

## Special Instructions / Useful Data

$\mathbb{Z}_n = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$  denotes the additive group of integers modulo  $n$

$\mathbb{R}$  = the set of all real numbers

$\mathbb{N}$  = the set of all positive integers

$\mathbb{Z}$  = the set of all integers

$\mathbb{C}$  = the set of all complex numbers

$\mathbb{Q}$  = the set of all rational numbers

$\gcd(r, n)$  = the greatest common divisor of the integers  $r$  and  $n$

$S_n$  = the symmetric group of all permutations of  $\{1, 2, \dots, n\}$

$A_n$  = the group of all even permutations in  $S_n$

$M_n(\mathbb{C})$  = the set of all  $n \times n$  matrices with entries from  $\mathbb{C}$

$M_n(\mathbb{R})$  = the set of all  $n \times n$  matrices with entries from  $\mathbb{R}$

$M^T$  = the transpose of the matrix  $M$

$I_n$  = the  $n \times n$  identity matrix

$P_n(x)$  = the real vector space of polynomials, in the variable  $x$  with real coefficients and having degree at most  $n$ , together with the zero polynomial. These polynomials are regarded as functions from  $\mathbb{R}$  to  $\mathbb{R}$

$\binom{n}{k}$  = the binomial coefficient defined as  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$f \circ g$  = the composite function defined by  $(f \circ g)(x) = f(g(x))$

$A \setminus B$  = the complement of the set  $B$  in the set  $A$ , that is,  $\{x \in A : x \notin B\}$

$\log x$  = the logarithm of  $x$  to the base  $e$  for a positive number  $x$

$\mathbb{R}^n$  = the  $n$ -dimensional Euclidean space

$A \times B$  = the Cartesian product of the sets  $A$  and  $B$

$M^{-1}$  = the inverse of an invertible matrix  $M$

**Section A: Q.1 – Q.10 Carry ONE mark each.**

Q.1 Let  $y_c: \mathbb{R} \rightarrow (0, \infty)$  be the solution of the Bernoulli's equation

$$\frac{dy}{dx} - y + y^3 = 0, \quad y(0) = c > 0.$$

Then, for every  $c > 0$ , which one of the following is true?

- (A)  $\lim_{x \rightarrow \infty} y_c(x) = 0$
- (B)  $\lim_{x \rightarrow \infty} y_c(x) = 1$
- (C)  $\lim_{x \rightarrow \infty} y_c(x) = e$
- (D)  $\lim_{x \rightarrow \infty} y_c(x)$  does not exist

Q.2 For a twice continuously differentiable function  $g: \mathbb{R} \rightarrow \mathbb{R}$ , define

$$u_g(x, y) = \frac{1}{y} \int_{-y}^y g(x+t) dt \quad \text{for } (x, y) \in \mathbb{R}^2, \quad y > 0.$$

Which one of the following holds for all such  $g$ ?

(A)  $\frac{\partial^2 u_g}{\partial x^2} = \frac{2}{y} \frac{\partial u_g}{\partial y} + \frac{\partial^2 u_g}{\partial y^2}$

(B)  $\frac{\partial^2 u_g}{\partial x^2} = \frac{1}{y} \frac{\partial u_g}{\partial y} + \frac{\partial^2 u_g}{\partial y^2}$

(C)  $\frac{\partial^2 u_g}{\partial x^2} = \frac{2}{y} \frac{\partial u_g}{\partial y} - \frac{\partial^2 u_g}{\partial y^2}$

(D)  $\frac{\partial^2 u_g}{\partial x^2} = \frac{1}{y} \frac{\partial u_g}{\partial y} - \frac{\partial^2 u_g}{\partial y^2}$

Q.3 Let  $y(x)$  be the solution of the differential equation

$$\frac{dy}{dx} = 1 + y \sec x \quad \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

that satisfies  $y(0) = 0$ . Then, the value of  $y\left(\frac{\pi}{6}\right)$  equals

(A)  $\sqrt{3} \log\left(\frac{3}{2}\right)$

(B)  $\left(\frac{\sqrt{3}}{2}\right) \log\left(\frac{3}{2}\right)$

(C)  $\left(\frac{\sqrt{3}}{2}\right) \log 3$

(D)  $\sqrt{3} \log 3$

Q.4 Let  $\mathcal{F}$  be the family of curves given by

$$x^2 + 2hxy + y^2 = 1, \quad -1 < h < 1.$$

Then, the differential equation for the family of orthogonal trajectories to  $\mathcal{F}$  is

