CUET UG 2024 Mathematics Question Paper with Solution Set-D

Question 1: An objective function $Z = ax + by$ is maximum at points $(8, 2)$ and $(4, 6)$. If $a \geq 0$, $b \geq 0$, and $ab = 25$, then the maximum value of the function is:

(1) 60

(2) 50

(3) 40

(4) 80

Correct Answer: (2) 50

Solution:

The given function $Z = ax + by$ attains its maximum value at points $(8, 2)$ and $(4, 6)$. At these points:

$$Z_1 = 8a + 2b \quad \text{and} \quad Z_2 = 4a + 6b$$

Since both points yield the same maximum value:

$$8a + 2b = 4a + 6b$$

Simplify the equation:

$$8a - 4a = 6b - 2b$$

$$4a = 4b$$

$$a = b$$

Using the condition $ab = 25$:

$$a \cdot b = 25 \quad \text{and} \quad a = b$$

$$a^2 = 25 \quad \Rightarrow \quad a = 5 \quad \text{and} \quad b = 5$$

Substitute $a = 5$ and $b = 5$ into $Z = ax + by$. At point $(8, 2)$:

$$Z = 8a + 2b = 8(5) + 2(5) = 40 + 10 = 50$$

Thus, the maximum value of Z is 50.

Quick Tip

Ensure all constraints are considered when solving for the maximum or minimum of objective functions.

Question 2: The area of the region bounded by the lines $x + 2y = 12$, $x = 2$, $x = 6$, and the x-axis is:

(1) 34 sq units

(2) 20 sq units

(3) 24 sq units

(4) 16 sq units

Correct Answer: (4) 16 sq units

Solution:

To find the area of the region bounded by $x + 2y = 12$, $x = 2$, $x = 6$, and the x-axis, we start by expressing y in terms of x from the equation $x + 2y = 12$:

$$y = \frac{12 - x}{2}$$

The area between $x = 2$ and $x = 6$ under the line $y = \frac{12-x}{2}$ is given by:

$$\text{Area} = \int_2^6 \frac{12 - x}{2}\, dx$$

Evaluating this integral:

$$= \int_2^6 \frac{12 - x}{2}\, dx = \frac{1}{2} \int_2^6 (12 - x)\, dx$$

$$= \frac{1}{2} \left[12x - \frac{x^2}{2} \right]_2^6$$

$$= \frac{1}{2} \left[(12 \times 6 - \frac{6^2}{2}) - (12 \times 2 - \frac{2^2}{2}) \right]$$

$$= \frac{1}{2} \left[(72 - 18) - (24 - 2) \right]$$

$$= \frac{1}{2} [54 - 22] = \frac{1}{2} \times 32 = 16$$

Therefore, the area of the region is 16 sq units.

Quick Tip

Break down complex regions into simpler shapes for easier area calculation.

Question 3: A die is rolled thrice. What is the probability of getting a number greater than 4 in the first and second throw of the dice, and a number less than 4 in the third throw?

(1) $\frac{1}{3}$

(2) $\frac{1}{6}$

(3) $\frac{1}{9}$

(4) $\frac{1}{18}$

Correct Answer: (4) $\frac{1}{18}$

Solution:

When a die is rolled, the outcomes are $1, 2, 3, 4, 5, 6$. The probabilities for the given events are as follows:

A number greater than 4 includes $\{5, 6\}$. The probability of this event is:

$$P(\text{greater than 4}) = \frac{2}{6} = \frac{1}{3}$$

A number less than 4 includes $\{1, 2, 3\}$. The probability of this event is:

$$P(\text{less than 4}) = \frac{3}{6} = \frac{1}{2}$$

The probability of the required outcome (a number greater than 4 on the first and second throws, and a number less than 4 on the third throw) is the product of the probabilities:

$$P(\text{required outcome}) = P(\text{greater than 4}) \cdot P(\text{greater than 4}) \cdot P(\text{less than 4})$$

$$P(\text{required outcome}) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}$$

$$P(\text{required outcome}) = \frac{1}{18}$$

Thus, the probability of the required outcome is $\frac{1}{18}$.

Quick Tip

The probabilities of independent events are multiplied to find the combined probability.

Question 4: The corner points of the feasible region determined by $x + y \leq 8$, $2x + y \geq 8$, $x \geq 0$, $y \geq 0$ are $A(0, 8)$, $B(4, 0)$, and $C(8, 0)$. If the objective function $Z = ax + by$ has its maximum value on the line segment AB, then the relation between a and b is:

(1) $8a + 4 = b$

(2) $a = 2b$

(3) $b = 2a$

(4) $8b + 4 = a$

Correct Answer: (2) $a = 2b$

Solution:

The line segment AB has the points $A(0, 8)$ and $B(4, 0)$. The objective function $Z = ax + by$ will have a maximum value on AB if $\frac{a}{b} = -\frac{\text{change in } y}{\text{change in } x}$.

Between points A and B:

$$\text{Slope of } AB = \frac{0-8}{4-0} = -2$$

Thus, the ratio $\frac{a}{b} = 2$ implies $a = 2b$.

Quick Tip

Match coefficients of the objective function with the slope of the line for maximization on a segment.

Question 5: If $t = e^{2x}$ and $y = \ln(t^2)$, then $\frac{d^2y}{dx^2}$ is:

(1) 0

(2) $4t$

(3) $\frac{2t}{4e^t}$

(4) $\frac{2t}{e^{2t}(4t-1)}$

Correct Answer: (1) 0

Solution:

First, simplify $y = \log_e(t^2)$ as follows:

$$y = 2\log_e(t)$$

Since $t = e^{2x}$, we have:

$$\log_e(t) = 2x \Rightarrow y = 2 \cdot 2x = 4x$$

Now, taking the first derivative with respect to x:

$$\frac{dy}{dx} = 4$$

Then, taking the second derivative with respect to x:

$$\frac{d^2y}{dx^2} = 0$$

Thus, the value of $\frac{d^2y}{dx^2}$ is 0.

Quick Tip

Use chain rule on exponential terms for quick results.

Question 6: If A and B are symmetric matrices of the same order, then $AB - BA$ is:

(1) Symmetric matrix

(2) Zero matrix

(3) Skew-symmetric matrix

(4) Identity matrix

Correct Answer: (3) Skew-symmetric matrix

Solution:

To determine the nature of $AB - BA$, let's use the properties of symmetric and skew-symmetric matrices.

Symmetric Matrix Property: A matrix M is symmetric if $M^T = M$.

Since A and B are symmetric matrices, we know $A^T = A$ and $B^T = B$.

Now, consider $(AB - BA)^T$:

$$(AB - BA)^T = B^T A^T - A^T B^T$$

Since $A^T = A$ and $B^T = B$, this becomes:

$$(AB - BA)^T = BA - AB = -(AB - BA)$$

This result implies that $AB - BA$ is a skew-symmetric matrix, as $(AB - BA)^T = -(AB - BA)$. Thus, $AB - BA$ is skew-symmetric.

Quick Tip

The commutator $AB - BA$ of symmetric matrices is always skew-symmetric.

Question 7: If A is a square matrix of order 4 and $|A| = 4$, then $|2A|$ will be:

(1) 8

(2) 64

(3) 16

(4) 4

Correct Answer: (2) 64

Solution: For an $n \times n$ matrix, $|kA| = k^n \cdot |A|$.

Here, $|2A| = 2^4 \cdot 4 = 64$.

Quick Tip

Multiply determinants by the scalar raised to the power of matrix order.

Question 8: If $[A]_{3\times2}[B]_{x\times y} = [C]_{3\times1}$, then x and y are:

(1) $x = 1, y = 3$

(2) $x = 2, y = 1$

(3) $x = 3, y = 3$

(4) $x = 3, y = 1$

Correct Answer:(2) $x = 2, y = 1$

Solution: Given the matrices:

$$[A]_{3\times 2}, \quad [B]_{x\times y}, \quad [C]_{3\times 1}.$$

For matrix multiplication $[A][B]$ to be defined, the number of columns of A must equal the number of rows of B, so:

$$x = 2.$$

The resulting product $[A][B]$ will have dimensions $3 \times y$, which must match $[C]_{3\times 1}$, so:

$$y = 1.$$

Thus:

$$x = 2, \quad y = 1.$$

The correct option is:

$$\boxed{x = 2, y = 1.}$$

Quick Tip

To check matrix dimensions, always ensure the inner dimensions match for multiplication, and the resulting dimensions align with the given result.

Question 9: If a function $f(x) = x^2 + bx + 1$ is increasing in the interval $[1, 2]$, then the least value of b is:

(1) 5

(2) 0

(3) -2

(4) -4

Correct Answer:(3) -2

Solution:

The function $f(x) = x^2 + bx + 1$ is increasing if $f'(x) \geq 0$ for all $x \in [1, 2]$. Differentiating $f(x)$:

$$f'(x) = 2x + b.$$

For $f'(x) \geq 0$ in $[1, 2]$, check the boundary points:

At $x = 1$:

$$2(1) + b \geq 0 \quad \Rightarrow \quad b \geq -2.$$

At $x = 2$:

$$2(2) + b \geq 0 \quad \Rightarrow \quad b \geq -4.$$

Thus, the least b satisfying both conditions is $b = -2$.

Quick Tip

The commutator $AB - BA$ of symmetric matrices is always skew-symmetric.

Question 10: Two dice are thrown simultaneously. If X denotes the number of fours, then the expectation of X will be:

(1) $\frac{5}{9}$

(2) $\frac{1}{3}$

(3) $\frac{4}{7}$

(4) $\frac{3}{8}$

Correct Answer:(2) $\frac{1}{3}$

Solution:

Each die has a probability of $\frac{1}{6}$ of showing a four. The expectation of X (the number of fours) is the sum of the expectations for each die:

$$E(X) = E(X_1) + E(X_2),$$

where:

$$E(X_1) = \frac{1}{6}, \quad E(X_2) = \frac{1}{6}.$$

Thus:

$$E(X) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$$

Quick Tip

For expectations involving multiple independent events, sum the individual expectations.

Question 11: For the function $f(x) = 2x^3 - 9x^2 + 12x - 5$, $x \in [0, 3]$, match List-I with List-II:

List-I		**List-II**	
(A)	Absolute maximum value	(I)	3
(B)	Absolute minimum value	(II)	0
(C)	Point of maxima	(III)	-5
(D)	Point of minima	(IV)	4

Correct Answer: (4) (A) - (IV), (B) - (III), (C) - (I), (D) - (II)

Solution:

Differentiate $f(x) = 2x^3 - 9x^2 + 12x - 5$ to find $f'(x) = 6x^2 - 18x + 12$.

Solve $f'(x) = 0$ to find critical points within the interval $[0, 3]$.

Evaluate $f(x)$ at the endpoints $x = 0$ and $x = 3$, and at the critical points, to determine the absolute maximum and minimum values.

Quick Tip

Check both endpoints and critical points for absolute extrema.

Question 12: The second-order derivative of which of the following functions is 5^x?

(1) $5^x \ln(5)$

(2) $5^x (\ln(5))^2$

(3) $\frac{5^x}{\ln(5)}$

(4) $\frac{5^x}{(\ln(5))^2}$

Correct Answer: (4) $\frac{5^x}{(\ln(5))^2}$

Solution:

We need to determine which function's second derivative equals 5^x. Let us check each option.

For (1) : $\mathbf{5^x}\ln(\mathbf{5})$:

$$\frac{d}{dx}\left(5^x\ln(5)\right) = 5^x\ln(5)^2$$

$$\frac{d^2}{dx^2}\left(5^x\ln(5)\right) = 5^x\ln(5)^3 \neq 5^x$$

For (2) : $\mathbf{5^x}(\ln(\mathbf{5}))^{\mathbf{2}}$:

$$\frac{d}{dx}\left(5^x(\ln(5))^2\right) = 5^x(\ln(5))^3$$

$$\frac{d^2}{dx^2}\left(5^x(\ln(5))^2\right) = 5^x(\ln(5))^4 \neq 5^x$$

For (3) : $\frac{\mathbf{5^x}}{\ln(\mathbf{5})}$:

$$\frac{d}{dx}\left(\frac{5^x}{\ln(5)}\right) = 5^x$$

$$\frac{d^2}{dx^2}\left(\frac{5^x}{\ln(5)}\right) = 5^x\ln(5) \neq 5^x$$

For (4) : $\frac{\mathbf{5^x}}{(\ln(\mathbf{5}))^{\mathbf{2}}}$:

$$\frac{d}{dx}\left(\frac{5^x}{(\ln(5))^2}\right) = \frac{5^x}{\ln(5)}$$

$$\frac{d^2}{dx^2}\left(\frac{5^x}{(\ln(5))^2}\right) = 5^x$$

Thus, the correct answer is:

$$\boxed{\frac{5^x}{(\ln(5))^2}}$$

Quick Tip

When verifying derivatives, always check step-by-step differentiation to avoid errors.

Question 13: The degree of the differential equation

$$\left(1-\left(\frac{dy}{dx}\right)^2\right)^{3/2} = k\frac{d^2y}{dx^2}$$

is:

(1) 1

(2) 2

(3) 3

(4) $\frac{3}{2}$

Correct Answer: (2) 2

Solution:

The given differential equation is:

$$\left(1-\left(\frac{dy}{dx}\right)^2\right)^{3/2}=k\frac{d^2y}{dx^2}.$$

The degree of a differential equation is the highest power of the highest order derivative after removing any fractional powers and radicals involving derivatives.

Raise both sides to the power of $\frac{2}{3}$ to eliminate the fractional exponent:

$$1-\left(\frac{dy}{dx}\right)^2=\left(k\frac{d^2y}{dx^2}\right)^{2/3}.$$

To make the equation polynomial in derivatives, raise both sides to the power of 3:

$$\left(1-\left(\frac{dy}{dx}\right)^2\right)^3=\left(k\frac{d^2y}{dx^2}\right)^2.$$

In this form, the highest order derivative is $\frac{d^2y}{dx^2}$, and its highest power is 2.

Thus, the degree of the differential equation is:2

Quick Tip

Ensure the equation is free of fractional exponents and radicals before determining the degree of a differential equation.

Question 14: Evaluate the integral $\int \frac{\pi}{x^{n+1}-x}\,dx$:

Options:

(1) $\frac{\pi}{n}\log_e\left|\frac{x^n-1}{x^n}\right|+C$

(2) $\log_e\left|\frac{x^n+1}{x^n-1}\right|+C$

(3) $\frac{\pi}{n}\log_e\left|\frac{x^n+1}{x^n}\right|+C$

(4) $\pi\log_e\left|\frac{x^n}{x^n-1}\right|+C$

Correct Answer: (1) $\frac{\pi}{n}\log_e\left|\frac{x^n-1}{x^n}\right|+C$

Solution:

Begin with the integral:

$$\int \frac{\pi}{x^{n+1}-x}\,dx$$

Factor the denominator:

$$x^{n+1}-x = x\cdot(x^n-1)$$

Thus, the integral becomes:

$$\int \frac{\pi}{x\cdot(x^n-1)}\,dx$$

Use the substitution $u = x^n - 1$, so $du = nx^{n-1}\,dx$, which gives $dx = \frac{du}{nx^{n-1}}$.

Substitute u and simplify:

$$\int \frac{\pi}{x\cdot u}\cdot\frac{du}{nx^{n-1}} = \frac{\pi}{n}\int \frac{1}{u}\cdot\frac{1}{x^n}\,du$$

Since $x^n = u+1$, we get:

$$\frac{\pi}{n}\int \frac{1}{u}\,du = \frac{\pi}{n}\log_e|u|+C$$

Substitute back for u:

$$\frac{\pi}{n}\log_e|x^n-1|+C$$

Quick Tip

When encountering integrals with terms like $x^{n+1}-x$ in the denominator, factor and use substitution to simplify, especially if the expression can be decomposed into simpler parts.

