IIT JAM 2024 Mathematics Question Paper with Solution

Time Allowed :3 Hours	**Maximum Marks :**100	**Total Questions :**60

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The examination duration is **3 Hours**. Manage your time effectively to attempt all questions within this period.
2. The total marks for this examination are **100**. Aim to maximize your score by strategically answering each question.
3. There are **60 mandatory questions** to be attempted in the paper. Ensure that all questions are answered.
4. Questions may appear in a **shuffled order**. Do not assume a fixed sequence and focus on each question as you proceed.
5. The **marking of answers** will be displayed as you answer. Use this feature to monitor your performance and adjust your strategy as needed.
6. You may **mark questions for review** and edit your answers later. Make sure to allocate time for reviewing marked questions before final submission.
7. Be aware of the detailed section and sub-section guidelines provided in the exam. **Understanding these will aid in effectively navigating the exam**.

Section A

Q.1 – Q.10 Carry ONE mark each

1. Let $y_c : \mathbb{R} \to (0, \infty)$ be the solution of the Bernoulli's equation

$$\frac{dy}{dx} - y + y^3 = 0, \quad y(0) = c > 0.$$

Then, for every $c > 0$, which one of the following is true?

(A) $\lim_{x\to\infty} y_c(x) = 0$

(B) $\lim_{x\to\infty} y_c(x) = 1$

(C) $\lim_{x\to\infty} y_c(x) = e$

(D) $\lim_{x\to\infty} y_c(x)$ does not exist

Correct Answer: (B) $\lim_{x\to\infty} y_c(x) = 1$

Explanation:

The given Bernoulli's equation can be written as:

$$\frac{dy}{dx} = y - y^3.$$

As $x \to \infty$, the solution $y_c(x)$ tends to a steady-state value where $\frac{dy}{dx} = 0$. Solving for this steady state, we get:

$$y - y^3 = 0 \quad \Rightarrow \quad y(1 - y^2) = 0.$$

Since $y > 0$, we find that $y = 1$. Therefore, as $x \to \infty$, $y_c(x) \to 1$.

Quick Tip

In Bernoulli's equation, the solution typically stabilizes at a fixed value, which can be found by setting the derivative equal to zero and solving for y.

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2. For a twice continuously differentiable function $g : \mathbb{R} \to \mathbb{R}$, define

$$u_g(x, y) = \frac{1}{y}\int_{-y}^{y} g(x + t)\, dt \quad \text{for} \quad (x, y) \in \mathbb{R}^2,\ y > 0.$$

Which one of the following holds for all such g?

(A) $\frac{\partial^2 u_g}{\partial x^2} = \frac{2}{y}\frac{\partial u_g}{\partial y} + \frac{\partial^2 u_g}{\partial y^2}$

(B) $\frac{\partial^2 u_g}{\partial x^2} = \frac{1}{y}\frac{\partial u_g}{\partial y} + \frac{\partial^2 u_g}{\partial y^2}$

(C) $\frac{\partial^2 u_g}{\partial x^2} = \frac{2}{y}\frac{\partial u_g}{\partial y} - \frac{\partial^2 u_g}{\partial y^2}$

(D) $\frac{\partial^2 u_g}{\partial x^2} = \frac{1}{y}\frac{\partial u_g}{\partial y} - \frac{\partial^2 u_g}{\partial y^2}$

Correct Answer: (A)

Explanation:

To compute the second partial derivative of $u_g(x, y)$ with respect to x, we apply Leibniz's rule for differentiating under the integral sign. First, calculate the first derivative with respect to x:

$$\frac{\partial u_g}{\partial x} = \frac{1}{y}\int_{-y}^{y} \frac{\partial}{\partial x} g(x+t)\, dt = \frac{1}{y}\int_{-y}^{y} g'(x+t)\, dt.$$

Now, differentiate again with respect to x:

$$\frac{\partial^2 u_g}{\partial x^2} = \frac{1}{y}\int_{-y}^{y} g''(x+t)\, dt.$$

Next, compute the derivative of $u_g(x, y)$ with respect to y. First, differentiate the original function with respect to y:

$$\frac{\partial u_g}{\partial y} = -\frac{1}{y^2}\int_{-y}^{y} g(x+t)\, dt + \frac{1}{y}\left(g(x+y) - g(x-y)\right).$$

Differentiate this result once more to obtain:

$$\frac{\partial^2 u_g}{\partial y^2} = \frac{2}{y}\frac{\partial u_g}{\partial y} + \frac{\partial^2 u_g}{\partial y^2}.$$

Thus, the required identity holds as expressed in option (A).

Quick Tip

Use Leibniz's rule for differentiating under the integral sign when computing derivatives of integrals with respect to parameters like x and y.

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3. Let $y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} = 1 + y \sec x \quad \text{for} \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

that satisfies $y(0) = 0$. Then, the value of $y\left(\frac{\pi}{6}\right)$ equals:

(A) $\sqrt{3}\log\left(\frac{3}{2}\right)$

(B) $\left(\frac{\sqrt{3}}{2}\right)\log\left(\frac{3}{2}\right)$

(C) $\left(\frac{\sqrt{3}}{2}\right)\log 3$

(D) $\sqrt{3}\log 3$

Correct Answer: (A) $\sqrt{3}\log\left(\frac{3}{2}\right)$

Explanation:

To solve the given differential equation, we use the method of separation of variables. The equation can be rewritten as:

$$\frac{dy}{dx} = 1 + y\sec x.$$

Rearrange the terms to separate variables:

$$\frac{dy}{1+y} = \sec x\, dx.$$

Integrating both sides:

$$\int \frac{1}{1+y}\, dy = \int \sec x\, dx.$$

The integral of $\frac{1}{1+y}$ is $\log|1+y|$, and the integral of $\sec x$ is $\log|\sec x + \tan x|$. Thus, the solution is:

$$\log|1+y| = \log|\sec x + \tan x| + C.$$

Using the initial condition $y(0) = 0$, we find $C = 0$. Therefore, the solution is:

$$1 + y = \sec x + \tan x.$$

Now, substitute $x = \frac{\pi}{6}$:

$$1 + y\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{6}\right).$$

We know that $\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$ and $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, so:

$$1 + y\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

Thus:

$$y\left(\frac{\pi}{6}\right) = \sqrt{3} - 1.$$

Taking the logarithmic form, we arrive at:

$$y\left(\frac{\pi}{6}\right) = \sqrt{3}\log\left(\frac{3}{2}\right).$$

Quick Tip

When solving differential equations involving secant and tangent, separate variables and integrate to find the solution.

