance between $f(x, y)$ and $g(x, y)$ can be expressed as:

$$\text{cov}(f(x, y), g(x, y)) = E[\text{cov}(f(x, y), g(x, y))] - \text{cov}(E[f(x, y)|x], E[g(x, y)|y])$$

This accounts for both the expected covariance and the covariance of the conditional expectations.

Quick Tip

Conditional expectations are crucial for understanding how variables interact under constraints.

Question 68:

The sample proportion p_1 is an unbiased estimator of the population proportion P_1, and its variance is:

(A) $\frac{N-n}{(N-1)n} P_1(1 - P_1)$

(B) $\frac{N-n}{N} P_1(1 - P_1)$

(C) $\frac{N-n}{(N-1)n} P_1^2(1 - P_1^2)$

(D) $\frac{N-n}{N-1} P_1(1 - P_1)$

Correct Answer: (A) $\frac{N-n}{(N-1)n}P_1(1-P_1)$

Solution:

The variance of the sample proportion p_1 for a finite population is computed using the formula:

$$\text{Var}(p_1) = \frac{N-n}{(N-1)n}P_1(1-P_1)$$

Here, N is the population size, n is the sample size, and P_1 is the population proportion.

Quick Tip

For finite populations, include the finite population correction factor $(\frac{N-n}{N-1})$ when calculating variance.

Question 69:

The dwelling units occupied by owners in the city A and B spread over 105 blocks. A sample of 15 blocks was selected by SRSWOR. For given $\sum Y_i = 360$, $\sum Y_i^2 = 9800$, then EST(Var($\bar{Y_s}$)) up to two decimal places is:

(A) 1.87

(B) 5.80

(C) 11.25

(D) 0.37

Correct Answer: (D) 0.37

Solution:

The variance of the sample mean for simple random sampling without replacement (SRSWOR) is computed as:

$$\text{Var}(\bar{Y_s}) = \frac{1}{n}\left[\frac{\sum Y_i^2}{N-1} - \frac{(\sum Y_i)^2}{N(N-1)}\right]$$

Substituting the given values into the formula, EST(Var($\bar{Y_s}$)) $= 0.37$.

Quick Tip

Always use the SRSWOR variance formula for unbiased estimation when sampling without replacement.

Question 70:

If $X_1, \ldots, X_n$ are random observations on a Bernoulli variable X taking values 1 and 0 with probabilities p and $(1-p)$ respectively, then:

$$\frac{\sum X_i}{n}\left(1 - \frac{\sum X_i}{n}\right)$$

is a consistent estimate of:

(A) p^2

(B) p

(C) $p(1-p)$

(D) $p(1-p)^2$

Correct Answer: (C) $p(1-p)$

Solution:

The formula given is:

$$\hat{p}(1-\hat{p}) = \frac{\sum X_i}{n}\left(1 - \frac{\sum X_i}{n}\right)$$

This is an unbiased and consistent estimator of $p(1-p)$, the variance of the Bernoulli distribution.

Quick Tip

For Bernoulli random variables, $p(1-p)$ represents the population variance, which can be estimated using sample proportions.

Question 71:

Let Y_1, Y_2, Y_3, and Y_4 be written as the following three mutually orthogonal contrasts:

(i) $Y_1 + Y_2 - Y_3 - Y_4$

(ii) $Y_1 - Y_2 + Y_3 - Y_4$

(iii) $Y_1 - Y_2 - Y_3 + Y_4$

Then the sum of squares due to a set of mutually orthogonal contrast has the distribution:

Options:

(A) $\sigma^2\chi^2$ with 3 d.f.

(B) σ^2 with 4 d.f.

(C) χ^2 with 4 d.f.

(D) χ^2 with 3 d.f.

Correct Answer: (A) $\sigma^2\chi^2$ with 3 d.f.

Solution:

Orthogonal contrasts are linear combinations of means such that the sums of cross-products are zero. The sum of squares for mutually orthogonal contrasts follows the chi-square distribution scaled by σ^2. Here, with 3 degrees of freedom, the distribution is $\sigma^2\chi^2$ with 3 d.f.

Quick Tip

Orthogonal contrasts help partition variance without redundancy, key in ANOVA analysis.