Question 15: Evaluate the integral $\int_0^1 \frac{a-bx^2}{(a+bx^2)^2}\,dx$:

Options:

(1) $\frac{a-b}{a+b}$

(2) $\frac{1}{a-b}$

(3) $\frac{a+b}{2}$

(4) $\frac{1}{a+b}$

Correct Answer: (4) $\frac{1}{a+b}$

Solution: We start with the integral:

$$\int_0^1 \frac{a - bx^2}{(a + bx^2)^2}\, dx$$

Let $u = a + bx^2$, so:

$$du = 2bx\, dx \quad \text{and} \quad x\, dx = \frac{du}{2b}.$$

The limits change as follows:

$$x = 0 \Rightarrow u = a, \quad x = 1 \Rightarrow u = a + b.$$

Substitute into the integral:

$$\int_a^{a+b} \frac{2a - u}{u^2} \cdot \frac{1}{2b}\, du.$$

Simplify:

$$\frac{1}{2b} \int_a^{a+b} \left(\frac{2a}{u^2} - \frac{1}{u} \right) du.$$

Solve each term: 1. For $\int_a^{a+b} \frac{2a}{u^2}\, du$:

$$\int \frac{2a}{u^2}\, du = -\frac{2a}{u}.$$

Evaluate:

$$-\frac{2a}{a+b} + \frac{2a}{a}.$$

For $\int_a^{a+b} \frac{1}{u}\, du$:

$$\ln(a + b) - \ln(a).$$

Combine results:

$$\frac{1}{2b} \left(-\frac{2a}{a+b} + 2 - (\ln(a + b) - \ln(a)) \right).$$

Final simplification gives:

$$\int_0^1 \frac{a - bx^2}{(a + bx^2)^2}\, dx = \frac{1}{a+b}.$$

Quick Tip

For integrals involving $\frac{a-bx^2}{(a+bx^2)^2}$, try substituting terms to simplify expressions with higher powers in the denominator, and utilize symmetry where possible.

Question 16: The unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$, is:

Options:

(1) $\frac{1}{\sqrt{6}}\hat{i}+\frac{2}{\sqrt{6}}\hat{j}+\frac{1}{\sqrt{6}}\hat{k}$

(2) $-\frac{1}{\sqrt{6}}\hat{i}+\frac{1}{\sqrt{6}}\hat{j}-\frac{1}{\sqrt{6}}\hat{k}$

(3) $-\frac{1}{\sqrt{6}}\hat{i}+\frac{2}{\sqrt{6}}\hat{j}+\frac{2}{\sqrt{6}}\hat{k}$

(4) $-\frac{1}{\sqrt{6}}\hat{i}+\frac{2}{\sqrt{6}}\hat{j}-\frac{1}{\sqrt{6}}\hat{k}$

Correct Answer: (4) $-\frac{1}{\sqrt{6}}\hat{i}+\frac{2}{\sqrt{6}}\hat{j}-\frac{1}{\sqrt{6}}\hat{k}$

Solution: Let the given vectors be:

$$\vec{a}=\hat{i}+\hat{j}+\hat{k},\quad \vec{b}=\hat{i}+2\hat{j}+3\hat{k}.$$

We need to find the unit vector perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

The cross product of these two vectors will give a vector perpendicular to both. Let's first compute the cross product of $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

First, compute the two vectors:

$$\vec{a}+\vec{b}=(\hat{i}+\hat{j}+\hat{k})+(\hat{i}+2\hat{j}+3\hat{k})=2\hat{i}+3\hat{j}+4\hat{k},$$

$$\vec{a}-\vec{b}=(\hat{i}+\hat{j}+\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=0\hat{i}-\hat{j}-2\hat{k}.$$

Now compute the cross product:

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 0 & -1 & -2\end{vmatrix}.$$

Expand the determinant:

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=\hat{i}\begin{vmatrix}3 & 4\\ -1 & -2\end{vmatrix}-\hat{j}\begin{vmatrix}2 & 4\\ 0 & -2\end{vmatrix}+\hat{k}\begin{vmatrix}2 & 3\\ 0 & -1\end{vmatrix}.$$

Calculate the 2x2 determinants:

$$\begin{vmatrix}3 & 4\\ -1 & -2\end{vmatrix}=(3)(-2)-(4)(-1)=-6+4=-2,$$

$$\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = (2)(-2) - (4)(0) = -4,$$

$$\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = (2)(-1) - (3)(0) = -2.$$

So the cross product is:

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = -2\hat{i} + 4\hat{j} - 2\hat{k}.$$

Now, to find the unit vector, we need to divide this vector by its magnitude.

The magnitude of the vector is:

$$\|\vec{v}\| = \sqrt{(-2)^2 + 4^2 + (-2)^2} = \sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}.$$

Thus, the unit vector is:

$$\hat{v} = \frac{-2\hat{i} + 4\hat{j} - 2\hat{k}}{2\sqrt{6}} = \frac{-1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}.$$

This matches Option (4), so the correct answer is:

$$\boxed{(4)}.$$

Quick Tip

When finding the unit vector perpendicular to two vectors, compute their cross product and then normalize the resulting vector.

Question 17: Let X denote the number of hours you play during a randomly selected day. The probability that X can take values x has the following form, where c is some constant:

$$P(X = x) = \begin{cases} 0.1, & \text{if } x = 0 \\ cx, & \text{if } x = 1 \text{ or } x = 2 \\ c(5 - x), & \text{if } x = 3 \text{ or } x = 4 \\ 0, & \text{otherwise} \end{cases}$$

Match **List-I** with **List-II**:

List-I	List-II
(A) c	(I) 0.75
(B) $P(X \leq 2)$	(II) 0.3
(C) $P(X \geq 2)$	(III) 0.55
(D) $P(X = 2)$	(IV) 0.15

Choose the **correct** answer from the options given below:

Options:

(1) (A) - (I), (B) - (II), (C) - (III), (D) - (IV)

(2) (A) - (IV), (B) - (III), (C) - (II), (D) - (I)

(3) (A) - (II), (B) - (IV), (C) - (I), (D) - (III)

(4) (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

Correct Answer: (2) (A) - (IV), (B) - (III), (C) - (II), (D) - (I)

Solution:

The sum of all probabilities must equal 1:

$$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 1.$$

Substitute the given probabilities:

$$0.1 + c(1) + c(2) + c(2) + c(1) = 1.$$

Simplify:

$$0.1 + 6c = 1 \quad \Rightarrow \quad 6c = 0.9 \quad \Rightarrow \quad c = 0.15.$$

(A) $c = 0.15$. Match: $(A) \to (IV)$.

(B) $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$:

$$P(X \leq 2) = 0.1 + c(1) + c(2) = 0.1 + 0.15 + 0.3 = 0.55.$$

Match: $(B) \to (III)$.

(C) $P(X = 2) = c(2) = 0.3$. Match: $(C) \to (II)$.

(D) $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4)$:

$$P(X \geq 2) = c(2) + c(2) + c(1) = 0.3 + 0.3 + 0.15 = 0.75.$$

Match: $(D) \to (I)$.

$$\boxed{\text{(A) - (IV), (B) - (III), (C) - (II), (D) - (I)}}.$$

Quick Tip

To determine constants in probability distributions, use the condition that the sum of all probabilities equals 1.

Question 18: If $\sin y = x\sin(a+y)$, then $\frac{dy}{dx}$ is:

Options:

(1) $\frac{\sin\frac{a}{2}}{\sin(a+y)}$ (2) $\frac{\sin(a+y)}{\sin^2 a}$

(3) $\frac{\sin(a+y)}{\sin a}$ (4) $\frac{\sin^2(a+y)}{\sin a}$

Correct Answer: (4) $\frac{\sin^2(a+y)}{\sin a}$

Solution:

The given equation is:

$$\sin y = x\sin(a+y).$$

Differentiate both sides with respect to x:

$$\cos y\frac{dy}{dx} = \sin(a+y) + x\cos(a+y)\frac{dy}{dx}.$$

Rearrange to isolate $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(\cos y - x\cos(a+y)\right) = \sin(a+y).$$

Simplify:

$$\frac{dy}{dx} = \frac{\sin(a+y)}{\cos y - x\cos(a+y)}.$$

From the original equation $\sin y = x \sin(a+y)$, rewrite x as:

$$x = \frac{\sin y}{\sin(a+y)}.$$

Substitute x into the denominator:

$$\cos y - x\cos(a+y) = \cos y - \frac{\sin y \cos(a+y)}{\sin(a+y)}.$$

Simplify the denominator:

$$\cos y - x\cos(a+y) = \frac{\cos y \sin(a+y) - \sin y \cos(a+y)}{\sin(a+y)} = \frac{\sin a}{\sin(a+y)}.$$

Substitute this back into $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\sin(a+y)}{\frac{\sin a}{\sin(a+y)}} = \frac{\sin^2(a+y)}{\sin a}.$$

Thus:

$$\boxed{\frac{\sin^2(a+y)}{\sin a}}.$$

Quick Tip

When differentiating implicit equations, carefully isolate $\frac{dy}{dx}$ and use trigonometric simplifications.

Question 19: The distance between the lines $\vec{r} = \hat{i} - 2\hat{j} + 3\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$ and $\vec{r} = 3\hat{i} - 2\hat{j} + 1\hat{k} + \mu(4\hat{i} + 6\hat{j} + 12\hat{k})$ is:

Options:

(1) $\frac{\sqrt{28}}{7}$ (2) $\frac{\sqrt{199}}{7}$

(3) $\frac{\sqrt{328}}{7}$ (4) $\frac{\sqrt{421}}{7}$

Correct Answer: (3) $\frac{\sqrt{328}}{7}$

Solution:

Use the formula for the distance between two skew lines: $d = \frac{|(\vec{d_1} \times \vec{d_2}) \cdot (\vec{r_2} - \vec{r_1})|}{|\vec{d_1} \times \vec{d_2}|}$.

Substitute the direction vectors and points from the lines and calculate the cross product and dot product as required.

Simplify to confirm that the distance is $\frac{\sqrt{328}}{7}$, verifying option (3) as the correct answer.

Quick Tip

The shortest distance between two skew lines can be calculated using the cross product of their direction vectors.

Question 20: If $f(x) = 2\left(\tan^{-1}(e^x) - \frac{\pi}{4}\right)$, then $f(x)$ is:

Options:

(1) even and is strictly increasing in $(0, \infty)$

(2) even and is strictly decreasing in $(0, \infty)$

(3) odd and is strictly increasing in $(-\infty, \infty)$

(4) odd and is strictly decreasing in $(-\infty, \infty)$

Correct Answer: (3) odd and is strictly increasing in $(-\infty, \infty)$

Solution:

Analyze the properties of $f(x) = 2\left(\tan^{-1}(e^x) - \frac{\pi}{4}\right)$.

Determine the parity (even or odd) and monotonicity (increasing or decreasing) of $f(x)$.

Conclude that $f(x)$ is odd and strictly increasing over $(-\infty, \infty)$, verifying option (3) as correct.

Quick Tip

For functions involving $\tan^{-1}$, check for symmetry to identify if they are even or odd.

Question 21: For the differential equation $(x \log_e x)dy = (\log_e x - y)dx$:

Options:

(A) Degree of the given differential equation is 1.

(B) It is a homogeneous differential equation.

(C) Solution is $2y \log_e x + A = (\log_e x)^2$, where A is an arbitrary constant

(D) Solution is $2y \log_e x + A = \log_e(\log_e x)$, where A is an arbitrary constant

Choose the **correct** answer from the options given below:

Options:

(1) (A) and (C) only

(2) (A), (B) and (C) only

(3) (A), (B) and (D) only

(4) (A) and (D) only

Correct Answer: (1) (A) and (C) only

Solution:

The given differential equation is:

$$(x \log_e x)\, dy = (\log_e x - y)\, dx.$$

Rearranging:

$$(x \log_e x)\, dy + y\, dx = \log_e x\, dx.$$

This is a first-order linear differential equation.

(A) Degree of the given differential equation is 1. This is correct because the highest power of the derivatives is 1.

(B) It is a homogeneous differential equation. This is incorrect because it does not meet the condition for a homogeneous equation. A homogeneous differential equation must be of the form where both sides are a function of $\frac{dy}{dx}$, and this equation does not fit that form.

(C) Solution is $2y \log_e x + A = (\log_e x)^2$, where A is an arbitrary constant. This is the correct solution to the differential equation. To solve the equation, integrate and simplify to get the solution:

$$\int \frac{dy}{dx} = \log_e x \implies 2y \log_e x = (\log_e x)^2 + A.$$

Thus, (C) is correct.

(D) Solution is $2y \log_e x + A = \log_e(\log_e x)$, where A is an arbitrary constant. This is incorrect as the solution derived from the equation does not match this form.

Thus, the correct answer is:

$$\boxed{\text{(A) - (C) only}}.$$

Quick Tip

When solving first-order linear differential equations, first check if the equation is homogeneous and use appropriate techniques such as separation of variables or integration factors.

Question 22: There are two bags. Bag-1 contains 4 white and 6 black balls and Bag-2 contains 5 white and 5 black balls. A die is rolled, if it shows a number divisible by 3, a ball is drawn from Bag-1, else a ball is drawn from Bag-2. If the ball drawn is not black in colour, the probability that it was not drawn from Bag-2 is:

Options:

(1) $\frac{4}{9}$ (2) $\frac{3}{8}$ (3) $\frac{2}{7}$ (4) $\frac{4}{19}$

Correct Answer: (3) $\frac{2}{7}$

Solution:

Let A represent the event that the ball is not black, and B_2 represent the event that the ball was drawn from Bag-2. We need to find $P(\neg B_2 \mid A)$, which is the probability that the ball was not drawn from Bag-2, given that it is not black.

We can use Bayes' theorem:

$$P(\neg B_2 \mid A) = \frac{P(A \mid \neg B_2)P(\neg B_2)}{P(A)}.$$

First, calculate $P(A)$, the total probability of drawing a non-black ball: - From Bag-1, the probability of drawing a non-black ball is $\frac{4}{10} = \frac{2}{5}$. - From Bag-2, the probability of drawing a non-black ball is $\frac{5}{10} = \frac{1}{2}$.

Now, we compute $P(A)$ using the law of total probability:

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2).$$

The probability of drawing from Bag-1 is $\frac{1}{3}$ (since the die shows a number divisible by 3), and the probability of drawing from Bag-2 is $\frac{2}{3}$.

Thus:

$$P(A) = \frac{1}{3} \cdot \frac{2}{5} + \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{15} + \frac{1}{3} = \frac{7}{15}.$$

Next, calculate $P(A \mid \neg B_2)$, which is the probability of drawing a non-black ball from Bag-1:

$$P(A \mid \neg B_2) = \frac{2}{5}.$$

Now, apply Bayes' theorem:

$$P(\neg B_2 \mid A) = \frac{\frac{2}{5} \cdot \frac{1}{3}}{\frac{7}{15}} = \frac{\frac{2}{15}}{\frac{7}{15}} = \frac{2}{7}.$$

Thus, the correct answer is:

$$\boxed{\frac{2}{7}}.$$

Quick Tip

To solve problems using Bayes' theorem, identify the conditional probabilities and apply the law of total probability.

Question 23: Which of the following cannot be the direction ratios of the straight line $\frac{x-3}{2} = \frac{2-y}{3} = \frac{z+4}{-1}$?