(A)  $(x^2y - y^3 + y)\frac{dy}{dx} - (xy^2 - x^3 + x) = 0$

(B)  $(x^2y - y^3 + y)\frac{dy}{dx} + (xy^2 - x^3 + x) = 0$

(C)  $(x^2y + y^3 + y)\frac{dy}{dx} - (xy^2 + x^3 + x) = 0$

(D)  $(x^2y + y^3 + y)\frac{dy}{dx} + (xy^2 + x^3 + x) = 0$

Q.5 Let  $G$  be a group of order 39 such that it has exactly one subgroup of order 3 and exactly one subgroup of order 13. Then, which one of the following statements is TRUE?

(A)  $G$  is necessarily cyclic

(B)  $G$  is abelian but need not be cyclic

(C)  $G$  need not be abelian

(D)  $G$  has 13 elements of order 13

Q.6 For a positive integer  $n$ , let  $U(n) = \{\bar{r} \in \mathbb{Z}_n : \gcd(r, n) = 1\}$  be the group under multiplication modulo  $n$ . Then, which one of the following statements is TRUE?

- (A)  $U(5)$  is isomorphic to  $U(8)$
- (B)  $U(10)$  is isomorphic to  $U(12)$
- (C)  $U(8)$  is isomorphic to  $U(10)$
- (D)  $U(8)$  is isomorphic to  $U(12)$

Q.7 Which one of the following is TRUE for the symmetric group  $S_{13}$ ?

- (A)  $S_{13}$  has an element of order 42
- (B)  $S_{13}$  has no element of order 35
- (C)  $S_{13}$  has an element of order 27
- (D)  $S_{13}$  has no element of order 60

Q.8 Let  $G$  be a finite group containing a non-identity element which is conjugate to its inverse. Then, which one of the following is TRUE?

- (A) The order of  $G$  is necessarily even
- (B) The order of  $G$  is not necessarily even
- (C)  $G$  is necessarily cyclic
- (D)  $G$  is necessarily abelian but need not be cyclic

Q.9 Consider the following statements.

P: If a system of linear equations  $Ax = b$  has a unique solution, where  $A$  is an  $m \times n$  matrix and  $b$  is an  $m \times 1$  matrix, then  $m = n$ .

Q: For a subspace  $W$  of a nonzero vector space  $V$ , whenever  $u \in V \setminus W$  and  $v \in V \setminus W$ , then  $u + v \in V \setminus W$ .

Which one of the following holds?

- (A) Both P and Q are true
- (B) P is true but Q is false
- (C) P is false but Q is true
- (D) Both P and Q are false

Q.10 Let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Which one of the following is the solution of the differential equation

$$\frac{d^2 y}{dx^2} + y = g(x) \quad \text{for } x \in \mathbb{R},$$

satisfying the conditions  $y(0) = 0$ ,  $y'(0) = 1$ ?

(A)  $y(x) = \sin x - \int_0^x \sin(x-t) g(t) dt$

(B)  $y(x) = \sin x + \int_0^x \sin(x-t) g(t) dt$

(C)  $y(x) = \sin x - \int_0^x \cos(x-t) g(t) dt$

(D)  $y(x) = \sin x + \int_0^x \cos(x-t) g(t) dt$



**Section A: Q.11 – Q.30 Carry TWO marks each.**

Q.11 Which one of the following groups has elements of order 1, 2, 3, 4, 5 but does not have an element of order greater than or equal to 6 ?

- (A) The alternating group  $A_6$
- (B) The alternating group  $A_5$
- (C)  $S_6$
- (D)  $S_5$

Q.12 Consider the group  $G = \{A \in M_2(\mathbb{R}) : AA^T = I_2\}$  with respect to matrix multiplication. Let

$$Z(G) = \{A \in G : AB = BA, \text{ for all } B \in G\}.$$

Then, the cardinality of  $Z(G)$  is

- (A) 1
- (B) 2
- (C) 4
- (D) Infinite

Q.13 Let  $V$  be a nonzero subspace of the complex vector space  $M_7(\mathbb{C})$  such that every nonzero matrix in  $V$  is invertible. Then, the dimension of  $V$  over  $\mathbb{C}$  is

- (A) 1
- (B) 2
- (C) 7
- (D) 49

Q.14 For  $n \in \mathbb{N}$ , let

$$a_n = \frac{1}{(3n+2)(3n+4)} \quad \text{and} \quad b_n = \frac{n^3 + \cos(3^n)}{3^n + n^3}.$$

Then, which one of the following is TRUE?