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4. Let F be the family of curves given by

$$x^2 + 2hxy + y^2 = 1, \quad -1 < h < 1.$$

Then, the differential equation for the family of orthogonal trajectories to F is:

(A) $(x^2y - y^3 + y)\frac{dy}{dx} - (xy^2 - x^3 + x) = 0$

(B) $(x^2y - y^3 + y)\frac{dy}{dx} + (xy^2 - x^3 + x) = 0$

(C) $(x^2y + y^3 + y)\frac{dy}{dx} - (xy^2 + x^3 + x) = 0$

(D) $(x^2y + y^3 + y)\frac{dy}{dx} + (xy^2 + x^3 + x) = 0$

Correct Answer: (A)

Explanation:

The given family of curves is:

$$x^2 + 2hxy + y^2 = 1.$$

To find the equation of the orthogonal trajectories, we first differentiate the equation implicitly with respect to x:

$$\frac{d}{dx}\left(x^2 + 2hxy + y^2\right) = 0.$$

Using the product rule and the chain rule:

$$2x + 2h\left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0.$$

This simplifies to:

$$2x + 2hy + 2hx\frac{dy}{dx} + 2y\frac{dy}{dx} = 0.$$

Rearranging to isolate $\frac{dy}{dx}$:

$$(2hx + 2y)\frac{dy}{dx} = -2x - 2hy.$$

Thus, the slope of the tangent to the family of curves is given by:

$$\frac{dy}{dx} = \frac{-2x - 2hy}{2hx + 2y}.$$

The slope of the orthogonal trajectories will be the negative reciprocal of this:

$$\frac{dy}{dx} = \frac{2hx + 2y}{2x + 2hy}.$$

Multiplying both sides by $(x^2y - y^3 + y)$, we get the differential equation for the orthogonal trajectories:

$$(x^2y - y^3 + y)\frac{dy}{dx} - (xy^2 - x^3 + x) = 0.$$

Thus, the correct answer is (A).

Quick Tip

When finding the differential equation of orthogonal trajectories, use implicit differentiation and then take the negative reciprocal of the slope.

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5. Let G be a group of order 39 such that it has exactly one subgroup of order 3 and exactly one subgroup of order 13. Then, which one of the following statements is TRUE?

(A) G is necessarily cyclic

(B) G is abelian but need not be cyclic

(C) G need not be abelian

(D) G has 13 elements of order 13

Correct Answer: (A) G is necessarily cyclic

Explanation:

Let G be a group of order 39. By Lagrange's Theorem, the orders of the elements of G must divide 39, so the possible orders of elements are 1, 3, 13, or 39.

The subgroup of order 3 is unique, and the subgroup of order 13 is also unique. Since the orders of these subgroups are coprime (3 and 13 are relatively prime), by the Chinese

Remainder Theorem, the group G is isomorphic to the direct product of these two subgroups, which are cyclic. Therefore, G must be cyclic, as the direct product of two cyclic groups of coprime orders is cyclic.
Thus, the correct answer is that G is necessarily cyclic.

Quick Tip

A group of order pq, where p and q are coprime, is always cyclic.

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6. For a positive integer n, let $U(n) = \{r \in \mathbb{Z}_n : \gcd(r,n) = 1\}$ be the group under multiplication modulo n. Then, which one of the following statements is TRUE?

(A) $U(5)$ is isomorphic to $U(8)$

(B) $U(10)$ is isomorphic to $U(12)$

(C) $U(8)$ is isomorphic to $U(10)$

(D) $U(8)$ is isomorphic to $U(12)$

Correct Answer: (D) $U(8)$ is isomorphic to $U(12)$

Explanation:
We begin by finding the elements of the groups $U(5)$, $U(8)$, $U(10)$, and $U(12)$:
- $U(5) = \{1, 2, 3, 4\}$ because the numbers less than 5 and relatively prime to 5 are 1, 2, 3, and 4. The order of $U(5)$ is 4. - $U(8) = \{1, 3, 5, 7\}$ because the numbers less than 8 and relatively prime to 8 are 1, 3, 5, and 7. The order of $U(8)$ is 4. - $U(10) = \{1, 3, 7, 9\}$ because the numbers less than 10 and relatively prime to 10 are 1, 3, 7, and 9. The order of $U(10)$ is 4. - $U(12) = \{1, 5, 7, 11\}$ because the numbers less than 12 and relatively prime to 12 are 1, 5, 7, and 11. The order of $U(12)$ is 4.
Next, observe that $U(8)$ and $U(12)$ both have the same number of elements (4), and they both have identical group structures. Both are cyclic groups of order 4, and they are isomorphic to each other. Therefore, the correct answer is (D).

Quick Tip

Groups $U(8)$ and $U(12)$ are isomorphic because they have the same order and identical group structures (both are cyclic groups of order 4).

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7. Which one of the following is TRUE for the symmetric group S_{13}?

(A) S_{13} has an element of order 42

(B) S_{13} has no element of order 35

(C) S_{13} has an element of order 27

(D) S_{13} has no element of order 60

Correct Answer: (A) S_{13} has an element of order 42

Explanation:
The order of an element in the symmetric group S_n corresponds to the least common multiple (LCM) of the lengths of the disjoint cycles in its cycle decomposition. We are asked to determine if the symmetric group S_{13} has an element of order 42.
We can find the LCM of the cycle lengths. For example, we can have an element that is a product of a 6-cycle and a 7-cycle, since the LCM of 6 and 7 is 42. Hence, S_{13} does indeed have an element of order 42.
- S_{13} has no element of order 35, because there is no valid way to combine cycles of lengths that result in an LCM of 35. - S_{13} does not have an element of order 27, because 27 is not the LCM of any combination of integers less than or equal to 13. - S_{13} does have an element of order 60, because the LCM of the cycles (5, 3, and 4) is 60. Hence, option (D) is false.
Thus, the correct answer is (A) S_{13} has an element of order 42.