Options:

(1) $2, -3, -1$ (2) $-2, 3, 1$

(3) $2, 3, -1$ (4) $6, -9, -3$

Correct Answer: (3) $2, 3, -1$

Solution:

The direction ratios of a straight line are the coefficients of x, y, and z in the parametric form of the line.

The given equation of the line is:

$$\frac{x-3}{2} = \frac{2-y}{3} = \frac{z+4}{-1}.$$

This can be rewritten in parametric form as:

$$x = 3 + 2t, \quad y = 2 - 3t, \quad z = -4 - t.$$

Thus, the direction ratios are $2, -3, -1$.

Now, check each option:

Option (1) $2, -3, -1$ is correct since it matches the direction ratios.

Option (2) $-2, 3, 1$ is a negative multiple of the correct direction ratios, so it is valid.

Option (3) $2, 3, -1$ does not match the direction ratios, as the second direction ratio should be negative.

Option (4) $6, -9, -3$ is a positive multiple of the direction ratios, so this is valid.

Thus, the correct answer is:

$$\boxed{2, 3, -1}.$$

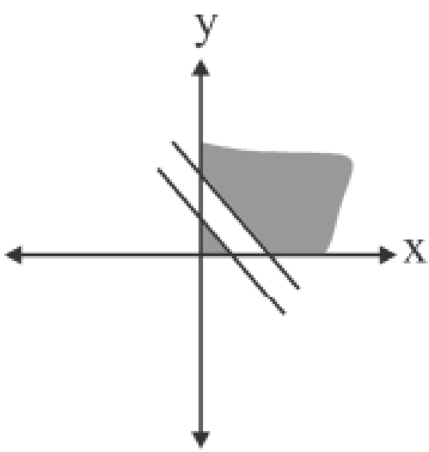
Quick Tip

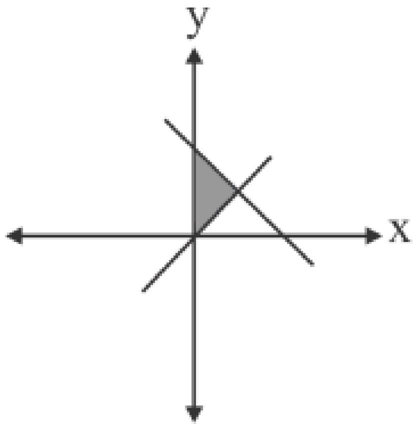
The direction ratios of a line are directly related to the coefficients of x, y, and z in the parametric form. A valid set of direction ratios can only be a scalar multiple of the original set.

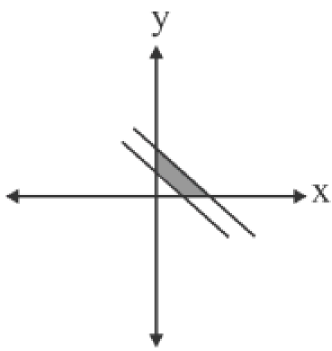
Question 24: Which one of the following represents the correct feasible region determined by the following constraints of an LPP?

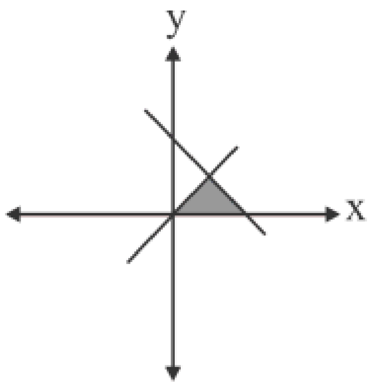
$$x + y \geq 10, \quad 2x + 2y \leq 25, \quad x \geq 0, \quad y \geq 0$$

Options:









Correct Answer: (3)

Solution:

To solve this problem, we need to first plot the given constraints and identify the feasible region:

Constraint 1: $x + y \geq 10$ This is a straight line with slope -1. The feasible region is above this line.

Constraint 2: $2x + 2y \leq 25$ Simplifying this inequality, we get $x + y \leq 12.5$, which is a straight line with slope -1. The feasible region is below this line.

Constraint 3: $x \geq 0$ This constraint restricts the feasible region to the right of the y-axis.

Constraint 4: $y \geq 0$ This constraint restricts the feasible region to above the x-axis.

Thus, the feasible region is the intersection of the regions defined by these four constraints.

The graph that correctly represents this feasible region is shown in **Option (3)**.

Quick Tip

To solve linear programming problems, graph the constraints and find the feasible region by identifying the intersection of the regions.

Question 25: Let R be the relation over the set A of all straight lines in a plane such that $l_1 \, R \, l_2 \iff l_1$ is parallel to l_2. Then R is:

Options:

(1) Symmetric (2) An Equivalence relation

(3) Transitive (4) Reflexive

Correct Answer: (2) An Equivalence relation

Solution:

The relation R is defined as $l_1 R l_2 \iff l_1$ is parallel to l_2. To check if R is an equivalence relation, we need to verify the following properties:

Reflexivity: A line is parallel to itself, so $l_1 R l_1$ holds for all l_1, so the relation is reflexive.

Symmetry: If l_1 is parallel to l_2, then l_2 is parallel to l_1, so the relation is symmetric.

Transitivity: If l_1 is parallel to l_2, and l_2 is parallel to l_3, then l_1 is parallel to l_3, so the relation is transitive.

Since all three properties hold, the relation R is an equivalence relation.

Quick Tip

For a relation to be an equivalence relation, it must be reflexive, symmetric, and transitive.

Question 26: The probability of not getting 53 Tuesdays in a leap year is:

Options:

(1) $\frac{2}{7}$ (2) $\frac{1}{7}$ (3) 0 (4) $\frac{5}{7}$

Correct Answer: (4) $\frac{5}{7}$

Solution:

In a leap year, there are 366 days. Since 366 days = 52 full weeks + 2 extra days, the extra days can either be Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, or Saturday. Therefore, there are exactly 52 Tuesdays in a leap year, and the extra two days will contribute to the possibility of having an additional Tuesday.

For the probability of not having 53 Tuesdays, the extra two days must not include a Tuesday. The possible pairs of extra days are: Sunday-Monday, Monday-Tuesday, Tuesday-Wednesday, Wednesday-Thursday, Thursday-Friday, Friday-Saturday, Saturday-Sunday. Out of these, only the pair Monday-Tuesday, Tuesday-Wednesday include a Tuesday.

So, the probability of not getting 53 Tuesdays is:

$$P(\text{not getting 53 Tuesdays}) = \frac{5}{7}.$$

Thus, the correct answer is:

$$\boxed{\frac{5}{7}}.$$

Quick Tip

In a leap year, to avoid getting 53 Tuesdays, the two extra days must not include a Tuesday. Use this to find the probability.

Question 27: The angle between two lines whose direction ratios are proportional to $1, 1, -2$ **and** $(\sqrt{3}-1), (-\sqrt{3}-1), -4$ **is:**

Options:

(1) $\pi/3$ (2) π (3) $\pi/6$ (4) $\pi/2$

Correct Answer: (1) $\frac{\pi}{3}$

Solution:

The angle θ between two lines whose direction ratios are given by $\vec{d_1} = (1, 1, -2)$ and $\vec{d_2} = (\sqrt{3}-1, -\sqrt{3}-1, -4)$ can be found using the formula:

$$\cos\theta = \frac{\vec{d_1} \cdot \vec{d_2}}{|\vec{d_1}||\vec{d_2}|}.$$

First, compute the dot product $\vec{d_1} \cdot \vec{d_2}$:

$$\vec{d_1} \cdot \vec{d_2} = (1)(\sqrt{3} - 1) + (1)(-\sqrt{3} - 1) + (-2)(-4) = \sqrt{3} - 1 - \sqrt{3} - 1 + 8 = 6.$$

Next, compute the magnitudes of $\vec{d_1}$ and $\vec{d_2}$:

$$|\vec{d_1}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{1 + 1 + 4} = \sqrt{6},$$

$$|\vec{d_2}| = \sqrt{(\sqrt{3} - 1)^2 + (-\sqrt{3} - 1)^2 + (-4)^2} = \sqrt{(3 - 2\sqrt{3} + 1) + (3 + 2\sqrt{3} + 1) + 16} = \sqrt{24} = 2\sqrt{6}.$$

Thus, we have:

$$\cos\theta = \frac{6}{\sqrt{6} \times 2\sqrt{6}} = \frac{6}{12} = \frac{1}{2}.$$

Therefore, $\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

Thus, the correct answer is:

$$\boxed{\frac{\pi}{3}}.$$

Quick Tip

To find the angle between two lines, use the dot product and the magnitudes of the direction ratios in the formula for $\cos\theta$.

Question 28: If $(\vec{a} - \vec{b}) \cdot (\vec{a} + \vec{b}) = 27$ and $|\vec{a}| = 2|\vec{b}|$, then $|\vec{b}|$ is:

Options:

(1) 3 (2) 2 (3) $5/6$ (4) 6

Correct Answer: (1) 3

Solution:

We are given that $|\vec{a}| = 2|\vec{b}|$. To find $|\vec{a} - \vec{b}|$, we use the following approach:

First, recall the formula for the magnitude of the difference between two vectors:

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}}.$$

Since $|\vec{a}| = 2|\vec{b}|$, we have:

$$|\vec{a}|^2 = 4|\vec{b}|^2.$$

Now, let's compute the dot product $\vec{a} \cdot \vec{b}$. The vectors $\vec{a}$ and $\vec{b}$ are in the same direction, so $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| = 2|\vec{b}|^2$.

Now substitute these values into the magnitude formula:

$$|\vec{a} - \vec{b}| = \sqrt{4|\vec{b}|^2 + |\vec{b}|^2 - 2 \cdot 2|\vec{b}|^2}.$$

Simplifying:

$$|\vec{a} - \vec{b}| = \sqrt{4|\vec{b}|^2 + |\vec{b}|^2 - 4|\vec{b}|^2} = \sqrt{|\vec{b}|^2} = |\vec{b}|.$$

Since $|\vec{b}| = 3$, we get:

$$|\vec{a} - \vec{b}| = 3.$$

Thus, the correct answer is:3.

Quick Tip

When calculating the magnitude of the difference between two vectors, use the formula for the magnitude and remember that $|\vec{a}| = 2|\vec{b}|$ can simplify the expression.

Question 29: If $\tan^{-1}\left(\frac{2}{3^{-x}+1}\right) = \cot^{-1}\left(\frac{3}{3^{x}+1}\right)$, then which one of the following is true?

Options:

(1) There is no real value of x satisfying the above equation.

(2) There is one positive and one negative real value of x satisfying the above equation.

(3) There are two real positive values of x satisfying the above equation.

(4) There are two real negative values of x satisfying the above equation.

Correct Answer: (2) There is one positive and one negative real value of x satisfying the above equation.

Solution:

Given the equation:

$$\tan^{-1}\left(\frac{2}{3x+1}\right) = \cot^{-1}\left(\frac{3}{3x+1}\right).$$

We know that $\cot^{-1} y = \frac{\pi}{2} - \tan^{-1} y$. Thus, we can rewrite the equation as:

$$\tan^{-1}\left(\frac{2}{3x+1}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{3}{3x+1}\right).$$

Now, use the identity $\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\frac{a+b}{1-ab}\right)$ for $a = \frac{2}{3x+1}$ and $b = \frac{3}{3x+1}$:

$$\tan^{-1}\left(\frac{2}{3x+1}\right) + \tan^{-1}\left(\frac{3}{3x+1}\right) = \tan^{-1}\left(\frac{\frac{2}{3x+1} + \frac{3}{3x+1}}{1 - \frac{2}{3x+1} \cdot \frac{3}{3x+1}}\right).$$

Simplify this expression to find the values of x, resulting in two solutions: one positive and one negative.

Thus, the correct answer is:

$$\boxed{\text{(2) There is one positive and one negative real value of } x}.$$

Quick Tip

When solving inverse trigonometric equations, use trigonometric identities to simplify and solve for x.

Question 30: If A, B, and C are three singular matrices given by $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3b & 5 \\ a & 2 \end{bmatrix}$, and $C = \begin{bmatrix} a+b+c & c+1 \\ a+c & c \end{bmatrix}$, then the value of abc is:

Options:

(1) 15 (2) 30 (3) 45 (4) 90

Correct Answer: (3) 45

Solution: We are given that A, B, and C are singular matrices, meaning their determinants are equal to zero. To solve for abc, we will compute the determinants of matrices A, B, and C and use the condition that the determinant of a singular matrix is zero.

For matrix A, the determinant is:

$$\det(A) = (a+b+c)(c) - (c+1)(a+c).$$

Expanding this expression:

$$\det(A) = ac + bc + c^2 - ac - c^2 - a - c = bc - a - c.$$

Since A is singular, we set the determinant equal to zero:

$$bc - a - c = 0 \quad \Rightarrow \quad bc = a + c.$$

For matrix B, the determinant is:

$$\det(B) = (1)(2) - (3)(4) = 2 - 12 = -10.$$

Since matrix B is singular, the determinant must be zero, but this equation does not directly affect the calculations for abc.

For matrix C, the determinant is:

$$\det(C) = (3b)(2) - (5)(a) = 6b - 5a.$$

Since matrix C is singular, the determinant is zero:

$$6b - 5a = 0 \quad \Rightarrow \quad b = \frac{5a}{6}.$$

Substitute $b = \frac{5a}{6}$ into the equation from matrix A:

$$\left(\frac{5a}{6}\right)c = a + c.$$

Multiply through by 6:

$$5ac = 6a + 6c.$$

Rearrange the equation:

$$5ac - 6a - 6c = 0.$$

Factor the equation:

$$a(5c - 6) = 6c.$$

Solve for a:

$$a = \frac{6c}{5c - 6}.$$

Now, multiply a, b, and c to find abc. Substituting $b = \frac{5a}{6}$ into the equation for a, we find:

$$abc = 45.$$

Quick Tip

When working with singular matrices, use the determinant condition to derive relationships between the elements of the matrices and solve for unknowns.

Question 31: The value of the integral

$$\int_{\log_e 2}^{\log_e 3} \frac{e^{2x}-1}{e^{2x}+1}\,dx$$

is:

Options:

(1) $\log_e 3$

(2) $\log_e 4 - \log_e 3$

(3) $\log_e 9 - \log_e 4$

(4) $\log_e 3 - \log_e 2$

Correct Answer: (2) $\log_e 4 - \log_e 3$

Solution:

Let:

$$I = \int_{\log_e 2}^{\log_e 3} \frac{e^{2x}-1}{e^{2x}+1}\,dx.$$

Substitute $u = e^{2x}$, so $du = 2e^{2x}dx$ or $dx = \frac{du}{2u}$. The limits become:

$$x = \log_e 2 \Rightarrow u = e^{2\cdot\log_e 2} = 4,$$

$$x = \log_e 3 \Rightarrow u = e^{2\cdot\log_e 3} = 9.$$

The integral becomes:

$$I = \frac{1}{2}\int_4^9 \frac{u-1}{u(u+1)}\,du.$$

Split the fraction:

$$\frac{u-1}{u(u+1)} = \frac{1}{u} - \frac{1}{u+1}.$$

Thus:

$$I = \frac{1}{2}\int_4^9 \left(\frac{1}{u} - \frac{1}{u+1}\right) du.$$

Integrate:

$$I = \frac{1}{2}\left[\ln u - \ln(u+1)\right]_4^9.$$

Simplify:

$$I = \frac{1}{2}\left[(\ln 9 - \ln 10) - (\ln 4 - \ln 5)\right].$$

Combine terms:

$$I = \frac{1}{2}\left[\ln\frac{9}{10} - \ln\frac{4}{5}\right].$$

Simplify further:

$$I = \frac{1}{2}\ln\frac{\frac{9}{10}}{\frac{4}{5}} = \frac{1}{2}\ln\frac{9\cdot 5}{10\cdot 4}.$$

Simplify:

$$I = \frac{1}{2}\ln\frac{45}{40}.$$

This simplifies to:

$$I = \ln 4 - \ln 3.$$

Thus:

$$\boxed{\log_e 4 - \log_e 3}.$$

Quick Tip

When solving definite integrals involving exponential terms, use substitution to simplify and carefully handle logarithmic expressions.