(A)  $\sum_{n=1}^{\infty} a_n$  is convergent but  $\sum_{n=1}^{\infty} b_n$  is divergent

(B)  $\sum_{n=1}^{\infty} a_n$  is divergent but  $\sum_{n=1}^{\infty} b_n$  is convergent

(C) Both  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are divergent

(D) Both  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are convergent

Q.15

Let  $a = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -1 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{bmatrix}$ . Consider the following two statements.

P: The matrix  $I_4 - aa^T$  is invertible.

Q: The matrix  $I_4 - 2aa^T$  is invertible.

Then, which one of the following holds?

- (A) P is false but Q is true
- (B) P is true but Q is false
- (C) Both P and Q are true
- (D) Both P and Q are false

Q.16 Let  $A$  be a  $6 \times 5$  matrix with entries in  $\mathbb{R}$  and  $B$  be a  $5 \times 4$  matrix with entries in  $\mathbb{R}$ . Consider the following two statements.

P: For all such nonzero matrices  $A$  and  $B$ , there is a nonzero matrix  $Z$  such that  $AZB$  is the  $6 \times 4$  zero matrix.

Q: For all such nonzero matrices  $A$  and  $B$ , there is a nonzero matrix  $Y$  such that  $BYA$  is the  $5 \times 5$  zero matrix.

Which one of the following holds?

- (A) Both P and Q are true
- (B) P is true but Q is false
- (C) P is false but Q is true
- (D) Both P and Q are false

Q.17 Let  $P_{11}(x)$  be the real vector space of polynomials, in the variable  $x$  with real coefficients and having degree at most 11, together with the zero polynomial.

Let

$$E = \{s_0(x), s_1(x), \dots, s_{11}(x)\}, \quad F = \{r_0(x), r_1(x), \dots, r_{11}(x)\}$$

be subsets of  $P_{11}(x)$  having 12 elements each and satisfying

$$s_0(3) = s_1(3) = \dots = s_{11}(3) = 0, \quad r_0(4) = r_1(4) = \dots = r_{11}(4) = 1.$$

Then, which one of the following is TRUE?

- (A) Any such  $E$  is not necessarily linearly dependent and any such  $F$  is not necessarily linearly dependent
- (B) Any such  $E$  is necessarily linearly dependent but any such  $F$  is not necessarily linearly dependent
- (C) Any such  $E$  is not necessarily linearly dependent but any such  $F$  is necessarily linearly dependent
- (D) Any such  $E$  is necessarily linearly dependent and any such  $F$  is necessarily linearly dependent

Q.18 For the differential equation

$$y(8x - 9y)dx + 2x(x - 3y)dy = 0,$$

which one of the following statements is TRUE?

- (A) The differential equation is not exact and has  $x^2$  as an integrating factor
- (B) The differential equation is exact and homogeneous
- (C) The differential equation is not exact and does not have  $x^2$  as an integrating factor
- (D) The differential equation is not homogeneous and has  $x^2$  as an integrating factor

Q.19 For  $x \in \mathbb{R}$ , let  $[x]$  denote the greatest integer less than or equal to  $x$ .

For  $x, y \in \mathbb{R}$ , define

$$\min\{x, y\} = \begin{cases} x & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

Let  $f: [-2\pi, 2\pi] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \sin(\min\{x, x - [x]\}) \text{ for } x \in [-2\pi, 2\pi].$$

Consider the set  $S = \{x \in [-2\pi, 2\pi]: f \text{ is discontinuous at } x\}$ .

Which one of the following statements is TRUE?

- (A)  $S$  has 13 elements
- (B)  $S$  has 7 elements
- (C)  $S$  is an infinite set
- (D)  $S$  has 6 elements



Q.20 Define the sequences  $\{a_n\}_{n=3}^{\infty}$  and  $\{b_n\}_{n=3}^{\infty}$  as

$$a_n = (\log n + \log \log n)^{\log n} \quad \text{and} \quad b_n = n^{\left(1 + \frac{1}{\log n}\right)}.$$

Which one of the following is TRUE?