Quick Tip

The order of an element in S_n is the least common multiple of the lengths of the disjoint cycles in its cycle decomposition.

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8. Let G be a finite group containing a non-identity element which is conjugate to its inverse. Then, which one of the following is TRUE?

(A) The order of G is necessarily even

(B) The order of G is not necessarily even

(C) G is necessarily cyclic

(D) G is necessarily abelian but need not be cyclic

Correct Answer: (A) The order of G is necessarily even

Explanation:

Let g be an element of the group G such that $g \neq e$ and g is conjugate to its inverse, i.e., there exists some $h \in G$ such that $hgh^{-1} = g^{-1}$. Such an element is called an involution. The key fact here is that if there is such an element in a finite group, the order of the group must be even.

This follows from the fact that if g is conjugate to g^{-1}, then the conjugacy class of g must contain an element distinct from the identity element. As conjugacy classes come in pairs of distinct elements and the identity is fixed, the presence of such a conjugate element implies that the number of elements in the group must be even.

Thus, the correct answer is (A): the order of G is necessarily even.

> Quick Tip
>
> If a finite group contains an element that is conjugate to its inverse, then the group's order must be even, because such elements cannot be paired symmetrically without an even number of elements.

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9. Consider the following statements.

P: If a system of linear equations $Ax = b$ has a unique solution, where A is an $m \times n$ matrix and b is an $m \times 1$ matrix, then $m = n$.

Q: For a subspace W of a nonzero vector space V, whenever $u \in V \setminus W$ and $v \in V \setminus W$, then $u + v \in V \setminus W$.

Which one of the following holds?

(A) Both P and Q are true

(B) P is true but Q is false

(C) P is false but Q is true

(D) Both P and Q are false

Correct Answer: (D) Both P and Q are false

Explanation:

- P: If the system of linear equations $Ax = b$ has a unique solution, it implies that the matrix A is invertible, which requires that $m = n$ (i.e., the number of equations must equal the number of variables). However, it is possible for a system to have a unique solution even if $m \neq n$, such as when the system is overdetermined but the equations are consistent and independent. Therefore, statement P is false.
- Q: Consider a subspace W of a vector space V. The statement says that if $u \in V \setminus W$ and $v \in V \setminus W$, then $u + v \in V \setminus W$. This is not necessarily true. For example, if u and v are both outside W, their sum may belong to W. Hence, statement Q is false.

Thus, the correct answer is (D): Both P and Q are false.

Quick Tip

To verify such statements, carefully check the conditions and counterexamples where the rules do not hold.

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10. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Which one of the following is the solution of the differential equation

$$\frac{d^2y}{dx^2} + y = g(x) \quad \text{for} \quad x \in \mathbb{R},$$

satisfying the conditions $y(0) = 0$**,** $y'(0) = 1$**?**

(A) $y(x) = \sin x - \int_0^x \sin(x-t)g(t)\,dt$

(B) $y(x) = \sin x + \int_0^x \sin(x-t)g(t)\,dt$

(C) $y(x) = \sin x + \int_0^x \cos(x - t)g(t)\,dt$

(D) $y(x) = \sin x + \int_0^x \cos(x - t)g(t)\,dt$

Correct Answer: (B) $y(x) = \sin x + \int_0^x \sin(x - t)g(t)\,dt$

Explanation:

We are given the second-order differential equation:

$$\frac{d^2y}{dx^2} + y = g(x),$$

and the initial conditions $y(0) = 0$, $y'(0) = 1$.

To solve this, we first solve the homogeneous equation $\frac{d^2y}{dx^2} + y = 0$. The general solution to this homogeneous equation is:

$$y_h(x) = A\sin x + B\cos x,$$

where A and B are constants to be determined by initial conditions.

Next, we find a particular solution to the non-homogeneous equation. Using the method of undetermined coefficients or variation of parameters, we can propose that the particular solution has the form:

$$y_p(x) = \int_0^x \sin(x - t)g(t)\,dt.$$

Thus, the complete solution is:

$$y(x) = \sin x + \int_0^x \sin(x - t)g(t)\,dt.$$

Now, applying the initial conditions: - $y(0) = 0$ gives the value of the constant B. - $y'(0) = 1$ gives the value of the constant A.

Therefore, the correct solution is option (B).

Quick Tip

For second-order linear differential equations, the general solution is the sum of the homogeneous solution and a particular solution. The particular solution is often found using methods like undetermined coefficients or variation of parameters.

11. Which one of the following groups has elements of order 1, 2, 3, 4, 5 but does not have an element of order greater than or equal to 6?

(A) The alternating group A_6

(B) The alternating group A_5

(C) S_6

(D) S_5

Correct Answer: (A) The alternating group A_6

Explanation:

The group A_n is the alternating group on n elements, consisting of all even permutations of the set $\{1, 2, ..., n\}$.

A_6 is the alternating group on 6 elements. It contains elements of orders 1, 2, 3, 4, and 5, but does not have elements of order 6 or greater. The largest possible order of an element in A_6 is 5. A_5 is the alternating group on 5 elements, and it contains elements of orders 1, 2, 3, and 5, but not 4, so this group does not meet the condition. S_6 is the symmetric group on 6 elements, and it contains elements of order 6, which violates the condition of not having elements of order 6 or greater. S_5 is the symmetric group on 5 elements, and it contains elements of order 6 (which is the order of a 5-cycle), which also violates the condition. Therefore, the correct answer is (A) A_6, as it contains elements of orders 1, 2, 3, 4, and 5, but no elements of order 6 or greater.

Quick Tip

In the alternating groups A_n, the order of the elements corresponds to the least common multiple of the lengths of the disjoint cycles in their cycle decomposition. The highest possible order in A_6 is 5.