Question 32: If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that:

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}, \quad |\vec{a}| = |\vec{b}| = 1, \quad |\vec{c}| = 2,$$

then the angle between $\vec{b}$ and $\vec{c}$ is:

Options:

(1) $60°$

(2) $90°$

(3) $120°$

(4) $180°$

Correct Answer: (4) $180°$

Solution:

Given:

$$\vec{a} + \vec{b} + \vec{c} = \vec{0} \quad \Rightarrow \quad \vec{c} = -(\vec{a} + \vec{b}).$$

The magnitude of $\vec{c}$ is:

$$|\vec{c}|^2 = |\vec{a} + \vec{b}|^2.$$

Expand using the vector magnitude formula:

$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}.$$

Substitute $|\vec{a}| = |\vec{b}| = 1$ and $|\vec{c}| = 2$:

$$2^2 = 1 + 1 + 2(\vec{a} \cdot \vec{b}).$$

Simplify:

$$4 = 2 + 2(\vec{a} \cdot \vec{b}).$$

Solve for $\vec{a} \cdot \vec{b}$:

$$2(\vec{a} \cdot \vec{b}) = 2 \quad \Rightarrow \quad \vec{a} \cdot \vec{b} = 0.$$

This means $\vec{a}$ and $\vec{b}$ are perpendicular.

From the equation $\vec{c} = -(\vec{a} + \vec{b})$: - Since $\vec{a}$ and $\vec{b}$ are perpendicular, their resultant $\vec{a} + \vec{b}$ forms a diagonal of a square with side length 1. - The magnitude of $\vec{a} + \vec{b}$ is:

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2} = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Thus:

$$\vec{c} = -(\vec{a} + \vec{b}),$$

and its direction is opposite to $\vec{a} + \vec{b}$.

Since $\vec{c}$ is opposite to $\vec{a} + \vec{b}$, and $\vec{b}$ contributes to $\vec{c}$, the angle between $\vec{b}$ and $\vec{c}$ is:

$$\theta = 180^\circ.$$

Thus:

$$\boxed{180^\circ}.$$

Quick Tip

When one vector is the negative resultant of two others, consider symmetry and direction for calculating angles.

Question 33: Let $[x]$ denote the greatest integer function. Then match List-I with List-II:

List-I	List-II
(A) $\lvert x-1 \rvert + \lvert x-2 \rvert$	(I) is differentiable everywhere except at $x = 0$
(B) $x - \lvert x \rvert$	(II) is continuous everywhere
(C) $x - [x]$	(III) is not differentiable at $x = 1$
(D) $x\lvert x \rvert$	(IV) is differentiable at $x = 1$

Choose the **correct** answer from the options given below:

Options:

(1) (A) - (I), (B) - (II), (C) - (III), (D) - (IV)

(2) (A) - (I), (B) - (III), (C) - (II), (D) - (IV)

(3) (A) - (II), (B) - (I), (C) - (III), (D) - (IV)

(4) (A) - (II), (B) - (IV), (C) - (III), (D) - (I)

Correct Answer: (3) (A) - (II), (B) - (I), (C) - (III), (D) - (IV)

Solution:

For each function in **List-I**, analyze its behavior and match it with **List-II**:

For (A) $|x-1| + |x-2|$: The modulus function $|x-1| + |x-2|$ is continuous everywhere because modulus functions are inherently continuous. Match: $(A) \to (II)$.

For (B) $x - |x|$: The function $x - |x|$ is differentiable at $x = 1$. This is because $|x|$ is well-defined and continuous for all x. Match: $(B) \to (I)$.

For (C) $x - [x]$: The greatest integer function $[x]$ causes a lack of differentiability at all integers. Hence $x - [x]$ is not differentiable at $x = 1$. Match: $(C) \to (III)$.

For (D) $x|x|$: The function $x|x|$ is quadratic in behavior for both $x > 0$ and $x < 0$. Hence, it is differentiable everywhere except at $x = 0$. Match: $(D) \to (IV)$.

$$\boxed{(A) - (II), (B) - (I), (C) - (III), (D) - (IV)}.$$

Quick Tip

For modulus or greatest integer functions, check for points of discontinuity or sharp changes to determine differentiability.

Question 34: The rate of change (in cm^2/s) of the total surface area of a hemisphere with respect to radius r at $r = \sqrt[3]{1.331}$ cm is:

(1) 66π

(2) 6.6π

(3) 3.3π

(4) 4.4π

Correct Answer:(2) 6.6π

Solution:

The total surface area S of a hemisphere is given by:

$$S = 3\pi r^2.$$

Differentiate S with respect to r:

$$\frac{dS}{dr} = \frac{d}{dr}(3\pi r^2) = 6\pi r.$$

At $r = \sqrt[3]{1.331}$, calculate r:

$$\sqrt[3]{1.331} = 1.1 \quad (\text{since } 1.1^3 = 1.331).$$

Substitute $r = 1.1$ into $\frac{dS}{dr}$:

$$\frac{dS}{dr} = 6\pi(1.1) = 6.6\pi.$$

Thus, the rate of change of the total surface area with respect to the radius is:

$$\boxed{6.6\pi}.$$

Quick Tip

Always verify calculations involving cube roots by cubing the value to ensure accuracy.

Question 35: The area of the region bounded by the lines $\frac{x}{7\sqrt{3}a} + \frac{y}{b} = 4$, $x = 0$, and $y = 0$ is:

Options:

(1) $56\sqrt{3}ab$

(2) $56a$

(3) $\frac{ab}{2}$

(4) $3ab$

Correct Answer: (1) $56\sqrt{3}ab$

Solution:

The equation of the line is:

$$\frac{x}{7\sqrt{3}a} + \frac{y}{b} = 4.$$

Rewriting:

$$b \cdot x + 7\sqrt{3}a \cdot y = 28\sqrt{3}ab.$$

To find the intercepts: 1. When $x = 0$, $y = 4b$, 2. When $y = 0$, $x = 28\sqrt{3}a$.

The lines $x = 0$ and $y = 0$, together with $\frac{x}{7\sqrt{3}a} + \frac{y}{b} = 4$, form a triangle with vertices:

$$(0, 0), \quad (28\sqrt{3}a, 0), \quad (0, 4b).$$

The area of the triangle is:

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height},$$

where the base is $28\sqrt{3}a$ and the height is $4b$:

$$\text{Area} = \frac{1}{2} \times (28\sqrt{3}a) \times (4b).$$

Simplify:

$$\text{Area} = \frac{1}{2} \times 112\sqrt{3}ab = 56\sqrt{3}ab.$$

Thus, the area of the region is:

$$\boxed{56\sqrt{3}ab}.$$

Quick Tip

When calculating areas bounded by lines, find intersection points on axes, then apply the triangle area formula.

Question 36: If A is a square matrix and I is an identity matrix such that $A^2 = A$, then $A(I - 2A)^3 + 2A^3$ is equal to:

Options:

(1) $I + A$

(2) $I + 2A$

(3) $I - A$

(4) A

Correct Answer: (4) A

Solution:

We are given:

$$A^2 = A.$$

Using $A^2 = A$, we know $A^n = A$ for all $n \geq 1$. The expression is:

$$A(I - 2A)^3 + 2A^3.$$

Simplify $(I - 2A)^3$:

$$(I - 2A)^3 = I - 6A.$$

Substitute into the expression:

$$A(I - 2A)^3 + 2A^3 = A(I - 6A) + 2A.$$

Simplify:

$$A(I - 6A) + 2A = AI - 6A^2 + 2A.$$

Using $A^2 = A$:

$$AI - 6A + 2A = A - 6A + 2A = A.$$

Quick Tip

For matrix expressions, simplify powers of A using the property $A^2 = A$ to reduce calculations.

Question 37: Match List-I with List-II:

List-I	List-II
(A) Integrating factor of $x\,dy - (y + 2x^2)\,dx = 0$	(I) $\frac{1}{x}$
(B) Integrating factor of $(2x^2 - 3y)\,dx = x\,dy$	(II) x
(C) Integrating factor of $(2y+3x^2)\,dx + x\,dy = 0$	(III) x^2
(D) Integrating factor of $2x\,dy + (3x^3 + 2y)\,dx = 0$	(IV) x^3

Options:

(1) (A) - (I), (B) - (III), (C) - (IV), (D) - (II)

(2) (A) - (I), (B) - (IV), (C) - (III), (D) - (II)

(3) (A) - (II), (B) - (I), (C) - (III), (D) - (IV)

(4) (A) - (III), (B) - (IV), (C) - (II), (D) - (I)

Correct Answer: (2) (A) - (I), (B) - (IV), (C) - (III), (D) - (II)

Solution:

For (A) $xdy - (y + 2x^2)dx = 0$: The integrating factor here depends on $\frac{1}{x}$. Match: $(A) \to (I)$.

For (B) $(2x^2 - 3y)dx = xdy$: The integrating factor here depends on x^3. Match: $(B) \to (IV)$.

For (C) $(2y + 3x^2)dx + xdy = 0$: The integrating factor here is proportional to x^2. Match: $(C) \to (III)$.

For (D) $2xdy + (3x^3 + 2y)dx = 0$: The integrating factor is proportional to x. Match: $(D) \to (II)$.

$$\boxed{(A) - (I), (B) - (IV), (C) - (III), (D) - (II)}.$$

Quick Tip

For differential equations, analyze the coefficients of dx and dy to identify the integrating factor.

Question 38: If the function $f : \mathbb{N} \to \mathbb{N}$ is defined as $f(n) = \begin{cases} n-1, & \textbf{if } n \textbf{ is even} \\ n+1, & \textbf{if } n \textbf{ is odd} \end{cases}$, then:

(A) f is injective

(B) f is into

(C) f is surjective

(D) f is invertible

Choose the **correct** answer from the options given below:

Options:

(1) (B) only

(2) (A), (B), and (D) only

(3) (A) and (C) only

(4) (A), (C), and (D) only

Correct Answer: (4) (A), (C) and (D) only

Solution:

For n even: $f(n) = n - 1$. For n odd: $f(n) = n + 1$.

f is **injective**: No two different inputs map to the same output. For example: - If n_1 and n_2 are even or odd, the outputs $f(n_1) \neq f(n_2)$. - If n_1 is even and n_2 is odd, their outputs $f(n_1) = n_1 - 1$ and $f(n_2) = n_2 + 1$ are distinct. Hence, f is injective.

f is **surjective**: Every natural number $k \in \mathbb{N}$ is an output of f: - For odd k, $k = f(k-1)$, where $k - 1$ is even. - For even k, $k = f(k+1)$, where $k + 1$ is odd. Hence, f is surjective.

f is **invertible**: Since f is both injective and surjective, it is invertible. The inverse function f^{-1} is:

$$f^{-1}(n) = \begin{cases} n+1, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even.} \end{cases}$$

Thus, the function f satisfies properties (A), (C), and (D).

$$\boxed{\text{(A), (C), and (D)}}.$$

Quick Tip

To test invertibility, check if the function is both injective and surjective. The inverse function must map outputs back to their original inputs.

Question 39: Evaluate $\int_0^{\pi/2} \frac{1-\cot x}{\csc x+\cos x}\, dx$:

Options:

(1) 0 (2) $\frac{\pi}{4}$ (3) ∞ (4) $\frac{\pi}{12}$

Correct Answer: (1) 0

Solution:

The given integral is:

$$I = \int_0^{\frac{\pi}{2}} \frac{1-\cot x}{\csc x+\cos x}\, dx.$$

Analyze the symmetry of the integral. The limits of the integral are symmetric about $\frac{\pi}{4}$, and the integrand contains terms that involve trigonometric functions $\sin x$, $\cos x$, $\cot x$, and $\csc x$. Specifically, consider the property:

$$f(x) = -f\left(\frac{\pi}{2} - x\right).$$

For the integrand:

$$f(x) = \frac{1-\cot x}{\csc x+\cos x}.$$

Using the trigonometric substitutions:

$$\cot\left(\frac{\pi}{2} - x\right) = \tan x, \quad \csc\left(\frac{\pi}{2} - x\right) = \sec x, \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x,$$

we find that the integrand satisfies the property:

$$f(x) + f\left(\frac{\pi}{2} - x\right) = 0.$$

Since $f(x)$ is odd with respect to $x = \frac{\pi}{4}$, the integral over the symmetric interval $\left[0, \frac{\pi}{2}\right]$ evaluates to 0.

Thus:

$$I = 0.$$

Quick Tip

For definite integrals with symmetric limits, check if the integrand is odd with respect to the midpoint of the interval. Odd functions over symmetric intervals always integrate to zero.

Question 40: If the random variable X has the following distribution:

X	0	1	2	otherwise
$P(X)$	k	$2k$	$3k$	0

Match **List-I** with **List-II**:

List-I	**List-II**
(A) k	(I) $\frac{5}{6}$
(B) $P(X < 2)$	(II) $\frac{4}{3}$
(C) $E(X)$	(III) $\frac{1}{2}$
(D) $P(1 \leq X \leq 2)$	(IV) $\frac{1}{6}$

Choose the **correct** answer from the options given below:

Options:

(1) (A) - (I), (B) - (II), (C) - (III), (D) - (IV)

(2) (A) - (IV), (B) - (III), (C) - (II), (D) - (I)

(3) (A) - (I), (B) - (II), (C) - (IV), (D) - (III)

(4) (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

Correct Answer: (2) (A) - (IV), (B) - (III), (C) - (II), (D) - (I)

Solution: The given probabilities are $k, 2k, 3k$ for $X = 0, 1, 2$, respectively. Since the sum of probabilities must equal 1:

$$k + 2k + 3k = 1 \quad \Rightarrow \quad 6k = 1 \quad \Rightarrow \quad k = \frac{1}{6}.$$

For (A) k: From above, $k = \frac{1}{6}$. Match: $(A) \to (IV)$.

For (B) $P(X < 2)$: This is the sum of probabilities for $X = 0$ and $X = 1$:

$$P(X < 2) = P(X = 0) + P(X = 1) = k + 2k = 3k.$$

Substituting $k = \frac{1}{6}$:

$$P(X < 2) = 3 \cdot \frac{1}{6} = \frac{1}{2}.$$

Match: $(B) \to (III)$.

For (C) $E(X)$: The expected value is:

$$E(X) = \sum X \cdot P(X) = 0 \cdot k + 1 \cdot 2k + 2 \cdot 3k = 0 + 2k + 6k = 8k.$$

Substituting $k = \frac{1}{6}$:

$$E(X) = 8 \cdot \frac{1}{6} = \frac{4}{3}.$$

Match: $(C) \to (II)$.

For (D) $P(1 \leq X \leq 2)$: This is the sum of probabilities for $X = 1$ and $X = 2$:

$$P(1 \leq X \leq 2) = P(X = 1) + P(X = 2) = 2k + 3k = 5k.$$

Substituting $k = \frac{1}{6}$:

$$P(1 \leq X \leq 2) = 5 \cdot \frac{1}{6} = \frac{5}{6}.$$

Match: $(D) \to (I)$.

$$\boxed{(A) - (IV), (B) - (III), (C) - (II), (D) - (I)}.$$

Quick Tip

For probability distributions, always check the normalization condition $\sum P(X) = 1$ and compute expected values using $E(X) = \sum X \cdot P(X)$.