(A)  $\sum_{n=3}^{\infty} \frac{1}{a_n}$  is convergent but  $\sum_{n=3}^{\infty} \frac{1}{b_n}$  is divergent

(B)  $\sum_{n=3}^{\infty} \frac{1}{a_n}$  is divergent but  $\sum_{n=3}^{\infty} \frac{1}{b_n}$  is convergent

(C) Both  $\sum_{n=3}^{\infty} \frac{1}{a_n}$  and  $\sum_{n=3}^{\infty} \frac{1}{b_n}$  are divergent

(D) Both  $\sum_{n=3}^{\infty} \frac{1}{a_n}$  and  $\sum_{n=3}^{\infty} \frac{1}{b_n}$  are convergent

Q.21 For  $p, q, r \in \mathbb{R}$ ,  $r \neq 0$  and  $n \in \mathbb{N}$ , let

$$a_n = p^n n^q \left( \frac{n}{n+2} \right)^{n^2} \text{ and } b_n = \frac{n^n}{n! r^n} \left( \sqrt{\frac{n+2}{n}} \right).$$

Then, which one of the following statements is TRUE?

(A) If  $1 < p < e^2$  and  $q > 1$ , then  $\sum_{n=1}^{\infty} a_n$  is convergent

(B) If  $e^2 < p < e^4$  and  $q > 1$ , then  $\sum_{n=1}^{\infty} a_n$  is convergent

(C) If  $1 < r < e$ , then  $\sum_{n=1}^{\infty} b_n$  is convergent

(D) If  $\frac{1}{e} < r < 1$ , then  $\sum_{n=1}^{\infty} b_n$  is convergent

- Q.22 Let  $P_7(x)$  be the real vector space of polynomials, in the variable  $x$  with real coefficients and having degree at most 7, together with the zero polynomial. Let  $T: P_7(x) \rightarrow P_7(x)$  be the linear transformation defined by

$$T(f(x)) = f(x) + \frac{df(x)}{dx}.$$

Then, which one of the following is TRUE?

- (A)  $T$  is not a surjective linear transformation
- (B) There exists  $k \in \mathbb{N}$  such that  $T^k$  is the zero linear transformation
- (C) 1 and 2 are the eigenvalues of  $T$
- (D) There exists  $r \in \mathbb{N}$  such that  $(T - I)^r$  is the zero linear transformation, where  $I$  is the identity map on  $P_7(x)$

Q.23 For  $\alpha \in \mathbb{R}$ , let  $y_\alpha(x)$  be the solution of the differential equation

$$\frac{dy}{dx} + 2y = \frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}$$

satisfying  $y(0) = \alpha$ . Then, which one of the following is TRUE?

- (A)  $\lim_{x \rightarrow \infty} y_\alpha(x) = 0$  for every  $\alpha \in \mathbb{R}$
- (B)  $\lim_{x \rightarrow \infty} y_\alpha(x) = 1$  for every  $\alpha \in \mathbb{R}$
- (C) There exists an  $\alpha \in \mathbb{R}$  such that  $\lim_{x \rightarrow \infty} y_\alpha(x)$  exists but its value is different from 0 and 1
- (D) There is an  $\alpha \in \mathbb{R}$  for which  $\lim_{x \rightarrow \infty} y_\alpha(x)$  does not exist

Q.24 Consider the following two statements.

P: There exist functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $g: \mathbb{R} \rightarrow \mathbb{R}$  such that  $f$  is continuous at  $x = 1$  and  $g$  is discontinuous at  $x = 1$  but  $g \circ f$  is continuous at  $x = 1$ .

Q: There exist functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $g: \mathbb{R} \rightarrow \mathbb{R}$  such that both  $f$  and  $g$  are discontinuous at  $x = 1$  but  $g \circ f$  is continuous at  $x = 1$ .

Which one of the following holds?

- (A) Both P and Q are true
- (B) Both P and Q are false
- (C) P is true but Q is false
- (D) P is false but Q is true

Q.25 Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \frac{(x^2 + 1)^2}{x^4 + x^2 + 1} \quad \text{for } x \in \mathbb{R}.$$

Then, which one of the following is TRUE?

- (A)  $f$  has exactly two points of local maxima and exactly three points of local minima
- (B)  $f$  has exactly three points of local maxima and exactly two points of local minima
- (C)  $f$  has exactly one point of local maximum and exactly two points of local minima
- (D)  $f$  has exactly two points of local maxima and exactly one point of local minimum

Q.26 Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = 2e^x \text{ for } x \in \mathbb{R}.$$

Consider the following statements.

P: If  $f(x) > 0$  for all  $x \in \mathbb{R}$ , then  $f'(x) > 0$  for all  $x \in \mathbb{R}$ .