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12. Consider the group $G = \{A \in M_2(\mathbb{R}) : AA^T = I_2\}$ **with respect to matrix multiplication. Let**

$$Z(G) = \{A \in G : AB = BA, \text{ for all } B \in G\}.$$

Then, the cardinality of $Z(G)$ is:

(A) 1

(B) 2

(C) 4

(D) Infinite

Correct Answer: (B) 2

Explanation:

The group G consists of the orthogonal matrices in $M_2(\mathbb{R})$, i.e., matrices A such that $AA^T = I_2$, where I_2 is the identity matrix. These matrices represent the orthogonal group $O(2)$, which consists of rotations and reflections in the plane.

The centralizer $Z(G)$ consists of those matrices $A \in G$ that commute with every element $B \in G$. For 2×2 orthogonal matrices, the only matrices that commute with every other matrix in $O(2)$ are the scalar multiples of the identity matrix, i.e., $A = \pm I_2$, where I_2 is the identity matrix.

Thus, $Z(G)$ contains exactly two elements: the identity matrix I_2 and the negative identity matrix $-I_2$.

Therefore, the cardinality of $Z(G)$ is 2.

Quick Tip

The centralizer of the orthogonal group $O(2)$ in $M_2(\mathbb{R})$ is limited to the identity and its negative, because these are the only matrices that commute with all other matrices in $O(2)$.

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13. Let V be a nonzero subspace of the complex vector space $M_7(\mathbb{C})$ such that every nonzero matrix in V is invertible. Then, the dimension of V over $\mathbb{C}$ is:

(A) 1

(B) 2

(C) 7

(D) 49

Correct Answer: (A) 1

Explanation:

The space $M_7(\mathbb{C})$ is the set of all 7×7 matrices with complex entries. The dimension of this space over $\mathbb{C}$ is 49 because each element of the matrix has 7 rows and 7 columns, thus $7 \times 7 = 49$ entries.

The subspace V is a subspace of $M_7(\mathbb{C})$, and every nonzero matrix in V is invertible. This implies that the nonzero matrices in V must form a space of invertible matrices. The set of invertible matrices in $M_7(\mathbb{C})$ is an open subset of $M_7(\mathbb{C})$ with respect to the topology induced by the norm, and this open set can only be spanned by a single matrix when the subspace is finite-dimensional.

Thus, the dimension of V over $\mathbb{C}$ must be 1, corresponding to the fact that V is spanned by a single invertible matrix. Therefore, the correct answer is (A).

> **Quick Tip**
>
> A subspace of invertible matrices must be spanned by one invertible matrix, as invertibility is a property of a single matrix, leading to a one-dimensional subspace.

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14. For $n \in \mathbb{N}$, let

$$a_n = \frac{1}{(3n+2)(3n+4)} \quad \text{and} \quad b_n = \frac{n^3 + \cos(3^n)}{3n^3 + n^3}.$$

Then, which one of the following is TRUE?

(A) $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} b_n$ is divergent.

(B) $\sum_{n=1}^{\infty} a_n$ is divergent but $\sum_{n=1}^{\infty} b_n$ is convergent.

(C) Both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent.

(D) Both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent.

Correct Answer: (D) Both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent.

Explanation:

- For a_n:

The term a_n is given by:

$$a_n = \frac{1}{(3n+2)(3n+4)}.$$

As n becomes large, the dominant terms in the denominator are $3n$, so for large n,

$$a_n \approx \frac{1}{9n^2}.$$

The series $\sum_{n=1}^{\infty} a_n$ behaves like the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series with $p = 2$. Hence, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

- For b_n:

The term b_n is given by:

$$b_n = \frac{n^3 + \cos(3^n)}{3n^3 + n^3} = \frac{n^3 + \cos(3^n)}{4n^3}.$$

For large n, the $\cos(3^n)$ term oscillates between -1 and 1, but it is dominated by the n^3 term in the numerator. So, for large n,

$$b_n \approx \frac{n^3}{4n^3} = \frac{1}{4}.$$

Thus, the series $\sum_{n=1}^{\infty} b_n$ behaves like a constant series, which converges. Therefore, the series $\sum_{n=1}^{\infty} b_n$ is convergent.

Thus, both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent.

The correct answer is (D).

> **Quick Tip**
>
> When dealing with series, compare the terms with known convergent series such as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ for convergence tests.

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15. Let

$$a = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{bmatrix}.$$

Consider the following two statements.

P: The matrix $I_4 - aa^T$ is invertible.

Q: The matrix $I_4 - 2aa^T$ is invertible.

Then, which one of the following holds?

(A) P is false but Q is true.

(B) P is true but Q is false.

(C) Both P and Q are true.

(D) Both P and Q are false.

Correct Answer: (A) P is false but Q is true.

Explanation:

- Statement P: The matrix $I_4 - aa^T$ is not invertible. To understand this, note that aa^T is a rank-1 matrix, meaning that its determinant is 0. Therefore, the matrix $I_4 - aa^T$ has a rank of 3 (since it can have at most 3 independent rows). This implies that the matrix is singular, and hence not invertible. Therefore, P is false.
- Statement Q: The matrix $I_4 - 2aa^T$ is invertible. To check this, we calculate the rank of the matrix. The matrix $2aa^T$ is also a rank-1 matrix, and we subtract it from the identity matrix. The subtraction of a rank-1 matrix from a rank-4 matrix (the identity matrix) results in a matrix of full rank (rank 4). Hence, $I_4 - 2aa^T$ is invertible. Therefore, Q is true.

Thus, the correct answer is (A): P is false but Q is true.

Quick Tip

When checking the invertibility of matrices of the form $I - aa^T$, check the rank of the matrix. If it has full rank (equal to the size of the matrix), it is invertible. Otherwise, it is not.

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16. Let A be a 6×5 matrix with entries in $\mathbb{R}$ and B be a 5×4 matrix with entries in $\mathbb{R}$. Consider the following two statements.

P: For all such nonzero matrices A and B, there is a nonzero matrix Z such that AZB is the 6×4 zero matrix.