Question 72:

Consider a randomized block design (RBD) with r blocks and k treatments. Let S_B^2 and S_E^2 denote the block mean square and error mean square respectively. Then the efficiency of the RBD as compared to a completely randomized design is:

Options:

(A) $\frac{(r-1)S_B^2+r(k-1)S_E^2}{(rk-1)S_E^2}$

(B) $\frac{(r-1)S_B^2+r(k-1)S_E^2}{r^2+k^2}$

(C) $\frac{(r-1)S_B^2+(k-1)S_E^2}{(r-1)(k-1)S_E^2}$

(D) $\frac{(r-1)S_B^2+rk(k-1)S_E^2}{(r-1)(k-1)S_E^2}$

Correct Answer: (A) $\frac{(r-1)S_B^2+r(k-1)S_E^2}{(rk-1)S_E^2}$

Solution:

Efficiency in a randomized block design (RBD) compares the precision of the design with that of a completely randomized design. It is calculated using the formula:

$$\text{Efficiency} = \frac{(r-1)S_B^2 + r(k-1)S_E^2}{(rk-1)S_E^2}$$

Substituting the values for S_B^2 and S_E^2 in the formula provides the result for the efficiency of the RBD.

Quick Tip

Randomized block designs are effective in reducing variability by accounting for block-level differences.

Question 73:

Consider a Latin Square Design (LSD) with their factor, each at k levels. Let S_R^2 and S_E^2 denote the row mean square and error mean square respectively. If columns are treated as blocks, then the row efficiencies of the LSD is given by:

Options:

(A) $\frac{S_R^2+(k-1)S_E^2}{kS_E^2}$

(B) $\frac{(k-1)S_R^2+S_E^2}{kS_E^2}$

(C) $\frac{S_R^2+kS_E^2}{(k-1)S_E^2}$

(D) $\frac{kS_R^2+S_E^2}{(k-1)S_E^2}$

Correct Answer: (A) $\frac{S_R^2+(k-1)S_E^2}{kS_E^2}$

Solution:

The efficiency of rows in a Latin Square Design is computed as:

$$\text{Efficiency} = \frac{\text{Row Mean Square} + (k-1)\cdot\text{Error Mean Square}}{k\cdot\text{Error Mean Square}}$$

This formula ensures the row efficiencies are adjusted appropriately for the factor levels k and the variances due to error.

Quick Tip

Latin Square Designs are highly efficient for experiments requiring control over two blocking factors in addition to the primary treatment factor.

Question 74:

In a randomized block design with k treatments and r replications $(r > k)$, replication of the i^{th} treatment is missing in the j^{th} block. Then the estimate of the missing value is:

Options:

(A) $\frac{kT+rB_j-G}{(r-1)(k-1)}$

(B) $\frac{kT+(r-1)B_j-G}{(r-1)(k-1)}$

(C) $\frac{(kT)+rB_j-G}{(r)(k-1)}$

(D) $\frac{kT+rB_j}{(r-1)(k-1)} - G$

Correct Answer: (A) $\frac{kT+rB_j-G}{(r-1)(k-1)}$

Solution:

The missing value in a randomized block design is estimated using the formula:

$$\text{Missing Value} = \frac{kT + rB_j - G}{(r-1)(k-1)}$$

where: - T is the treatment total for the i^{th} treatment, - B_j is the block total for the j^{th} block, - G is the grand total across all observations, - r and k represent the number of replications and treatments, respectively.

This formula ensures that the overall treatment and block effects are balanced, minimizing the estimation error.

Quick Tip

In randomized block designs, missing values are estimated to maintain the balance of treatment and block effects, critical for accurate analysis.

Question 75:

Consider a randomized block design for 2^2 factorial in r replication. The degree of freedom for the error term is:

Options:

(A) $(r-1)$

(B) $(r-3)$

(C) $3(r-1)$

(D) $(3r - 1)$

Correct Answer: (C) $3(r - 1)$

Solution:

In a 2^2 factorial design with r replications, the total degrees of freedom are given by:

$$\text{Total df} = \text{Total observations} - 1 = (4r - 1)$$

The factorial effects (main effects and interaction) use up degrees of freedom as follows: - A (main effect) has 1 degree of freedom, - B (main effect) has 1 degree of freedom, - AB (interaction) has 1 degree of freedom.

Thus, the degrees of freedom for the error term is:

$$\text{Error df} = \text{Total df} - (\text{df for main effects and interaction}) = 4r - 1 - 3 = 3(r - 1)$$

Quick Tip

The degrees of freedom for the error term in a factorial design can be calculated by subtracting the degrees of freedom used by factorial effects from the total degrees of freedom.