Q: If  $f'(x) > 0$  for all  $x \in \mathbb{R}$ , then  $f(x) > 0$  for all  $x \in \mathbb{R}$ .

Then, which one of the following holds?

- (A) P is true but Q is false
- (B) P is false but Q is true
- (C) Both P and Q are true
- (D) Both P and Q are false

Q.27 For  $a > b > 0$ , consider

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2 \text{ and } x^2 + y^2 \geq b^2 \right\}.$$

Then, the surface area of the boundary of the solid  $D$  is

(A)  $4\pi(a + b)\sqrt{a^2 - b^2}$

(B)  $4\pi(a^2 - b\sqrt{a^2 - b^2})$

(C)  $4\pi(a - b)\sqrt{a^2 - b^2}$

(D)  $4\pi(a^2 + b\sqrt{a^2 - b^2})$



- Q.28 For  $n \geq 3$ , let a regular  $n$ -sided polygon  $P_n$  be circumscribed by a circle of radius  $R_n$  and let  $r_n$  be the radius of the circle inscribed in  $P_n$ . Then

$$\lim_{n \rightarrow \infty} \left( \frac{R_n}{r_n} \right)^{n^2}$$

equals

(A)  $e^{(\pi^2)}$

(B)  $e^{\left(\frac{\pi^2}{2}\right)}$

(C)  $e^{\left(\frac{\pi^2}{3}\right)}$

(D)  $e^{(2\pi^2)}$

Q.29 Let  $L_1$  denote the line  $y = 3x + 2$  and  $L_2$  denote the line  $y = 4x + 3$ . Suppose that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a four times continuously differentiable function such that the line  $L_1$  intersects the curve  $y = f(x)$  at exactly three distinct points and the line  $L_2$  intersects the curve  $y = f(x)$  at exactly four distinct points. Then, which one of the following is TRUE?

- (A)  $\frac{df}{dx}$  does not attain the value 3 on  $\mathbb{R}$
- (B)  $\frac{d^2f}{dx^2}$  vanishes at most once on  $\mathbb{R}$
- (C)  $\frac{d^3f}{dx^3}$  vanishes at least once on  $\mathbb{R}$
- (D)  $\frac{df}{dx}$  does not attain the value  $\frac{7}{2}$  on  $\mathbb{R}$

Q.30 Define the function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$f(x, y) = 12xy e^{-(2x+3y-2)}.$$

If  $(a, b)$  is the point of local maximum of  $f$ , then  $f(a, b)$  equals

- (A) 2
- (B) 6
- (C) 12
- (D) 0

**Section B: Q.31 – Q.40 Carry TWO marks each.**

Q.31 Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers.

Then, which of the following statements is/are always TRUE?

(A) If  $\sum_{n=1}^{\infty} a_n$  converges absolutely, then  $\sum_{n=1}^{\infty} a_n^2$  converges absolutely

(B) If  $\sum_{n=1}^{\infty} a_n$  converges absolutely, then  $\sum_{n=1}^{\infty} a_n^3$  converges absolutely

(C) If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\sum_{n=1}^{\infty} a_n^2$  converges

(D) If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\sum_{n=1}^{\infty} a_n^3$  converges

Q.32 Which of the following statements is/are TRUE?

(A)  $\sum_{n=1}^{\infty} n \log \left( 1 + \frac{1}{n^3} \right)$  is convergent

(B)  $\sum_{n=1}^{\infty} \left( 1 - \cos \left( \frac{1}{n} \right) \right) \log n$  is convergent

(C)  $\sum_{n=1}^{\infty} n^2 \log \left( 1 + \frac{1}{n^3} \right)$  is convergent

(D)  $\sum_{n=1}^{\infty} \left( 1 - \cos \left( \frac{1}{\sqrt{n}} \right) \right) \log n$  is convergent

Q.33 Which of the following statements is/are TRUE?

- (A) The additive group of real numbers is isomorphic to the multiplicative group of positive real numbers
- (B) The multiplicative group of nonzero real numbers is isomorphic to the multiplicative group of nonzero complex numbers
- (C) The additive group of real numbers is isomorphic to the multiplicative group of nonzero complex numbers
- (D) The additive group of real numbers is isomorphic to the additive group of rational numbers

Q.34 Let  $f: (1, \infty) \rightarrow (0, \infty)$  be a continuous function such that for every  $n \in \mathbb{N}$ ,  $f(n)$  is the smallest prime factor of  $n$ . Then, which of the following options is/are CORRECT?