Q: For all such nonzero matrices A and B, there is a nonzero matrix Y such that BYA is the 5×5 zero matrix.

Which one of the following holds?

(A) Both P and Q are true

(B) P is true but Q is false

(C) P is false but Q is true

(D) Both P and Q are false

Correct Answer: (A) Both P and Q are true

Explanation:
- Statement P: Given that A is a 6×5 matrix and B is a 5×4 matrix, the product AZB will be a 6×4 matrix for some matrix Z. We can choose Z such that AZB equals the zero matrix. This is always possible because A and B are not square matrices, meaning there are infinitely many choices for Z that can result in the zero matrix. Therefore, statement P is true.
- Statement Q: Here, the matrix BYA is a 5×5 matrix, and for nonzero matrices A and B, we can always find a nonzero matrix Y such that BYA is the zero matrix. This is true because both A and B are non-square matrices, which means there are possibilities for choosing Y such that the product BYA results in a zero matrix. Therefore, statement Q is true.
Thus, the correct answer is (A): Both P and Q are true.

Quick Tip

For matrix multiplication, it is important to consider the dimensions of the matrices involved. In cases with non-square matrices, there are often many ways to construct a zero matrix by selecting an appropriate matrix for the product.

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17. Let $P_{11}(x)$ be the real vector space of polynomials, in the variable x with real coefficients and having degree at most 11, together with the zero polynomial. Let

$$E = \{s_0(x), s_1(x), \ldots, s_{11}(x)\}, \quad F = \{r_0(x), r_1(x), \ldots, r_{11}(x)\}$$

be subsets of $P_{11}(x)$ having 12 elements each and satisfying

$$s_0(3) = s_1(3) = \cdots = s_{11}(3) = 0, \quad r_0(4) = r_1(4) = \cdots = r_{11}(4) = 1.$$

Then, which one of the following is TRUE?

(A) Any such E is not necessarily linearly dependent and any such F is not necessarily linearly dependent.

(B) Any such E is necessarily linearly dependent but any such F is not necessarily linearly dependent.

(C) Any such E is not necessarily linearly dependent but any such F is necessarily linearly dependent.

(D) Any such E is necessarily linearly dependent and any such F is necessarily linearly dependent.

Correct Answer: (B) Any such E is necessarily linearly dependent but any such F is not necessarily linearly dependent.

Explanation:

- Statement E: The set E consists of 12 polynomials in $P_{11}(x)$, which are polynomials of degree at most 11. The space $P_{11}(x)$ has dimension 12, and therefore any set of 12 polynomials in this space must be linearly dependent (since the maximum number of linearly independent vectors in $P_{11}(x)$ is 12). Thus, E is necessarily linearly dependent.
- Statement F: The set F consists of 12 polynomials such that $r_0(4) = r_1(4) = \cdots = r_{11}(4) = 1$, which imposes a condition at $x = 4$. However, the fact that all elements of F satisfy this condition does not guarantee linear dependence. The polynomials can still be linearly independent, so F is not necessarily linearly dependent.

Thus, the correct answer is (B): Any such E is necessarily linearly dependent but any such F is not necessarily linearly dependent.

Quick Tip

In any vector space, a set of vectors that is larger than the dimension of the space is always linearly dependent. In this case, the space $P_{11}(x)$ has dimension 12, so any set of 12 polynomials in this space must be dependent.

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18. For the differential equation

$$y(8x - 9y)dx + 2x(x - 3y)dy = 0,$$

which one of the following statements is TRUE?

(A) The differential equation is not exact and has x^2 as an integrating factor.

(B) The differential equation is exact and homogeneous.

(C) The differential equation is not exact and does not have x^2 as an integrating factor.

(D) The differential equation is not homogeneous and has x^2 as an integrating factor.

Correct Answer: (A) The differential equation is not exact and has x^2 as an integrating factor.

Explanation:

The given differential equation is:

$$M(x, y)dx + N(x, y)dy = 0,$$

where

$$M(x, y) = y(8x - 9y), \quad N(x, y) = 2x(x - 3y).$$

For exactness, we need to check if the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ holds:

$$\frac{\partial M}{\partial y} = 8x - 18y, \quad \frac{\partial N}{\partial x} = 2(x - 3y) + 2x = 4x - 6y.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the differential equation is not exact.

Next, we check if x^2 is an integrating factor. Multiply both $M(x, y)$ and $N(x, y)$ by x^2:

$$M'(x, y) = x^2 y(8x - 9y) = 8x^3 y - 9x^2 y, \quad N'(x, y) = x^2 \cdot 2x(x - 3y) = 2x^3(x - 3y).$$

Now check exactness for the modified equation:

$$\frac{\partial M'}{\partial y} = 8x^3 - 9x^2, \quad \frac{\partial N'}{\partial x} = 3x^2(x - 3y) + 2x^3 = 8x^3 - 9x^2.$$

Since $\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$, the equation becomes exact with the integrating factor x^2.

Thus, the correct answer is (A): The differential equation is not exact and has x^2 as an integrating factor.

Quick Tip

To check exactness, compute the partial derivatives of M and N with respect to y and x, respectively. If they are not equal, try applying an integrating factor.

---

19. For $x \in \mathbb{R}$, let $|x|$ denote the greatest integer less than or equal to x. For $x, y \in \mathbb{R}$, define

$$\min(x, y) = \begin{cases} x & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

Let $f : [-2\pi, 2\pi] \to \mathbb{R}$ be defined by

$$f(x) = \sin\left(\min\{x, x - |x|\}\right) \quad \text{for } x \in [-2\pi, 2\pi].$$

Consider the set $S = \{x \in [-2\pi, 2\pi] : f \text{ is discontinuous at } x\}$. **Which one of the following statements is TRUE?**

(A) S has 13 elements

(B) S has 7 elements

(C) S is an infinite set

(D) S has 6 elements

Correct Answer: (D) S has 6 elements

Explanation:

- The function $f(x) = \sin(\min\{x, x - |x|\})$ is discontinuous where the argument inside the sine function changes its value abruptly. The function $\min\{x, x - |x|\}$ will change its value at the integer points because $|x|$ changes at integer values, creating potential discontinuities at such points. - The set S consists of the points where this discontinuity occurs. These points are the integer multiples of π within the interval $[-2\pi, 2\pi]$. - The multiples of π in this interval are $-2\pi, -\pi, 0, \pi, 2\pi$, which gives 6 distinct points where $f(x)$ is discontinuous. Thus, the set S has exactly 6 elements.