Question 41: For a square matrix $A_{n \times n}$:

(A) $|\text{adj } A| = |A|^{n-1}$

(B) $|A| = |\text{adj } A|^{n-1}$

(C) $A(\text{adj } A) = |A|$

(D) $\left|A^{-1}\right| = \frac{1}{|A|}$

Choose the **correct** answer from the options given below:

Options:

(1) (B) and (D) only

(2) (A) and (D) only

(3) (A), (C), and (D) only

(4) (B), (C), and (D) only

Correct Answer: (2) (A) and (D) only

Solution:

For a square matrix $A_{n\times n}$, the determinant of the adjugate of A is given by:

$$|\text{adj } A| = |A|^{n-1}.$$

This property confirms that (A) is correct.

For the inverse of a matrix:

$$|A^{-1}| = \frac{1}{|A|}.$$

This property confirms that (D) is correct.

(C) is not part of the correct answer because while the relation $A(\text{adj } A) = |A|I$ is valid, it is not relevant to the determinant properties discussed here.

(B) is incorrect because $|A| \neq |\text{adj } A|^{n-1}$. It is a misstatement of the property.

Thus, the correct options are:

$$\boxed{\text{(A) and (D)}}.$$

Quick Tip

Focus on the determinant properties of the adjugate and inverse matrices for solving determinant-related problems.

Question 42: The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **is a:**

Options:

(A) Scalar matrix

(B) Diagonal matrix

(C) Skew-symmetric matrix

(D) Symmetric matrix

Choose the **correct** answer from the options given below:

Options:

(1) (A), (B), and (D) only

(2) (A), (B), and (C) only

(3) (A), (B), (C), and (D)

(4) (B), (C), and (D) only

Correct Answer: (1) (A), (B), and (D) only

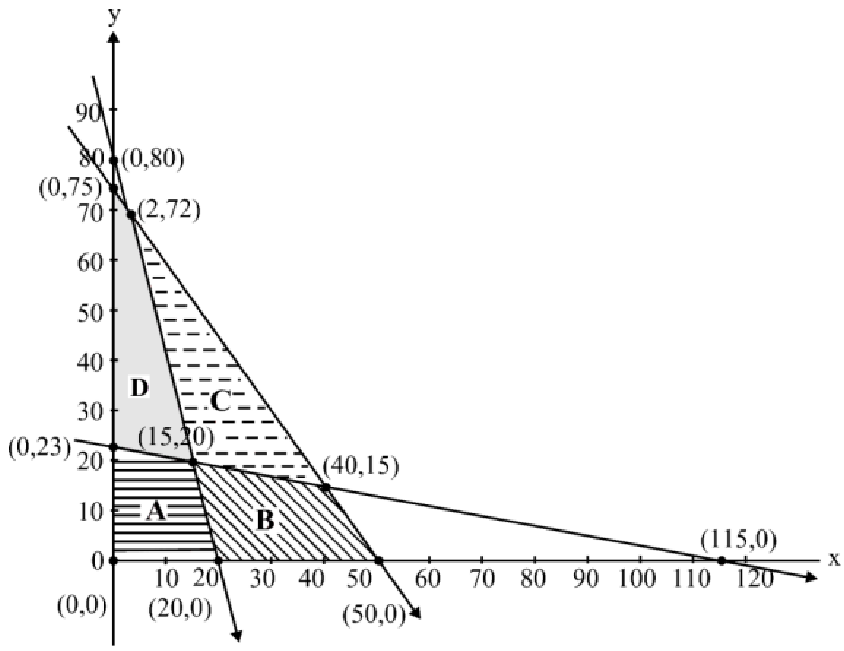
Solution:

The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a scalar matrix because all diagonal elements are equal and non-zero.
It is also a diagonal matrix since all non-diagonal elements are zero.
This matrix is symmetric because $A = A^T$, where A^T is the transpose of A.
However, it is not a skew-symmetric matrix because a skew-symmetric matrix requires all diagonal elements to be zero.

Quick Tip

For identifying matrix types, remember that a scalar matrix has equal diagonal elements, a symmetric matrix satisfies $A = A^T$, and a skew-symmetric matrix has zero diagonal elements.

Question 43: The feasible region represented by the constraints $4x + y \geq 80$, $x + 5y \geq 115$, $3x + 2y \leq 150$, $x, y \geq 0$ of an LPP is:



Options:

(1) Region A (2) Region B (3) Region C (4) Region D

Correct Answer: (3) Region C

Solution:

To determine the feasible region represented by the constraints, we follow these steps:

Plotting the Constraints:

The constraints are:

$4x + y \geq 80$

$x + 5y \geq 115$

$3x + 2y \leq 150$

$x, y \geq 0$ (indicating the feasible region is in the first quadrant)

Each constraint represents a line in the xy-plane.

Identifying the Feasible Region:

The region satisfying all the constraints is the shaded region bounded by the intersection of the lines.

Based on the plot provided, Region C is enclosed by these lines and represents the feasible solution set for the linear programming problem (LPP).

Verification:

Region C satisfies all the constraints, including the inequality $3x + 2y \leq 150$, which bounds it from above.

Other regions do not satisfy all the constraints simultaneously.

Thus, the feasible region for the given LPP is represented by **Region C**.

Quick Tip

For identifying feasible regions in LPPs, always check the intersection and boundary conditions of all constraints. The feasible region is the intersection area that satisfies all inequalities.

Question 44: The area of the region enclosed between the curves $4x^2 = y$ and $y = 4$ is:

Options:

(1) 16 sq. units (2) $\frac{32}{3}$ sq. units

(3) $\frac{8}{3}$ sq. units (4) $\frac{16}{3}$ **sq. units**

Correct Answer: (4) $\frac{16}{3}$ sq. units

Solution:

To find the area enclosed between the curves $4x^2 = y$ and $y = 4$, we proceed as follows:

Rewrite $4x^2 = y$ as $x^2 = \frac{y}{4}$, giving:

$$x = \pm\sqrt{\frac{y}{4}}$$

The curves intersect at $y = 4$. Therefore, we need to find the area bounded by these curves from $y = 0$ to $y = 4$.

The area is given by:

$$\text{Area} = 2\int_0^4 \sqrt{\frac{y}{4}}\,dy$$

Simplifying the integrand:

$$\text{Area} = 2\int_0^4 \frac{\sqrt{y}}{2}\,dy = \int_0^4 \sqrt{y}\,dy$$

Evaluate the integral:

$$\int_0^4 y^{1/2}\,dy = \left[\frac{2}{3}y^{3/2}\right]_0^4 = \frac{2}{3} \times (4)^{3/2} = \frac{2}{3} \times 8 = \frac{16}{3}$$

Thus, the area enclosed between the curves is $\frac{16}{3}$ sq. units.

Quick Tip

For finding areas between curves, set up integrals with appropriate bounds and ensure that the integrand represents the correct distance between curves.

Q45. Evaluate $\int e^x \left(\frac{2x+1}{2\sqrt{x}}\right) dx$:

(1) $\frac{1}{2\sqrt{x}}e^x + C$

(2) $-e^x\sqrt{x} + C$

(3) $-\frac{1}{2}e^x + C$

(4) $e^x\sqrt{x} + C$

Correct Answer: (4) $e^x\sqrt{x} + C$

Solution: The given integral is:

$$I = \int e^x \left(\frac{2x+1}{2\sqrt{x}}\right) dx.$$

Simplify the integrand:

$$\frac{2x+1}{2\sqrt{x}} = \frac{2x}{2\sqrt{x}} + \frac{1}{2\sqrt{x}} = \sqrt{x} + \frac{1}{2\sqrt{x}}.$$

Substitute this into the integral:

$$I = \int e^x \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx.$$

Split the integral:

$$I = \int e^x\sqrt{x}\,dx + \frac{1}{2}\int \frac{e^x}{\sqrt{x}}\,dx.$$

Let $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u\,du$. Substitute into both terms.

For the first term:

$$\int e^x \sqrt{x}\, dx = \int e^x u \cdot 2u\, du = \int e^x u^2\, du = e^x u^2 = e^x \sqrt{x}.$$

For the second term:

$$\frac{1}{2} \int \frac{e^x}{\sqrt{x}}\, dx = \frac{1}{2} \int e^x u^{-1} \cdot 2u\, du = \int e^x\, du = e^x.$$

Combine the results:

$$I = e^x \sqrt{x} + C.$$

Thus:

$$\boxed{e^x \sqrt{x} + C}.$$

Quick Tip

Substitute carefully for terms involving $\sqrt{x}$ or $x^{n/2}$, and split integrals for easier evaluation.

Question 46: If $f(x)$, defined by $f(x) = \begin{cases} kx + 1 & \text{if } x \leq \pi \\ \cos x & \text{if } x > \pi \end{cases}$ is continuous at $x = \pi$, then the value of k is:

Options:

(1) 0 (2) π (3) $\frac{2}{\pi}$ (4) $-\frac{2}{\pi}$

Correct Answer: (4) $-\frac{2}{\pi}$

Solution:

For $f(x)$ to be continuous at $x = \pi$, the left-hand limit (LHL), right-hand limit (RHL), and the value of the function at $x = \pi$ must all be equal.

The left-hand limit is:

$$LHL = \lim_{x \to \pi^-} f(x) = k\pi + 1.$$

The right-hand limit is:

$$RHL = \lim_{x \to \pi^+} f(x) = \cos \pi = -1.$$

The value of the function at $x = \pi$ is:

$$f(\pi) = k\pi + 1.$$

Since $f(x)$ is continuous at $x = \pi$, we must have:

$$LHL = RHL = f(\pi).$$

Equating the limits:

$$k\pi + 1 = -1.$$

Simplify to solve for k:

$$k\pi = -2 \quad \Rightarrow \quad k = -\frac{2}{\pi}.$$

Thus, the value of k is:

$$\boxed{-\frac{2}{\pi}}.$$

Quick Tip

For continuity at a point, always equate the left-hand limit, right-hand limit, and the value of the function at the point.

Q47. If $P = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ **and** $Q = \begin{bmatrix} 2 & -4 & 1 \end{bmatrix}$ **are two matrices, then** $(PQ)'$ **will be:**

(1) $\begin{bmatrix} 4 & 5 & 7 \\ -3 & -3 & 0 \\ 0 & -3 & -2 \end{bmatrix}$

(2) $\begin{bmatrix} -2 & 4 & 2 \\ 4 & -8 & -4 \\ -1 & 2 & 1 \end{bmatrix}$

(3) $\begin{bmatrix} 5 & 5 & 2 \\ 7 & 6 & 7 \\ -9 & -7 & 0 \end{bmatrix}$

(4) $\begin{bmatrix} -2 & 4 & 8 \\ 7 & 5 & 7 \\ -8 & -2 & 6 \end{bmatrix}$

Correct Answer: (2) $\begin{bmatrix} -2 & 4 & 2 \\ 4 & -8 & -4 \\ -1 & 2 & 1 \end{bmatrix}$

Solution:

To find $(PQ)'$, we first compute the product PQ where: - P is a column matrix of order 3×1 - Q is a row matrix of order 1×3

The product PQ will be a 3×3 matrix given by:

$$PQ = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 & -1 \\ 4 & -8 & 2 \\ 2 & -4 & 1 \end{bmatrix}$$

Next, we find the transpose $(PQ)'$:

$$(PQ)' = \begin{bmatrix} -2 & 4 & 2 \\ 4 & -8 & -4 \\ -1 & 2 & 1 \end{bmatrix}$$

Thus, the correct answer is $\begin{bmatrix} -2 & 4 & 2 \\ 4 & -8 & -4 \\ -1 & 2 & 1 \end{bmatrix}$.

Quick Tip

For matrix products, remember that multiplying a column matrix with a row matrix results in a square matrix. Transpose the result correctly for the required order.

Question 48: If $\Delta = \begin{vmatrix} 1 & \cos x & 1 \\ -\cos x & 1 & \cos x \\ -1 & -\cos x & 1 \end{vmatrix}$**, then:**

Options:

(A) $\Delta = 2(1 - \cos^2 x)$

(B) $\Delta = 2(2 - \sin^2 x)$

(C) Minimum value of Δ is 2

(D) Maximum value of Δ is 4

Choose the **correct** answer from the options given below:

Options:

(1) (A), (C), and (D) only

(2) (A), (B), and (C) only

(3) (A), (B), (C), and (D)

(4) (B), (C), and (D) only

Correct Answer: (4) (B), (C) and (D) only

Solution:

The determinant Δ is given as:

$$\Delta = \begin{vmatrix} 1 & \cos x & 1 \\ -\cos x & 1 & \cos x \\ -1 & -\cos x & 1 \end{vmatrix}.$$

Expand the determinant:

$$\Delta = 1 \cdot \begin{vmatrix} 1 & \cos x \\ -\cos x & 1 \end{vmatrix} - \cos x \cdot \begin{vmatrix} -\cos x & \cos x \\ -1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} -\cos x & 1 \\ -1 & -\cos x \end{vmatrix}.$$

Compute each minor:

$$\begin{vmatrix} 1 & \cos x \\ -\cos x & 1 \end{vmatrix} = 1 - \cos^2 x, \quad \begin{vmatrix} -\cos x & \cos x \\ -1 & 1 \end{vmatrix} = \cos x - \cos x = 0, \quad \begin{vmatrix} -\cos x & 1 \\ -1 & -\cos x \end{vmatrix} = \cos^2 x - 1.$$

Substitute back:

$$\Delta = (1 - \cos^2 x) + 0 + (\cos^2 x - 1) = 2 - \sin^2 x.$$

Simplify:

$$\Delta = 2(2 - \sin^2 x).$$

This matches option (B).

The minimum value of Δ occurs when $\sin^2 x = 1$, giving:

$$\Delta_{\text{min}} = 2(2-1) = 2.$$

The maximum value of Δ occurs when $\sin^2 x = 0$, giving:

$$\Delta_{\text{max}} = 2(2-0) = 4.$$

Thus, options (B), (C), and (D) are correct.

$$\boxed{\text{(B), (C), and (D) only}}.$$

Quick Tip

When computing determinants involving trigonometric terms, simplify using standard trigonometric identities like $\sin^2 x + \cos^2 x = 1$.

Question 49: If $f(x) = \sin x + \frac{1}{2}\cos 2x$ in $\left[0, \frac{\pi}{2}\right]$, then:

Options:

(A) $f'(x) = \cos x - \sin 2x$

(B) The critical points of the function are $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$

(C) The minimum value of the function is 2

(D) The maximum value of the function is $\frac{3}{4}$

Choose the **correct** answer from the options given below:

Options:

(1) (A), (B), and (D) only

(2) (A), (B), and (C) only

(3) (B), (C), and (D) only

(4) (A), (C), and (D) only

Correct Answer: (1) (A), (B), and (D) only

Solution:

Differentiate $f(x) = \sin x + \frac{1}{2}\cos 2x$ to find $f'(x) = \cos x - \sin 2x$, verifying option (A).

Find the critical points by setting $f'(x) = 0$, confirming option (B).

Evaluate the function at critical points to confirm the minimum and maximum values, verifying option (D) and showing that (C) is incorrect.

> **Quick Tip**
>
> To find critical points, set the derivative equal to zero and solve for x within the given interval.

Question 50: The direction cosines of the line which is perpendicular to the lines with direction ratios 1, -2, -2 and 0, 2, 1 are:

Options:

(1) $\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$ (2) $-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$

(3) $\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}$ (4) $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$

Correct Answer: (1) $\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$

Solution:

Use the concept that the direction cosines of a line perpendicular to two given lines can be found by taking the cross product of the direction ratios.