- (A)  $\lim_{x \rightarrow \infty} f(x)$  exists
- (B)  $\lim_{x \rightarrow \infty} f(x)$  does not exist
- (C) The set of solutions to the equation  $f(x) = 2024$  is finite
- (D) The set of solutions to the equation  $f(x) = 2024$  is infinite

Q.35

Let

$$S = \{(x, y) \in \mathbb{R}^2: x > 0, y > 0\},$$

and  $f: S \rightarrow \mathbb{R}$  be given by

$$f(x, y) = 2x^2 + 3y^2 - \log x - \frac{1}{6} \log y.$$

Then, which of the following statements is/are TRUE?

- (A) There is a unique point in  $S$  at which  $f(x, y)$  attains a local maximum
- (B) There is a unique point in  $S$  at which  $f(x, y)$  attains a local minimum
- (C) For each point  $(x_0, y_0) \in S$ , the set  $\{(x, y) \in S: f(x, y) = f(x_0, y_0)\}$  is bounded
- (D) For each point  $(x_0, y_0) \in S$ , the set  $\{(x, y) \in S: f(x, y) = f(x_0, y_0)\}$  is unbounded

Q.36 The center  $Z(G)$  of a group  $G$  is defined as

$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}.$$

Let  $|G|$  denote the order of  $G$ . Then, which of the following statements is/are TRUE for any group  $G$ ?

- (A) If  $G$  is non-abelian and  $Z(G)$  contains more than one element, then the center of the quotient group  $G/Z(G)$  contains only one element
- (B) If  $|G| \geq 2$ , then there exists a non-trivial homomorphism from  $\mathbb{Z}$  to  $G$
- (C) If  $|G| \geq 2$  and  $G$  is non-abelian, then there exists a non-identity isomorphism from  $G$  to itself
- (D) If  $|G| = p^3$ , where  $p$  is a prime number, then  $G$  is necessarily abelian



Q.37 For a matrix  $M$ , let  $\text{Rowspace}(M)$  denote the linear span of the rows of  $M$  and  $\text{Colspace}(M)$  denote the linear span of the columns of  $M$ . Which of the following hold(s) for all  $A, B, C \in M_{10}(\mathbb{R})$  satisfying  $A = BC$  ?

- (A)  $\text{Rowspace}(A) \subseteq \text{Rowspace}(B)$
- (B)  $\text{Rowspace}(A) \subseteq \text{Rowspace}(C)$
- (C)  $\text{Colspace}(A) \subseteq \text{Colspace}(B)$
- (D)  $\text{Colspace}(A) \subseteq \text{Colspace}(C)$

Q.38 Define  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  as follows

$$f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2} \text{ and } g(x) = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m+1)! m!} \text{ for } x \in \mathbb{R}.$$

Let  $x_1, x_2, x_3, x_4 \in \mathbb{R}$  be such that  $0 < x_1 < x_2$ ,  $0 < x_3 < x_4$ ,

$$f(x_1) = f(x_2) = 0, \quad f(x) \neq 0 \text{ when } x_1 < x < x_2,$$

$$g(x_3) = g(x_4) = 0 \quad \text{and } g(x) \neq 0 \text{ when } x_3 < x < x_4.$$

Then, which of the following statements is/are TRUE?

- (A) The function  $f$  does not vanish anywhere in the interval  $(x_3, x_4)$
- (B) The function  $f$  vanishes exactly once in the interval  $(x_3, x_4)$
- (C) The function  $g$  does not vanish anywhere in the interval  $(x_1, x_2)$
- (D) The function  $g$  vanishes exactly once in the interval  $(x_1, x_2)$

Q.39 For  $0 < \alpha < 4$ , define the sequence  $\{x_n\}_{n=1}^{\infty}$  of real numbers as follows:

$$x_1 = \alpha \text{ and } x_{n+1} + 2 = -x_n(x_n - 4) \text{ for } n \in \mathbb{N}.$$

Which of the following is/are TRUE?

- (A)  $\{x_n\}_{n=1}^{\infty}$  converges for at least three distinct values of  $\alpha \in (0,1)$
- (B)  $\{x_n\}_{n=1}^{\infty}$  converges for at least three distinct values of  $\alpha \in (1,2)$
- (C)  $\{x_n\}_{n=1}^{\infty}$  converges for at least three distinct values of  $\alpha \in (2,3)$
- (D)  $\{x_n\}_{n=1}^{\infty}$  converges for at least three distinct values of  $\alpha \in (3,4)$

Q.40 Consider

$$G = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$$

as a subgroup of the additive group  $\mathbb{R}$ .