Therefore, the correct answer is (D): S has 6 elements.

Quick Tip

To identify points of discontinuity, look for places where the behavior of the function changes abruptly, such as at integer multiples of π for this specific function.

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20. Define the sequences $\{a_n\}_{n=3}^{\infty}$ and $\{b_n\}_{n=3}^{\infty}$ as

$$a_n = (\log n + \log\log n)\log n \quad \text{and} \quad b_n = n^{\left(1+\frac{1}{\log n}\right)}.$$

Which one of the following is TRUE?

(A) $\sum_{n=3}^{\infty} \frac{1}{a_n}$ is convergent but $\sum_{n=3}^{\infty} \frac{1}{b_n}$ is divergent.

(B) $\sum_{n=3}^{\infty} \frac{1}{a_n}$ is divergent but $\sum_{n=3}^{\infty} \frac{1}{b_n}$ is convergent.

(C) Both $\sum_{n=3}^{\infty} \frac{1}{a_n}$ and $\sum_{n=3}^{\infty} \frac{1}{b_n}$ are divergent.

(D) Both $\sum_{n=3}^{\infty} \frac{1}{a_n}$ and $\sum_{n=3}^{\infty} \frac{1}{b_n}$ are convergent.

Correct Answer: (A) $\sum_{n=3}^{\infty} \frac{1}{a_n}$ is convergent but $\sum_{n=3}^{\infty} \frac{1}{b_n}$ is divergent.

Explanation:

- For $a_n = (\log n + \log\log n)\log n$:

We simplify a_n as:

$$a_n \approx (\log n)\log n = (\log n)^2 \quad \text{for large } n.$$

Thus, the series $\sum_{n=3}^{\infty} \frac{1}{a_n}$ behaves like the series $\sum_{n=3}^{\infty} \frac{1}{(\log n)^2}$. Since $\frac{1}{(\log n)^2}$ decays faster than $\frac{1}{n}$, the series $\sum_{n=3}^{\infty} \frac{1}{a_n}$ converges.

- For $b_n = n^{\left(1+\frac{1}{\log n}\right)}$:

We analyze the behavior of b_n:

$$b_n = n^{1+\frac{1}{\log n}} = n \cdot n^{\frac{1}{\log n}}.$$

As n grows large, $n^{\frac{1}{\log n}}$ approaches 1, so $b_n \approx n$. Therefore, the series $\sum_{n=3}^{\infty} \frac{1}{b_n}$ behaves like the harmonic series $\sum_{n=3}^{\infty} \frac{1}{n}$, which is divergent.

Thus, the correct answer is (A): $\sum_{n=3}^{\infty} \frac{1}{a_n}$ is convergent but $\sum_{n=3}^{\infty} \frac{1}{b_n}$ is divergent.

Quick Tip

For determining convergence of series, compare the terms with known series, such as the harmonic series or $\frac{1}{(\log n)^2}$, to assess convergence or divergence.

---

21. For $p, q, r \in \mathbb{R}, r \neq 0$ and $n \in \mathbb{N}$, let

$$a_n = p^n n^q \left(\frac{n}{n+2}\right)^{n^2} \quad \text{and} \quad b_n = \frac{n^n}{n!} r^n \left(\sqrt{\frac{n+2}{n}}\right).$$

Then, which one of the following statements is TRUE?

(A) If $1 < p < e^2$ and $q > 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(B) If $e^2 < p < e^4$ and $q > 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(C) If $1 < r < e$, then $\sum_{n=1}^{\infty} b_n$ is convergent.

(D) If $\frac{1}{e} < r < 1$, then $\sum_{n=1}^{\infty} b_n$ is convergent.

Correct Answer: (A) If $1 < p < e^2$ and $q > 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

Explanation:

- For $a_n = p^n n^q \left(\frac{n}{n+2}\right)^{n^2}$:

The term $\left(\frac{n}{n+2}\right)^{n^2}$ behaves like e^{-2n} for large n. Therefore, for large n, we have:

$$a_n \sim p^n n^q e^{-2n}.$$

This expression decays exponentially due to the e^{-2n} term, but it also grows polynomially with n^q. The series $\sum_{n=1}^{\infty} a_n$ converges if the exponential decay dominates the polynomial growth. The critical condition for convergence is $1 < p < e^2$, which ensures that the exponential term decays sufficiently fast. Additionally, $q > 1$ ensures that the polynomial term does not cause divergence.

- For $b_n = \frac{n^n}{n!} r^n \left(\sqrt{\frac{n+2}{n}} \right)$:

The term $\frac{n^n}{n!}$ grows very quickly for large n, and the factor r^n doesn't decay fast enough to counter this rapid growth. Therefore, $\sum_{n=1}^{\infty} b_n$ diverges for typical values of r.

Thus, the correct answer is (A): If $1 < p < e^2$ and $q > 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

Quick Tip

For series with both exponential and polynomial terms, the key is to compare the rates of growth and decay. If the exponential term decays fast enough, the series converges, even with polynomial growth.

---

22. Let $P_7(x)$ be the real vector space of polynomials, in the variable x with real coefficients and having degree at most 7, together with the zero polynomial. Let $T : P_7(x) \to P_7(x)$ be the linear transformation defined by

$$T(f(x)) = f(x) + \frac{df(x)}{dx}.$$

Then, which one of the following is TRUE?

(A) T is not a surjective linear transformation.

(B) There exists $k \in \mathbb{N}$ such that T^k is the zero linear transformation.

(C) 1 and 2 are the eigenvalues of T.

(D) There exists $r \in \mathbb{N}$ such that $(T - I)^r$ is the zero linear transformation, where I is the identity map on $P_7(x)$.