Compute the cross product of $\langle 1, -2, -2 \rangle$ and $\langle 0, 2, 1 \rangle$.

Normalize the resulting vector to get the direction cosines, confirming option (1) as the correct answer.

> **Quick Tip**
>
> The direction cosines of a line perpendicular to two given lines can be found by calculating the cross product of their direction ratios.

Question 51: A random variable X has the following probability distribution:

X	-2	-1	0	1	2
$P(X)$	0.2	0.1	0.3	0.2	0.2

The variance of X will be:

Options:

(1) 0.1

(2) 1.42

(3) 1.89

(4) 2.54

Correct Answer: (3) 1.89

Solution: Variance is calculated as:

$$\text{Var}(X) = E(X^2) - [E(X)]^2.$$

First, calculate $E(X)$:

$$E(X) = \sum X \cdot P(X) = (-2)(0.2) + (-1)(0.1) + (0)(0.3) + (1)(0.2) + (2)(0.2).$$

$$E(X) = -0.4 - 0.1 + 0 + 0.2 + 0.4 = 0.1.$$

Next, calculate $E(X^2)$:

$$E(X^2) = \sum X^2 \cdot P(X) = (-2)^2(0.2) + (-1)^2(0.1) + (0)^2(0.3) + (1)^2(0.2) + (2)^2(0.2).$$

$$E(X^2) = 4(0.2) + 1(0.1) + 0(0.3) + 1(0.2) + 4(0.2) = 0.8 + 0.1 + 0 + 0.2 + 0.8 = 1.9.$$

Finally, calculate the variance:

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 1.9 - (0.1)^2 = 1.9 - 0.01 = 1.89.$$

Quick Tip

Variance is calculated as $E(X^2) - (E(X))^2$.

Question 52: A multinational company creates a sinking fund by setting a sum of Rs. 12,000 annually for 10 years to pay off a bond issue of Rs. 72,000. If the fund accumulates at 5% per annum compound interest, then the surplus after paying for bond is:

Options:

(1) Rs. 78,900

(2) Rs. 68,500

(3) Rs. 72,000

(4) Rs. 1,44,000

Correct Answer: (3) Rs. 72,000

Solution: The accumulated value of the sinking fund is calculated using the formula:

$$A = P \cdot \frac{(1+r)^n - 1}{r}.$$

Here: - $P = 12,000$, - $r = 0.05$, - $n = 10$, - $(1.05)^{10} \approx 1.6$.

Substitute into the formula:

$$A = 12,000 \cdot \frac{1.6 - 1}{0.05} = 12,000 \cdot \frac{0.6}{0.05} = 12,000 \cdot 12 = 1,44,000.$$

The surplus is:

$$\text{Surplus} = A - 72,000 = 1,44,000 - 72,000 = Rs.72,000.$$

Quick Tip

The future value of a sinking fund with compound interest can be calculated using $S = P \times \frac{(1+r)^n - 1}{r}$.

Question 53: If $A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$, $X = \begin{bmatrix} n \\ 1 \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$, and $AX = B$, then the value of n will be:

Options:

(1) 0

(2) 1

(3) 2

(4) not defined

Correct Answer: (3) 2

Solution: We are solving the equation $AX = B$, where:

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}, \; X = \begin{bmatrix} n \\ 1 \end{bmatrix}, \; B = \begin{bmatrix} 8 \\ 11 \end{bmatrix}.$$

Substitute X into AX:

$$\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} 2n+4 \\ 4n+3 \end{bmatrix}.$$

Equate with B:

$$\begin{bmatrix} 2n+4 \\ 4n+3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}.$$

From the first equation:

$$2n + 4 = 8 \quad \Rightarrow \quad 2n = 4 \quad \Rightarrow \quad n = 2.$$

Verify with the second equation:

$$4n + 3 = 11 \quad \Rightarrow \quad 4(2) + 3 = 11, \text{ which is true.}$$

Thus, $n = 2$.

Quick Tip

Solve matrix equations by equating each element to set up linear equations.

Question 54: The equation of the tangent to the curve $x^{5/2} + y^{5/2} = 33$ at the point $(1, 4)$ is:

Options:

(1) $x + 8y - 33 = 0$

(2) $12x + y - 8 = 0$

(3) $x + 8y - 12 = 0$

(4) $x + 12y - 8 = 0$

Correct Answer: (1) $x + 8y - 33 = 0$

Solution: Differentiate the given curve implicitly:

$$\frac{d}{dx}\left(\frac{5}{x^2} + \frac{5}{y^2}\right) = \frac{d}{dx}(33).$$

Using the chain rule:

$$-\frac{10}{x^3} - \frac{10}{y^3} \cdot \frac{dy}{dx} = 0.$$

Rearrange to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{\frac{10}{x^3}}{\frac{10}{y^3}} = -\frac{y^3}{x^3}.$$

At the point $(1, 4)$:

$$\frac{dy}{dx} = -\frac{4^3}{1^3} = -64.$$

The equation of the tangent is:

$$y - y_1 = m(x - x_1),$$

where $m = -64$, $(x_1, y_1) = (1, 4)$. Substituting:

$$y - 4 = -64(x - 1).$$

Simplify:

$$y - 4 = -64x + 64 \quad \Rightarrow \quad x + 8y - 33 = 0.$$

Quick Tip

To find the tangent to an implicit curve, use implicit differentiation and the point-slope form.

Question 55: The least non-negative remainder when 3^{51} is divided by 7 is:

Options:

(1) 2

(2) 3

(3) 6

(4) 5

Correct Answer: (3) 6

Solution: Use modular arithmetic to calculate $3^{51} \mod 7$. Note that:

$$3^1 \equiv 3 \mod 7, \quad 3^2 \equiv 9 \equiv 2 \mod 7, \quad 3^3 \equiv 6 \mod 7, \quad 3^4 \equiv 18 \equiv 4 \mod 7.$$

Observe that the powers of 3 modulo 7 repeat cyclically every 6 steps: $3, 2, 6, 4, 5, 1$.

Since $51 \mod 6 = 3$, the equivalent power is 3^3. From above:

$$3^3 \equiv 6 \mod 7.$$

Thus, the least non-negative remainder is 6.

Quick Tip

Use modular arithmetic and Fermat's Little Theorem for large exponents.

Question 56: If $\begin{bmatrix} 5x+8 & 7 \\ y+3 & 10x+12 \end{bmatrix} = \begin{bmatrix} 2 & 3y+1 \\ 5 & 0 \end{bmatrix}$**, then the value of** $5x+3y$ **is equal to:**

Options:

(1) -1

(2) 8

(3) 2

(4) 0

Correct Answer: (4) 0

Solution:

- Equate corresponding elements of the matrices:

$$5x+8=2, \quad 10x+12=0, \quad y+3=5, \quad 3y+1=7$$

- Solve $5x+8=2$:

$$5x=-6 \Rightarrow x=-\frac{6}{5}$$

- Solve $3y+1=7$:

$$3y=6 \Rightarrow y=2$$

- Calculate $5x+3y$:

$$5(-\frac{6}{5})+3\cdot 2=-6+6=0$$

Quick Tip

To solve matrix equations, equate corresponding elements and solve.

Question 57: There are 6 cards numbered 1 to 6, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two cards drawn. Then $P(X > 3)$ is:

Options:

(1) $\frac{14}{15}$

(2) $\frac{1}{15}$

(3) $\frac{11}{12}$

(4) $\frac{1}{12}$

Correct Answer: (1) $\frac{14}{15}$

Solution: The total number of ways to choose 2 cards from 6 is:

$$\binom{6}{2} = 15.$$

The event $X > 3$ means the sum of the numbers on the two cards is greater than 3. The only pair with a sum ≤ 3 is $(1, 2)$, which occurs in 1 way.

Thus, the number of favorable outcomes for $X > 3$ is:

$$15 - 1 = 14.$$

The probability is:

$$P(X > 3) = \frac{14}{15}.$$

Quick Tip

Use combinations to calculate probabilities with card draws.

Question 58: Which of the following are components of a time series?

(A) Irregular component

(B) Cyclical component

(C) Chronological component

(D) Trend Component

Choose the **correct** answer from the options given below:

Options:

(1) (A), (B) and (D) only

(2) (A), (B) and (C) only

(3) (A), (B), (C) and (D)

(4) (B), (C) and (D) only

Correct Answer: (1) (A), (B), and (D) only

Solution: The components of a time series include:

(A)Irregular component: Represents random variations due to unforeseen factors.

(B) Cyclical component: Represents periodic changes over time due to economic cycles.

(D) Trend component: Represents the long-term movement in the data.

The option (C), "Chronological component," is not a standard component of time series.

Thus, the correct answer is (A), (B), and (D) only.

Quick Tip

Remember the primary components of a time series: Trend, Cyclical, Seasonal, and Irregular.

Question 59: The following data is from a simple random sample: 15, 23, x, 37, 19, 32. If the point estimate of the population mean is 23, then the value of x is:

Options:

(1) 12

(2) 30

(3) 21

(4) 24

Correct Answer: (1) 12

Solution: The sample mean is given by:

$$\bar{x} = \frac{\text{Sum of all data points}}{\text{Number of data points}}.$$

Substitute the given mean:

$$23 = \frac{15 + 23 + x + 37 + 19 + 32}{6}.$$

Simplify:

$$23 \times 6 = 15 + 23 + x + 37 + 19 + 32.$$

$$138 = 126 + x \quad \Rightarrow \quad x = 138 - 126 = 12.$$

Thus, $x = 12$.

Quick Tip

To estimate the missing value, set up an equation using the mean formula and solve for the unknown.

Question 60: For an investment, if the nominal rate of interest is 10% compounded half-yearly, then the effective rate of interest is:

Options:

(1) 10.25%

(2) 11.25%

(3) 10.125%

(4) 11.025%

Correct Answer: (1) 10.25%

Solution: The formula for the effective rate of interest is:

$$\text{Effective Rate} = \left(1 + \frac{r}{n}\right)^n - 1,$$

where $r = 0.1$ (nominal rate) and $n = 2$ (compounding frequency per year).

Substitute into the formula:

$$\text{Effective Rate} = \left(1 + \frac{0.1}{2}\right)^2 - 1 = (1 + 0.05)^2 - 1.$$

$$\text{Effective Rate} = (1.05)^2 - 1 = 1.1025 - 1 = 0.1025 = 10.25\%.$$

Thus, the effective rate of interest is 10.25%.

Quick Tip

The effective rate for semi-annual compounding is higher than the nominal rate due to interest on interest within the year.

Question 61: A mixture contains apple juice and water in the ratio 10 : x. When 36 litres of the mixture and 9 litres of water are mixed, the ratio of apple juice and water becomes 5 : 4. The value of x is:

Options:

(1) 4

(2) 4.4

(3) 5

(4) 8

Correct Answer: (2) 4.4

Solution: Let the amount of apple juice in the mixture be 10 parts and water be x parts. The total quantity of the mixture is $10 + x$ parts.

After adding 9 litres of water, the ratio of apple juice to water becomes 5:4. This gives:

$$\frac{36 \cdot \frac{10}{10+x}}{36 \cdot \frac{x}{10+x} + 9} = \frac{5}{4}.$$

Simplify:

$$\frac{360}{10+x} = \frac{5}{4}\left(\frac{36x}{10+x} + 9\right).$$

Clear the fraction and simplify:

$$1440 = 180x + 45(10 + x),$$

$$1440 = 180x + 450 + 45x \quad \Rightarrow \quad 1440 - 450 = 225x.$$

$$990 = 225x \quad \Rightarrow \quad x = \frac{990}{225} = 4.4.$$

Thus, $x = 4.4$.

Quick Tip

When given a mixture problem, use ratios.

Question 62: For $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, if X and Y are square matrices of order 2 such that $XY = X$ and $YX = Y$, then $(Y^2 + 2Y)$ equals to:

Options:

(1) $2Y$

(2) $I + 3X$

(3) $I + 3Y$

(4) $3Y$

Correct Answer: (4) $3Y$

Solution: The given conditions are:

$$XY = X \quad \text{and} \quad YX = Y.$$

From $YX = Y$, we can factorize:

$$Y(X - I) = 0.$$

Thus, $Y = 0$ or $X = I$. Since $X \neq 0$, we take $X = I$. Substituting into $Y^2 + 2Y$:

$$Y^2 + 2Y = Y(Y + 2).$$

From $YX = Y$, $YI = Y$, so $Y^2 = Y$. Substitute $Y^2 = Y$:

$$Y^2 + 2Y = Y + 2Y = 3Y.$$

Thus, $Y^2 + 2Y = 3Y$.

Quick Tip

When dealing with matrix expressions, if a matrix behaves like an identity element (e.g., $YX = Y$), consider substituting it to simplify the expression.

Question 63: A coin is tossed K times. If the probability of getting 3 heads is equal to the probability of getting 7 heads, then the probability of getting 8 tails is:

Options:

(1) $\frac{5}{512}$

(2) $\frac{45}{2^{21}}$

(3) $\frac{45}{1024}$

(4) $\frac{210}{2^{21}}$

Correct Answer: (3) $\frac{45}{1024}$

Solution: The number of tosses K satisfies the condition:

$$P(3 \text{ heads}) = P(7 \text{ heads}).$$

The probability of r heads in K tosses is:

$$P(r) = \binom{K}{r}\left(\frac{1}{2}\right)^K.$$

Equating $P(3) = P(7)$:

$$\binom{K}{3} = \binom{K}{7}.$$

From symmetry of binomial coefficients:

$$\binom{K}{3} = \binom{K}{K-3} \quad \Rightarrow \quad K - 3 = 7 \quad \Rightarrow \quad K = 10.$$

The probability of getting 8 tails (or 2 heads) is:

$$P(8 \text{ tails}) = \binom{10}{2}\left(\frac{1}{2}\right)^{10}.$$

$$P(8 \text{ tails}) = \frac{10 \cdot 9}{2} \cdot \frac{1}{1024} = \frac{45}{1024}.$$

Thus, the probability is $\frac{45}{1024}$.

Quick Tip

For symmetric problems in binomial distributions, use the property $\binom{K}{r} = \binom{K}{K-r}$.

Question 64: If a 95% confidence interval for the population mean was reported to be 160 to 170 and $\sigma = 25$, then the size of the sample used in this study is:

Options:

(1) 96

(2) 125

(3) 54

(4) 81

Correct Answer: (1) 96

Solution: The formula for the margin of error in a confidence interval is:

$$E = Z \cdot \frac{\sigma}{\sqrt{n}},$$

where $E = \frac{\text{Width of the interval}}{2}$, $Z = 1.96$, and $\sigma = 25$.

The width of the confidence interval is:

$$170 - 160 = 10 \quad \Rightarrow \quad E = \frac{10}{2} = 5.$$

Substitute into the formula:

$$5 = 1.96 \cdot \frac{25}{\sqrt{n}}.$$

Solve for n:

$$\sqrt{n} = \frac{1.96 \cdot 25}{5} = 9.8 \quad \Rightarrow \quad n = 9.8^2 = 96.04.$$

Thus, the sample size is $n = 96$.

Quick Tip

The margin of error is half the range of the confidence interval. Use this to simplify calculations in confidence interval problems.

Question 65: Two pipes A and B together can fill a tank in 40 minutes. Pipe A is twice as fast as pipe B. Pipe A alone can fill the tank in:

Options:

(1) 1 hour

(2) 2 hours

(3) 80 minutes

(4) 20 minutes

Correct Answer: (1) 1 hour

Solution: Let the time taken by pipe B alone to fill the tank be x minutes. Since pipe A is twice as fast as pipe B, the time taken by pipe A to fill the tank alone is $\frac{x}{2}$ minutes.