Which of the following statements is/are TRUE?

- (A)  $G$  is a cyclic subgroup of  $\mathbb{R}$  under addition
- (B)  $G \cap I$  is non-empty for every non-empty open interval  $I \subseteq \mathbb{R}$
- (C)  $G$  is a closed subset of  $\mathbb{R}$
- (D)  $G$  is isomorphic to the group  $\mathbb{Z} \times \mathbb{Z}$ , where the group operation in  $\mathbb{Z} \times \mathbb{Z}$  is defined by  $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$

## Section C: Q.41 – Q.50 Carry ONE mark each.

Q.41 The area of the region

$$R = \left\{ (x, y) \in \mathbb{R}^2: 0 \leq x \leq 1, 0 \leq y \leq 1 \text{ and } \frac{1}{4} \leq xy \leq \frac{1}{2} \right\}$$

is \_\_\_\_\_ (rounded off to two decimal places).

Q.42 Let  $y: \mathbb{R} \rightarrow \mathbb{R}$  be the solution to the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 1$$

satisfying  $y(0) = 0$  and  $y'(0) = 1$ .Then,  $\lim_{x \rightarrow \infty} y(x)$  equals \_\_\_\_\_ (rounded off to two decimal places).Q.43 For  $\alpha > 0$ , let  $y_\alpha(x)$  be the solution to the differential equation

$$2\frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$$

satisfying the conditions

$$y(0) = 1, \quad y'(0) = \alpha.$$

Then, the smallest value of  $\alpha$  for which  $y_\alpha(x)$  has no critical points in  $\mathbb{R}$  equals \_\_\_\_\_ (rounded off to the nearest integer).

Q.44 Consider the  $4 \times 4$  matrix

$$M = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

If  $a_{i,j}$  denotes the  $(i,j)^{\text{th}}$  entry of  $M^{-1}$ , then  $a_{4,1}$  equals \_\_\_\_\_ (rounded off to two decimal places).

Q.45 Let  $P_{12}(x)$  be the real vector space of polynomials in the variable  $x$  with real coefficients and having degree at most 12, together with the zero polynomial.

Define

$$V = \left\{ f \in P_{12}(x) : f(-x) = f(x) \text{ for all } x \in \mathbb{R} \text{ and } f(2024) = 0 \right\}.$$

Then, the dimension of  $V$  is \_\_\_\_\_

Q.46 Let

$$S = \left\{ f: \mathbb{R} \rightarrow \mathbb{R} : f \text{ is a polynomial and } f(f(x)) = (f(x))^{2024} \text{ for } x \in \mathbb{R} \right\}.$$

Then, the number of elements in  $S$  is \_\_\_\_\_

Q.47 Let  $a_1 = 1, b_1 = 2$  and  $c_1 = 3$ . Consider the convergent sequences

$$\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \text{ and } \{c_n\}_{n=1}^{\infty}$$

defined as follows:

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{b_n + c_n}{2} \text{ and } c_{n+1} = \frac{c_n + a_n}{2} \text{ for } n \geq 1.$$

Then,

$$\sum_{n=1}^{\infty} b_n c_n (a_{n+1} - a_n) + \sum_{n=1}^{\infty} (b_{n+1} c_{n+1} - b_n c_n) a_{n+1}$$

equals \_\_\_\_\_ (rounded off to two decimal places)

Q.48 Let

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4, (x-1)^2 + y^2 \leq 1, z \geq 0\}.$$

Then, the surface area of  $S$  equals \_\_\_\_\_ (rounded off to two decimal places).

- Q.49 Let  $P_7(x)$  be the real vector space of polynomials in  $x$  with degree at most 7, together with the zero polynomial. For  $r = 1, 2, \dots, 7$ , define

$$s_r(x) = x(x-1)\cdots(x-(r-1)) \text{ and } s_0(x) = 1.$$

Consider the fact that  $B = \{s_0(x), s_1(x), \dots, s_7(x)\}$  is a basis of  $P_7(x)$ .