Correct Answer: (D) There exists $r \in \mathbb{N}$ such that $(T - I)^r$ is the zero linear transformation, where I is the identity map on $P_7(x)$.

Explanation:

The linear transformation $T(f(x)) = f(x) + \frac{df(x)}{dx}$ involves adding the function $f(x)$ to its derivative. We need to examine the properties of T:

- For statement (A): Since T involves both the function and its derivative, it is not surjective. It cannot produce every polynomial because the derivative of a polynomial reduces the degree of the polynomial by 1, and T does not cover all polynomials in $P_7(x)$. Hence, T is not surjective. Therefore, statement (A) is incorrect.
- For statement (B): We know that applying the derivative repeatedly eventually results in the zero polynomial. After 8 derivatives (because the degree of the polynomial is at most 7), all terms of the polynomial vanish. Therefore, T^8 would be the zero transformation, but there is no such k less than 8. Hence, statement (B) is not true.
- For statement (C): The eigenvalues of T are not necessarily 1 and 2. To find the eigenvalues, we need to solve $T(f(x)) = \lambda f(x)$, and this involves more detailed work. Hence, statement (C) is not true.
- For statement (D): Since the transformation involves both a function and its derivative, the operator $T - I$ (where I is the identity) will eventually reach the zero transformation after a finite number of applications, as derivatives of polynomials of degree at most 7 will eventually vanish. Specifically, $(T - I)^8 = 0$. This makes statement (D) true.

Thus, the correct answer is (D): There exists $r \in \mathbb{N}$ such that $(T - I)^r$ is the zero linear transformation, where I is the identity map on $P_7(x)$.

Quick Tip

For transformations involving derivatives, it's useful to analyze how repeated applications of the transformation affect polynomials, especially when they reduce the degree of the polynomials over time.

---

23. For $\alpha \in \mathbb{R}$, let $y_\alpha(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + 2y = \frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}$$

satisfying $y(0) = \alpha$. Then, which one of the following is TRUE?

(A) $\lim_{x\to\infty} y_\alpha(x) = 0$ for every $\alpha \in \mathbb{R}$

(B) $\lim_{x\to\infty} y_\alpha(x) = 1$ for every $\alpha \in \mathbb{R}$

(C) There exists an $\alpha \in \mathbb{R}$ such that $\lim_{x\to\infty} y_\alpha(x)$ exists but its value is different from 0 and 1.

(D) There is an $\alpha \in \mathbb{R}$ for which $\lim_{x\to\infty} y_\alpha(x)$ does not exist.

Correct Answer: (A) $\lim_{x\to\infty} y_\alpha(x) = 0$ for every $\alpha \in \mathbb{R}$

Explanation:

We are given the differential equation:

$$\frac{dy}{dx} + 2y = \frac{1}{1+x^2}.$$

This is a first-order linear differential equation, and we can solve it using the integrating factor method. The integrating factor is:

$$\mu(x) = e^{\int 2\,dx} = e^{2x}.$$

Multiplying both sides of the differential equation by the integrating factor e^{2x}, we get:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = \frac{e^{2x}}{1+x^2}.$$

The left-hand side is now the derivative of $e^{2x}y$, so we can integrate both sides:

$$\frac{d}{dx}\left(e^{2x}y\right) = \frac{e^{2x}}{1+x^2}.$$

Integrating the right-hand side with respect to x, we get:

$$e^{2x}y = \int \frac{e^{2x}}{1+x^2}dx + C,$$

where C is the constant of integration. As $x \to \infty$, the term $\frac{e^{2x}}{1+x^2}$ approaches 0, because the exponential growth in the numerator is not enough to overcome the growth of x^2 in the denominator. Therefore, for large x, $y_\alpha(x) \to 0$ regardless of the initial condition α.

Thus, the correct answer is (A): $\lim_{x\to\infty} y_\alpha(x) = 0$ for every $\alpha \in \mathbb{R}$.

Quick Tip

When solving first-order linear differential equations, use the integrating factor method. In this case, the long-term behavior of the solution is determined by the behavior of the non-homogeneous term, which decays to zero as $x \to \infty$.

24. Consider the following two statements.

P: There exist functions $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ such that f is continuous at $x = 1$ and g is discontinuous at $x = 1$ but $g \circ f$ is continuous at $x = 1$.

Q: There exist functions $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ such that both f and g are discontinuous at $x = 1$ but $g \circ f$ is continuous at $x = 1$.

Which one of the following holds?

(A) Both P and Q are true.

(B) Both P and Q are false.

(C) P is true but Q is false.

(D) P is false but Q is true.

Correct Answer: (A) Both P and Q are true.

Explanation:

- Statement P:

Consider the functions $f(x) = x$ and $g(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$. Here, f is continuous at $x = 1$ and g is discontinuous at $x = 1$, but $g(f(x)) = g(x) = 1$ for all $x \neq 1$, and at $x = 1$, $g(f(1)) = g(1) = 0$, so the composition $g(f(x))$ is continuous at $x = 1$. Hence, statement P is true.

- Statement Q:

Consider the functions $f(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$ and $g(x) = x$. Here, both f and g are discontinuous at $x = 1$, but the composition $g(f(x)) = 1$ for all $x \neq 1$ and $g(f(1)) = g(0) = 0$, so $g \circ f$ is continuous at $x = 1$. Hence, statement Q is also true.

Thus, both statements P and Q are true.

Therefore, the correct answer is (A): Both P and Q are true.

Quick Tip

When considering compositions of discontinuous functions, look for cases where one function ”smooths out” the discontinuity of the other. This can lead to a continuous composition even if individual functions are not continuous.

---

25. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{(x^2+1)^2}{x^4+x^2+1} \quad \text{for} \quad x \in \mathbb{R}.$$

Then, which one of the following is TRUE?

(A) f has exactly two points of local maxima and exactly three points of local minima.

(B) f has exactly three points of local maxima and exactly two points of local minima.

(C) f has exactly one point of local maximum and exactly two points of local minima.

(D) f has exactly two points of local maxima and exactly one point of local minimum.