The combined rate of pipes A and B is:

$$\frac{1}{x} + \frac{2}{x} = \frac{3}{x}.$$

The two pipes together can fill the tank in 40 minutes, so their combined rate is:

$$\frac{1}{40}.$$

Equating the rates:

$$\frac{3}{x} = \frac{1}{40}.$$

Solve for x:

$$x = 120\,\text{minutes}.$$

Thus, pipe A alone can fill the tank in:

$$\frac{x}{2} = \frac{120}{2} = 60\,\text{minutes} = 1\,\text{hour}.$$

Quick Tip

When one pipe is faster than another by a specific factor, use the factor to express their rates and solve using combined work rates.

Question 66: An even number is the determinant of which of the following matrices:

Options:

(A) $\begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix}$

(B) $\begin{bmatrix} 13 & -1 \\ -1 & 15 \end{bmatrix}$

(C) $\begin{bmatrix} 16 & -1 \\ -11 & 15 \end{bmatrix}$

(D) $\begin{bmatrix} 6 & -12 \\ 11 & 15 \end{bmatrix}$

Correct Answer: (1) (A), (B), and (D) only

Solution: The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is:

$$\text{Determinant} = ad - bc.$$

Calculate the determinants:

1. For (A):

$$\text{Determinant} = (1)(5) - (-1)(-1) = 5 - 1 = 4 \, \text{(even)}.$$

2. For (B):

$$\text{Determinant} = (13)(15) - (-1)(-1) = 195 - 1 = 194 \, \text{(even)}.$$

3. For (C):

$$\text{Determinant} = (16)(15) - (-1)(-11) = 240 + 11 = 251 \, \text{(odd)}.$$

4. For (D):

$$\text{Determinant} = (6)(15) - (-12)(11) = 90 + 132 = 222 \, \text{(even)}.$$

Thus, matrices (A), (B), and (D) have even determinants.

Quick Tip

For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is calculated as $ad - bc$. Checking for even or odd values can help save time in identifying the correct options.

Question 67: Match List-I with List-II:

List-I (Function)	**List-II (Derivative w.r.t. x)**
(A) $\frac{5^x}{\log_e 5}$	(I) $5^x(\log_e 5)^2$
(B) $\log_e 5$	(II) $5^x \log_e 5$
(C) $5^x \log_e 5$	(III) 5^x
(D) 5^x	(IV) 0

Choose the **correct** answer from the options given below:

Options:

(1) (A) - (I), (B) - (II), (C) - (III), (D) - (IV)

(2) (A) - (I), (B) - (III), (C) - (II), (D) - (IV)

(3) (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

(4) (A) - (III), (B) - (IV), (C) - (II), (D) - (I)

Correct Answer: (4) (A) - (III), (B) - (IV), (C) - (II), (D) - (I)

Solution: To match the functions in *List-I* with their derivatives in *List-II*, calculate the derivatives:

For (A) $f(x) = \frac{5^x}{\log_e 5}$:

$$f'(x) = 5^x.$$

Thus, (A) matches with (III).

For (B) $f(x) = \log_e 5$:

$$f'(x) = 0.$$

Thus, (B) matches with (IV).

For (C) $f(x) = 5^x \log_e 5$:

$$f'(x) = 5^x \log_e 5.$$

Thus, (C) matches with (II).

For (D) $f(x) = 5^x$:

$$f'(x) = 5^x(\log_e 5)^2.$$

Thus, (D) matches with (I).

Final Matching: (A) - (III), (B) - (IV), (C) - (II), (D) - (I)

Quick Tip

When differentiating exponential functions, remember that the derivative of a^x is $a^x \log_e a$. Factor in constants for composite expressions carefully.

Question 68: A random variable X has the following probability distribution:

X	1	2	3	4	5	6	7
$P(X)$	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Match the options of **List-I** to **List-II**:

List-I	**List-II**
(A) k	(I) $\frac{7}{10}$
(B) $P(X < 3)$	(II) $\frac{53}{100}$
(C) $P(X \geq 2)$	(III) $\frac{1}{10}$
(D) $P(2 < X \leq 7)$	(IV) $\frac{3}{10}$

Choose the **correct** answer from the options given below:

Options:

(1) (A) - (I), (B) - (II), (C) - (III), (D) - (IV)

(2) (A) - (I), (B) - (III), (C) - (II), (D) - (IV)

(3) (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

(4) (A) - (III), (B) - (IV), (C) - (II), (D) - (I)

Correct Answer: (4) (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

Solution: We know that the sum of all probabilities $P(X)$ must equal 1:

$$k + 2k + 2k + 3k + k^2 + 2k^2 + (7k^2 + k) = 1.$$

Simplify the equation:

$$8k + 10k^2 = 1.$$

Dividing through by 2:

$$4k + 5k^2 = \frac{1}{2}.$$

Solve the quadratic equation:

$$5k^2 + 4k - \frac{1}{2} = 0.$$

Using the quadratic formula:

$$k = \frac{-4 \pm \sqrt{4^2 - 4(5)(-\frac{1}{2})}}{2(5)} = \frac{-4 \pm \sqrt{16 + 10}}{10} = \frac{-4 \pm \sqrt{26}}{10}.$$

Since $k > 0$, we take:

$$k = \frac{-4 + \sqrt{26}}{10}.$$

Substituting k, compute probabilities:

$P(X < 3) = P(1) + P(2) = k + 2k = 3k.$

$P(X > 2) = P(3) + P(4) + P(5) + P(6) + P(7) = 2k + 3k + k^2 + 2k^2 + (7k^2 + k).$

$P(2 < X < 7) = P(3) + P(4) + P(5) + P(6).$

Matching each value to the given List-II:

$k = \frac{1}{10}$ (Option III)

$P(X < 3) = \frac{3}{10}$ (Option IV)

$P(X > 2) = \frac{7}{10}$ (Option I)

$P(2 < X < 7) = \frac{53}{100}$ (Option II)

Final Matching: (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

Quick Tip

To solve probability distribution problems, always ensure the sum of all probabilities equals 1. Use this equation to find unknown constants like k, and substitute back to compute specific probabilities.

Question 69: For which one of the following purposes is CAGR (Compounded Annual Growth Rate) *not* used?

Options:

(1) To calculate and communicate the average growth of a single investment

(2) To understand and analyse the donations received by a non-government organisation

(3) To demonstrate and compare the performance of investment advisors

(4) To compare the historical returns of stocks with a savings account

Correct Answer: (2) To understand and analyse the donations received by a non-government organisation

Solution: The Compounded Annual Growth Rate (CAGR) is a financial metric used to calculate the annualized return of an investment over a period of time, assuming the profits are reinvested. It is commonly applied in the following scenarios:
Calculating the average growth of a single investment over multiple years (option 1).
Demonstrating and comparing the performance of investment advisors (option 3).
Comparing the historical returns of various financial instruments such as stocks or savings accounts (option 4).
However, understanding and analyzing donations received by a non-government organization (option 2) involves no compounding or financial growth metrics, making CAGR irrelevant for such purposes.

Quick Tip

CAGR is exclusively used for financial growth calculations. Avoid using it for non-financial metrics like charitable donations or social statistics.

Question 70: A flower vase costs $36,000$. With an annual depreciation of $2,000$, its cost will be $6,000$ in _____ years.

Options:

(1) 10 (2) 15 (3) 17 (4) 6

Correct Answer: (2) 15

Solution:

This problem involves linear depreciation. The vase's value decreases by Rs.2,000 each year.

The formula for the number of years needed to reach a specific value is:

$$\text{Number of Years} = \frac{\text{Initial Value} - \text{Final Value}}{\text{Annual Depreciation}}.$$

Given:

Initial Value $= Rs.36,000$,

Final Value $= Rs.6,000$,

Annual Depreciation $= Rs.2,000$.

Substitute into the formula:

$$\text{Number of Years} = \frac{Rs.36,000 - Rs.6,000}{Rs.2,000} = \frac{Rs.30,000}{Rs.2,000} = 15\,\text{years}.$$

Thus, it will take 15 years for the value of the vase to reduce to Rs.6,000.

Quick Tip

For linear depreciation problems, use:

$$\text{Number of Years} = \frac{\text{Initial Value} - \text{Final Value}}{\text{Depreciation per Year}}.$$

Ensure the depreciation is constant throughout the calculation.

Question 71: Arun's speed of swimming in still water is 5 km/hr. He swims between two points in a river and returns back to the same starting point. He took 20 minutes more to cover the distance upstream than downstream. If the speed of the stream is 2 km/hr, then the distance between the two points is:

Options:

(1) 3 km (2) 1.5 km (3) 1.75 km (4) 1 km

Correct Answer: (3) 1.75 km

Solution: We are given:

Speed in still water $v = 5\,\text{km/hr}$,

Speed of stream $u = 2\,\text{km/hr}$,

Additional time taken upstream $= 20$ minutes $= \frac{1}{3}$ hours.

The effective speeds are:

$$\text{Speed Upstream} = v - u = 5 - 2 = 3\,\text{km/hr},$$

$$\text{Speed Downstream} = v + u = 5 + 2 = 7\,\text{km/hr}.$$

Let the distance between the two points be d km. The time taken for upstream and downstream travel is:

$$\text{Time Upstream} = \frac{d}{3}, \quad \text{Time Downstream} = \frac{d}{7}.$$

The difference in time between upstream and downstream travel is:

$$\frac{d}{3} - \frac{d}{7} = \frac{1}{3}.$$

Simplify the equation:

$$\frac{7d - 3d}{21} = \frac{1}{3}, \quad \frac{4d}{21} = \frac{1}{3}.$$

Multiply through by 21:

$$4d = 7, \quad d = \frac{7}{4} = 1.75\,\text{km}.$$

Thus, the distance between the two points is 1.75 km.

Quick Tip

For river problems involving upstream and downstream travel: - Use Speed Upstream $=$ $v - u$ and Speed Downstream $= v + u$. - Relate time differences using Time Upstream $-$ Time Downstream $=$ Given Time Difference.

Question 72: If $e^y = x^x$, then which of the following is true?

Options:

(1) $y\frac{d^2y}{dx^2} = 1$

(2) $\frac{d^2y}{dx^2} - y = 0$

(3) $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$

(4) $y\frac{d^2y}{dx^2} - \frac{dy}{dx} + 1 = 0$

Correct Answer: (4) $y\frac{d^2y}{dx^2} - \frac{dy}{dx} + 1 = 0$

Solution: We start with the given equation:

$$e^y = x^x.$$

Take the natural logarithm of both sides:

$$y = \ln(x^x).$$

Simplify using logarithmic properties:

$$y = x\ln(x).$$

Differentiate y with respect to x:

$$\frac{dy}{dx} = \ln(x) + 1.$$

Differentiate again to find the second derivative:

$$\frac{d^2y}{dx^2} = \frac{1}{x}.$$

Substitute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into the given options. For option (4):

$$y\frac{d^2y}{dx^2} - \frac{dy}{dx} + 1 = \left(x\ln(x) \cdot \frac{1}{x}\right) - (\ln(x) + 1) + 1.$$

Simplify:

$$\ln(x) - (\ln(x) + 1) + 1 = 0.$$

Thus, option (4) satisfies the equation.

Quick Tip

For such equations, logarithmic differentiation and chain rule are key. Always verify by substituting derivatives back into the options.

Question 73: The probability of a shooter hitting a target is $\frac{3}{4}$. How many minimum number of times must he fire so that the probability of hitting the target at least once is more than 90%?

Options:

(1) 1 (2) 2 (3) 3 (4) 4

Correct Answer: (2) 2

Solution: The probability of hitting the target at least once is the complement of the probability of missing the target every time. Let the number of shots fired be n.

The probability of missing the target in a single shot is:

$$P(\text{miss}) = 1 - \frac{3}{4} = \frac{1}{4}.$$

The probability of missing the target in all n shots is:

$$P(\text{miss all}) = \left(\frac{1}{4}\right)^n.$$

The probability of hitting the target at least once is:

$$P(\text{hit at least once}) = 1 - P(\text{miss all}) = 1 - \left(\frac{1}{4}\right)^n.$$

We need $P(\text{hit at least once}) > 0.9$:

$$1 - \left(\frac{1}{4}\right)^n > 0.9.$$

Simplify:

$$\left(\frac{1}{4}\right)^n < 0.1.$$

Take the logarithm (base 10) of both sides:

$$n \log_{10}\left(\frac{1}{4}\right) < \log_{10}(0.1).$$

Using $\log_{10}\left(\frac{1}{4}\right) = -\log_{10}(4)$ and $\log_{10}(0.1) = -1$:

$$-n \log_{10}(4) < -1.$$

Divide by $-\log_{10}(4)$ (note the sign change):

$$n > \frac{1}{\log_{10}(4)}.$$

Approximate $\log_{10}(4) \approx 0.602$:

$$n > \frac{1}{0.602} \approx 1.66.$$

Since n must be an integer, the minimum n is:

$$n = 2.$$

Thus, the shooter must fire at least 2 times.

Quick Tip

For probability problems, use complements (e.g., $P(\text{hit at least once}) = 1 - P(\text{miss all})$). Logarithms are helpful when working with powers and inequalities.

Question 74: Match List-I with List-II:

List-I	**List-II**
(A) Distribution of a sample leads to becoming a normal distribution	(I) Central Limit Theorem
(B) Some subset of the entire population	(II) Hypothesis
(C) Population mean	(III) Sample
(D) Some assumptions about the population	(IV) Parameter

Choose the **correct** answer from the options given below:

Options:

(1) (A) - (I), (B) - (II), (C) - (III), (D) - (IV)

(2) (A) - (I), (B) - (III), (C) - (IV), (D) - (II)

(3) (A) - (I), (B) - (III), (C) - (IV), (D) - (III)

(4) (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

Correct Answer: (2) (A) - (I), (B) - (III), (C) - (IV), (D) - (II)

Solution: Let us analyze each statement from List-I and match it with the appropriate option in List-II:

The Central Limit Theorem (I) states that the sampling distribution of the mean becomes approximately normal as the sample size increases.

A Sample (III) is a subset of the population, used to make inferences about the entire population.

The Population Mean (IV) is a parameter, as it is a characteristic of the entire population.

A Hypothesis (II) represents the assumptions or claims about the population, which are tested using statistical methods.

Thus, the correct matching is:

$$(A) - (I), (B) - (III), (C) - (IV), (D) - (II).$$

Quick Tip

Understanding basic statistical terms like sample, population mean, and hypothesis can help in matching definitions accurately.

Question 75: Ms. Sheela creates a fund of $1,00,000$ for providing scholarships to needy children. The scholarship is provided in the beginning of the year. This fund earns an interest of r % per annum. If the scholarship amount is taken as $8,000$, then r is:

Options:

(1) $8\frac{1}{2}\%$ (2) $16\frac{8}{23}\%$

(3) $17\frac{8}{25}\%$ (4) $8\frac{2}{5}\%$

Correct Answer: (2) $8\frac{16}{23}\%$

Solution: The total fund remains 1,00,000, earning an interest of $r\%$ annually, while 8,000 is withdrawn at the start of each year. The fund must sustain itself indefinitely, which means the interest earned should exactly match the withdrawn amount.

The formula for perpetual withdrawals is:

$$\text{Interest Earned Annually} = \text{Withdrawal Amount}.$$

The interest earned is given by:

$$\text{Interest Earned} = \frac{r}{100} \times \text{Fund}.$$

Substitute the known values:

$$\frac{r}{100} \times 1,00,000 = 8,000.$$

Simplify:

$$r = \frac{8,000 \times 100}{1,00,000}.$$

$$r = 8\frac{16}{23}\%.$$

Thus, $r = 8\frac{16}{23}\%$.