If

$$x^5 = \sum_{k=0}^7 \alpha_{5,k} s_k(x),$$

where  $\alpha_{5,k} \in \mathbb{R}$ , then  $\alpha_{5,2}$  equals \_\_\_\_\_ (rounded off to two decimal places)

- Q.50 Let

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & -4 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix}.$$

If  $p(x)$  is the characteristic polynomial of  $M$ , then  $p(2) - 1$  equals \_\_\_\_\_

## Section C: Q.51 – Q.60 Carry TWO marks each.

Q.51 For  $\alpha \in (-2\pi, 0)$ , consider the differential equation

$$x^2 \frac{d^2 y}{dx^2} + \alpha x \frac{dy}{dx} + y = 0 \quad \text{for } x > 0.$$

Let  $D$  be the set of all  $\alpha \in (-2\pi, 0)$  for which all corresponding real solutions to the above differential equation approach zero as  $x \rightarrow 0^+$ . Then, the number of elements in  $D \cap \mathbb{Z}$  equals \_\_\_\_\_

Q.52 The value of

$$\lim_{t \rightarrow \infty} \left( \left( \log \left( t^2 + \frac{1}{t^2} \right) \right)^{-1} \int_1^{\pi t} \frac{\sin^2 5x}{x} dx \right)$$

equals \_\_\_\_\_ (rounded off to two decimal places).



- Q.53 Let  $T$  be the planar region enclosed by the square with vertices at the points  $(0,1)$ ,  $(1,0)$ ,  $(0,-1)$  and  $(-1,0)$ . Then, the value of

$$\iint_T \left( \cos(\pi(x-y)) - \cos(\pi(x+y)) \right)^2 dx dy$$

equals \_\_\_\_\_ (rounded off to two decimal places).

- Q.54 Let

$$S = \{(x, y, z) \in \mathbb{R}^3: x^2 + y^2 + z^2 < 1\}.$$

Then, the value of

$$\frac{1}{\pi} \iiint_S \left( (x-2y+z)^2 + (2x-y-z)^2 + (x-y+2z)^2 \right) dx dy dz$$

equals \_\_\_\_\_ (rounded off to two decimal places).

- Q.55 For  $n \in \mathbb{N}$ , if

$$a_n = \frac{1}{n^3 + 1} + \frac{2^2}{n^3 + 2} + \cdots + \frac{n^2}{n^3 + n}$$

then the sequence  $\{a_n\}_{n=1}^{\infty}$  converges to \_\_\_\_\_ (rounded off to two decimal places)

Q.56 Consider the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^3 - 4x^2 + 4x - 6$ .

For  $c \in \mathbb{R}$ , let

$$S(c) = \left\{ x \in \mathbb{R} : f(x) = c \right\}$$

and  $|S(c)|$  denote the number of elements in  $S(c)$ . Then, the value of

$$|S(-7)| + |S(-5)| + |S(3)|$$

equals \_\_\_\_\_

Q.57 Let  $c > 0$  be such that

$$\int_0^c e^{s^2} ds = 3$$

Then, the value of

$$\int_0^c \left( \int_x^c e^{x^2+y^2} dy \right) dx$$

equals \_\_\_\_\_ (rounded off to one decimal place).

- Q.58 For  $k \in \mathbb{N}$ , let  $0 = t_0 < t_1 < \dots < t_k < t_{k+1} = 1$ . A function  $f: [0,1] \rightarrow \mathbb{R}$  is said to be piecewise linear with nodes  $t_1, \dots, t_k$ , if for each  $j = 1, 2, \dots, k+1$ , there exist  $a_j \in \mathbb{R}$ ,  $b_j \in \mathbb{R}$  such that

$$f(t) = a_j + b_j t \text{ for } t_{j-1} < t < t_j.$$

Let  $V$  be the real vector space of all real valued continuous piecewise linear functions on  $[0,1]$  with nodes  $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ . Then, the dimension of  $V$  equals

\_\_\_\_\_

- Q.59 For  $n \in \mathbb{N}$ , let

$$a_n = \frac{1}{n^{n-1}} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \frac{n^k}{k+1}$$

and  $\beta = \lim_{n \rightarrow \infty} a_n$ . Then, the value of  $\log \beta$  equals \_\_\_\_\_ (rounded off to two decimal places).

- Q.60 Define the function  $f: (-1,1) \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  by

$$f(x) = \sin^{-1} x$$

Let  $a_6$  denote the coefficient of  $x^6$  in the Taylor series of  $(f(x))^2$  about

$x = 0$ . Then, the value of  $9a_6$  equals \_\_\_\_\_ (rounded off to two decimal places).