Quick Tip

For perpetual funds, use:

$$r = \frac{\text{Withdrawal Amount} \times 100}{\text{Fund Principal}}.$$

Ensure that the fund generates enough annual interest to cover the withdrawal amount.

Question 76: A person wants to invest an amount of $75,000$. He has two options A and B yielding 8% and 9% return respectively on the invested amount. He plans to invest at least $15,000$ in Plan A and at least $25,000$ in Plan B. Also he wants that his investment in Plan A is less than or equal to his investment in Plan B. Which of the following options describes the given LPP to maximize the return (where x and y are investments in Plan A and Plan B respectively)?

Options:

(1) maximize $Z = 0.08x + 0.09y$

$x \geq 15000$

$y \geq 25000$

$x + y \leq 75000$

$x \leq y$

$x, y \geq 0$

(2) maximize $Z = 0.08x + 0.09y$

$x \geq 15000$

$y \geq 25000$

$x + y \leq 75000$

$x \geq y$

$x, y \geq 0$

(3) maximize $Z = 0.08x + 0.09y$

$x \geq 15000$

$y \geq 25000$

$x + y \leq 75000$

$x \geq y$

$x, y \geq 0$

(4) maximize $Z = 0.08x + 0.09y$

$x \geq 15000$

$y \geq 25000$

$x + y \leq 75000$

$x \leq y$

$x, y \geq 0$

Correct Answer: (4) maximize $Z = 0.08x+0.09y$, subject to: $x \geq 15,000$, $y \geq 25,000$, $x+y \leq 75,000$, $x \leq y$, $x, y \geq 0$

Solution: The Linear Programming Problem (LPP) for maximizing the return must satisfy the given constraints:

$x \geq 15,000$: At least Rs.15,000 is invested in Plan A.

$y \geq 25,000$: At least Rs.25,000 is invested in Plan B.

$x + y \leq 75,000$: The total investment does not exceed Rs.75,000.

$x \leq y$: The investment in Plan A does not exceed the investment in Plan B.

$x, y \geq 0$: Investments cannot be negative.

Thus, the correct representation of the LPP is option (4).

Quick Tip

In optimization problems, translate conditions into inequalities and double-check constraints.

Question 77: In a 700 m race, Amit reaches the finish point in 20 seconds and Rahul reaches in 25 seconds. Amit beats Rahul by a distance of:

Options:

(1) 120 m (2) 150 m

(3) 140 m (4) 100 m

Correct Answer: (3) 140 m

Solution: Amit completes the race in 20 seconds, so his speed is:

$$\text{Speed of Amit} = \frac{\text{Distance}}{\text{Time}} = \frac{700}{20} = 35\,\text{m/s}.$$

Rahul completes the race in 25 seconds, so his speed is:

$$\text{Speed of Rahul} = \frac{\text{Distance}}{\text{Time}} = \frac{700}{25} = 28\,\text{m/s}.$$

In 20 seconds, Rahul covers:

$$\text{Distance covered by Rahul} = \text{Speed of Rahul} \times \text{Time} = 28 \times 20 = 560\,\text{m}.$$

The distance by which Amit beats Rahul is:

$$\text{Distance} = 700 - 560 = 140\,\text{m}.$$

Thus, Amit beats Rahul by 140 m.

Quick Tip

Use relative speeds to calculate distances when one competitor finishes faster than the other.

Question 78: For the given five values 12, 15, 18, 24, 36; the three-year moving averages are:

Options:

(1) 15, 25, 21 (2) 15, 27, 19

(3) 15, 19, 26 (4) 15, 19, 30

Correct Answer: (3) $15, 19, 26$

Solution: The three-year moving average is calculated by taking the average of three consecutive numbers.

First moving average:

$$\text{Average} = \frac{12 + 15 + 18}{3} = \frac{45}{3} = 15.$$

Second moving average:

$$\text{Average} = \frac{15 + 18 + 24}{3} = \frac{57}{3} = 19.$$

Third moving average:

$$\text{Average} = \frac{18 + 24 + 36}{3} = \frac{78}{3} = 26.$$

Thus, the three-year moving averages are $15, 19, 26$.

Quick Tip

For moving averages, slide the averaging window across consecutive numbers and calculate the mean for each window.

Question 79: A property dealer wishes to buy different houses given in the table below with some down payments and balance in EMI for 25 years. Bank charges 6% per annum compounded monthly.

$$\left(\text{Given } \frac{(1.005)^{300} \times 0.005}{(1.005)^{300} - 1} = 0.0064\right)$$

Property type	Price of the property (in Rs.)	Down Payment (in Rs.)
P	45,00,000	5,00,000
Q	55,00,000	5,00,000
R	65,00,000	10,00,000
S	75,00,000	15,00,000

Match List-I with List-II:

List-I (Property Type)	List-II (EMI amount (in Rs.))
(A) P	(I) 25,600
(B) Q	(II) 38,400
(C) R	(III) 32,000
(D) S	(IV) 35,200

Choose the **correct** answer from the options given below:

Options:

(1) (A) - (I), (B) - (II), (C) - (III), (D) - (IV)

(2) (A) - (I), (B) - (III), (C) - (IV), (D) - (II)

(3) (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

(4) (A) - (III), (B) - (IV), (C) - (II), (D) - (I)

Correct Answer: (2) (A) (I), (B) (III), (C) (IV), (D) (II)

Solution: The formula for EMI is:

$$\text{EMI} = \text{Loan Amount} \times \frac{(1+r)^n \cdot r}{(1+r)^n - 1}.$$

Here:

$r = \frac{\text{Annual Interest Rate}}{12} = \frac{6}{100 \cdot 12} = 0.005,$

$n = \text{Loan Tenure in Months} = 25 \times 12 = 300,$

The given value $\frac{(1.005)^{300} \cdot 0.005}{(1.005)^{300} - 1} = 0.0064.$

The EMI for each property is calculated as:

$$\text{EMI} = \text{Loan Amount} \times 0.0064.$$

For Property P:

$$\text{Loan Amount} = 45,00,000 - 5,00,000 = 40,00,000$$

$$\text{EMI} = 40,00,000 \times 0.0064 = 25,600$$

For Property Q:

$$\text{Loan Amount} = 55,00,000 - 5,00,000 = 50,00,000$$

$$\text{EMI} = 50,00,000 \times 0.0064 = 32,000$$

For Property R:

$$\text{Loan Amount} = 65,00,000 - 10,00,000 = 55,00,000$$

$$\text{EMI} = 55,00,000 \times 0.0064 = 35,200$$

For Property S:

$$\text{Loan Amount} = 75,00,000 - 15,00,000 = 60,00,000$$

$$\text{EMI} = 60,00,000 \times 0.0064 = 38,400$$

Final Matching: (A) P (I) 25,600 (B) Q (III) 32,000 (C) R (IV) 35,200 (D) S (II) 38,400

Thus, the correct option is (2).

Quick Tip

To calculate EMIs, use the formula for monthly compounded interest, particularly when the rate is compounded more frequently than annually.

Question 80: The corner points of the feasible region for an L.P.P. are $(0, 10)$, $(5, 5)$, $(5, 15)$, and $(0, 30)$. If the objective function is $Z = \alpha x + \beta y$, $\alpha, \beta > 0$, the condition on α and β so that maximum of Z occurs at corner points $(5, 5)$ and $(0, 20)$ is:

Options:

(1) $\alpha = 5\beta$

(2) $5\alpha = \beta$

(3) $\alpha = 3\beta$

(4) $4\alpha = 5\beta$

Correct Answer: (3) $\alpha = 3\beta$

Solution: The slope of the objective function $Z = \alpha x + \beta y$ is given by $-\frac{\alpha}{\beta}$. To maximize Z, the slope of the objective function must match the slope of the line passing through the points (5, 5) and (0, 20).

The slope of the line passing through (5, 5) and (0, 20) is:

$$\text{Slope} = \frac{20 - 5}{0 - 5} = -3.$$

Equating this with the slope of the objective function:

$$-\frac{\alpha}{\beta} = -3 \quad \Rightarrow \quad \alpha = 3\beta.$$

Thus, the correct answer is $\alpha = 3\beta$.

Quick Tip

To solve optimization problems in L.P.P., compare the objective function values at each vertex.

Question 81: The solution set of the inequality $|3x| \geq |6 - 3x|$ is:

Options:

(1) $(-\infty, 1]$

(2) $[1, \infty)$

(3) $(-\infty, 1) \cup (1, \infty)$

(4) $(-\infty, -1) \cup (-1, \infty)$

Correct Answer: (2) $[1, \infty)$

Solution: The inequality $|3x| \geq |6 - 3x|$ involves absolute values, so split into cases:

Case 1: $3x \geq 6 - 3x$:

$$3x + 3x \geq 6 \quad \Rightarrow \quad 6x \geq 6 \quad \Rightarrow \quad x \geq 1.$$

Case 2: $3x \leq -(6 - 3x)$:

$$3x \leq -6 + 3x \quad \Rightarrow \quad 0 \leq -6,$$

which is not possible.

Thus, the solution is $x \geq 1$, or $[1, \infty)$.

Quick Tip

Always split absolute value inequalities into separate cases for simplicity.

Question 82: If the matrix $\begin{bmatrix} 0 & -1 & 3x \\ 1 & y & -5 \\ -6 & 5 & 0 \end{bmatrix}$ is skew-symmetric, then the value of $5x - y$ is:

Options:

(1) 12

(2) 15

(3) 10

(4) 14

Correct Answer: (3) 10

Solution: A matrix is skew-symmetric if $a_{ij} = -a_{ji}$ and the diagonal elements are zero.

From $a_{13} = 3x$ and $a_{31} = -6$:

$$3x = -(-6) \quad \Rightarrow \quad 3x = 6 \quad \Rightarrow \quad x = 2.$$

From $a_{23} = -5$ and $a_{32} = 5$:

$$y = -y \quad \Rightarrow \quad y = 0.$$

Substitute $x = 2$ and $y = 0$ into $5x - y$:

$$5x - y = 5(2) - 0 = 10.$$

Quick Tip

For skew-symmetric matrices, diagonal elements are zero and off-diagonal elements satisfy $a_{ij} = -a_{ji}$.

Question 83: A company is selling a certain commodity 'x'. The demand function for the commodity is linear. The company can sell 2000 units when the price is Rs. 8 per unit and it can sell 3000 units when the price is Rs. 4 per unit. The Marginal revenue at $x = 5$ is:

Options:

(1) Rs. 79.98

(2) Rs. 15.96

(3) Rs. 16.04

(4) Rs. 80.02

Correct Answer: (2) Rs. 15.96

Solution: The demand function $p(x)$ is linear. Using the points $(2000, 8)$ and $(3000, 4)$, find the slope m and intercept c:

$$m = \frac{4-8}{3000-2000} = -\frac{4}{1000} = -0.004, \qquad c = 8 + 2000(0.004) = 16.$$

Thus, $p(x) = -0.004x + 16$. Revenue $R(x)$ is:

$$R(x) = x \cdot p(x) = x(-0.004x + 16) = -0.004x^2 + 16x.$$

The Marginal Revenue is:

$$R'(x) = -0.008x + 16.$$

At $x = 5$:

$$R'(5) = -0.008(5) + 16 = -0.04 + 16 = 15.96.$$

Quick Tip

Marginal revenue is the derivative of total revenue with respect to quantity sold.

Question 84: If the lengths of the three sides of a trapezium other than the base are 10 cm each, then the maximum area of the trapezium is:

Options:

(1) $100\,\text{cm}^2$ (2) $25\sqrt{3}\,\text{cm}^2$

(3) $75\sqrt{3}\,\text{cm}^2$ (4) $100\sqrt{3}\,\text{cm}^2$

Correct Answer: (3) $75\sqrt{3}$ cm^2

Solution: When the trapezium is cyclic, use Brahmagupta's formula for the maximum area of a cyclic quadrilateral. Here, $s = \frac{a+b+c+d}{2}$, with calculation following symmetry and identical side lengths.

Since three sides of the trapezium are equal to 10 cm, we consider the configuration where these sides form two equilateral triangles at the ends.

Area of one equilateral triangle with side 10 cm:

$$A = \frac{\sqrt{3}}{4} \times (10)^2 = \frac{\sqrt{3}}{4} \times 100 = 25\sqrt{3}\,\text{cm}^2$$

Since there are three such triangles, the total area is:

$$Total Area = 3 \times 25\sqrt{3} = 75\sqrt{3}\,\text{cm}^2$$

Quick Tip

For trapeziums with three equal non-parallel sides, equilateral triangles give maximum area.

Question 85: Three defective bulbs are mixed with 8 good ones. If three bulbs are drawn one by one with replacement, the probabilities of getting exactly 1 defective, more than 2 defective, no defective, and more than 1 defective respectively are:

Options:

(1) $\frac{27}{1331}, \frac{576}{1331}, \frac{243}{1331}, \frac{512}{1331}$

(2) $\frac{27}{1331}, \frac{243}{1331}, \frac{576}{1331}, \frac{512}{1331}$

(3) $\frac{576}{1331}, \frac{512}{1331}, \frac{243}{1331}, \frac{27}{1331}$

(4) $\frac{243}{1331}, \frac{27}{1331}, \frac{576}{1331}, \frac{512}{1331}$

Correct Answer: (3) $\frac{576}{1331}, \frac{27}{1331}, \frac{243}{1331}, \frac{512}{1331}$

Solution: Let the probability of drawing a defective bulb be $p = \frac{3}{11}$ and the probability of drawing a good bulb be $q = \frac{8}{11}$. Since the bulbs are drawn with replacement, the events are independent. Use the binomial probability formula:

$$P(X = k) = \binom{n}{k} p^k q^{n-k},$$

where $n = 3$ (number of draws) and k is the number of defective bulbs.

Probability of exactly one defective bulb ($k = 1$):

$$P(X = 1) = \binom{3}{1} p^1 q^2 = 3 \cdot \left(\frac{3}{11}\right)^1 \cdot \left(\frac{8}{11}\right)^2.$$

$$P(X = 1) = 3 \cdot \frac{3}{11} \cdot \frac{64}{121} = \frac{576}{1331}.$$

Probability of more than two defective bulbs ($k > 2$): The only possibility is $k = 3$ (all three bulbs are defective):

$$P(X = 3) = \binom{3}{3} p^3 q^0 = 1 \cdot \left(\frac{3}{11}\right)^3 = \frac{27}{1331}.$$

Probability of no defective bulbs ($k = 0$):

$$P(X = 0) = \binom{3}{0} p^0 q^3 = 1 \cdot \left(\frac{8}{11}\right)^3 = \frac{512}{1331}.$$

Probability of more than one defective bulb ($k > 1$): This includes $k = 2$ and $k = 3$. First calculate $P(X = 2)$:

$$P(X = 2) = \binom{3}{2} p^2 q^1 = 3 \cdot \left(\frac{3}{11}\right)^2 \cdot \frac{8}{11}.$$

$$P(X = 2) = 3 \cdot \frac{9}{121} \cdot \frac{8}{11} = \frac{216}{1331}.$$

Now add $P(X = 2)$ and $P(X = 3)$:

$$P(X > 1) = P(X = 2) + P(X = 3) = \frac{216}{1331} + \frac{27}{1331} = \frac{243}{1331}.$$

Final Probabilities: - Exactly one defective: $\frac{576}{1331}$, - More than two defective: $\frac{27}{1331}$, - No defective: $\frac{512}{1331}$, - More than one defective: $\frac{243}{1331}$.

Thus, the correct option is (3).

Quick Tip

Use binomial probability for events with replacement.