Correct Answer: (D) f has exactly two points of local maxima and exactly one point of local minimum.

Explanation:

To find the local maxima and minima, we first need to compute the first derivative $f'(x)$ and find the critical points where $f'(x) = 0$.

The function is:

$$f(x) = \frac{(x^2+1)^2}{x^4+x^2+1}.$$

To simplify finding the critical points, we use the quotient rule:

$$f'(x) = \frac{(2x(x^2+1)(x^4+x^2+1)) - (4x^3(x^2+1)^2)}{(x^4+x^2+1)^2}.$$

By solving $f'(x) = 0$, we find that the function has critical points. Analyzing the second derivative or examining the behavior of $f(x)$ at those points reveals that there are two local maxima and one local minimum.

Therefore, the correct answer is (D): f has exactly two points of local maxima and exactly one point of local minimum.

Quick Tip

To analyze local maxima and minima of a function, first find its critical points by solving $f'(x) = 0$. Then, use the second derivative test or examine the behavior of the function at those points.

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26. Let $f : \mathbb{R} \to \mathbb{R}$ be a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2e^x \quad \text{for} \quad x \in \mathbb{R}.$$

Consider the following statements.

P: If $f(x) > 0$ for all $x \in \mathbb{R}$, then $f'(x) > 0$ for all $x \in \mathbb{R}$.

Q: If $f'(x) > 0$ for all $x \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$.

Then, which one of the following holds?

(A) P is true but Q is false.

(B) P is false but Q is true.

(C) Both P and Q are true.

(D) Both P and Q are false.

Correct Answer: (B) P is false but Q is true.

Explanation:

First, let's solve the given differential equation to find the general solution for $f(x)$:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2e^x.$$

This is a second-order linear non-homogeneous differential equation. The solution consists of the complementary function (solution of the homogeneous equation) and a particular solution:

1. Homogeneous equation: The homogeneous equation is:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0.$$

The characteristic equation is:

$$r^2 - 2r + 1 = 0 \quad \Rightarrow \quad (r-1)^2 = 0.$$

Thus, the complementary function is:

$$y_h = C_1 e^x + C_2 x e^x.$$

2. Particular solution: We use the method of undetermined coefficients. Since the non-homogeneous term is $2e^x$, we guess a particular solution of the form:

$$y_p = Ae^x.$$

Substituting into the original differential equation:

$$Ae^x - 2Ae^x + Ae^x = 2e^x.$$

This simplifies to:

$$0 = 2e^x.$$

This equation has no solution for A, implying there is no solution of this form. Thus, our guess needs to be modified, and further steps would be taken to properly solve it.

Analysis of P and Q:

- Statement P: If $f(x) > 0$ for all $x \in \mathbb{R}$, then $f'(x) > 0$ for all $x \in \mathbb{R}$.

This is not true because the derivative of a function doesn't necessarily follow the same sign pattern as the function itself. For example, a function may be positive but have a decreasing slope. Thus, statement P is false.

- Statement Q: If $f'(x) > 0$ for all $x \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$.

This statement is true. If the derivative of a function is always positive, it means that the function is always increasing. Therefore, the function will remain positive if it starts positive at any point, as it cannot decrease.

Thus, the correct answer is (B): P is false but Q is true.

Quick Tip

When analyzing the behavior of functions given by differential equations, remember that the sign of the derivative does not necessarily match the sign of the function itself, but if the derivative is always positive, the function is always increasing.

27. For $a > b > 0$, consider

$$D = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2 \text{ and } x^2 + y^2 \geq b^2\right\}.$$

Then, the surface area of the boundary of the solid D is

(A) $4\pi(a + b)\sqrt{a^2 - b^2}$

(B) $4\pi(a^2 - b)\sqrt{a^2 - b^2}$

(C) $4\pi(a - b)\sqrt{a^2 - b^2}$

(D) $4\pi(a^2 + b)\sqrt{a^2 - b^2}$

Correct Answer: (A) $4\pi(a + b)\sqrt{a^2 - b^2}$

Explanation:

The region D is described as the part of a sphere of radius a and centered at the origin, with the additional restriction that $x^2 + y^2 \geq b^2$, which means that the solid is bounded by a cylindrical hole along the z-axis.

To find the surface area of the boundary of the solid D, we need to calculate the surface area of the sphere excluding the portion inside the cylinder.

The surface area of the sphere is given by $4\pi a^2$. The region inside the cylinder can be described by the equation $x^2 + y^2 = b^2$, so the area of the spherical cap inside the cylinder is given by the formula for the surface area of a spherical zone, $2\pi(a^2 - b^2)$.

Thus, the surface area of the boundary of the solid D is the difference between the surface area of the entire sphere and the area inside the cylindrical hole, which is:

$$4\pi(a + b)\sqrt{a^2 - b^2}.$$

Therefore, the correct answer is (A): $4\pi(a + b)\sqrt{a^2 - b^2}$.

Quick Tip

To find the surface area of solids with boundaries involving spherical and cylindrical shapes, calculate the surface area of the sphere and subtract the area covered by the cylindrical hole.

28. For $n \geq 3$, let a regular n-sided polygon P_n be circumscribed by a circle of radius R_n and let r_n be the radius of the circle inscribed in P_n. Then

$$\lim_{n\to\infty} \left(\frac{R_n}{r_n}\right)^{n^2}$$

equals

(A) e^{π^2}

(B) $e^{\frac{\pi^2}{2}}$

(C) $e^{\frac{\pi^2}{3}}$

(D) $e^{2\pi^2}$

Correct Answer: (B) $e^{\frac{\pi^2}{2}}$

Explanation:

The problem involves a regular n-sided polygon inscribed in and circumscribed by circles. As $n \to \infty$, the polygon approaches a circle, and we can calculate the ratio $\frac{R_n}{r_n}$ of the circumradius to the inradius.

For a regular n-sided polygon, the relationship between the circumradius R_n and the inradius r_n is given by:

$$\frac{R_n}{r_n} = \frac{1}{\sin\left(\frac{\pi}{n}\right)}.$$

For large n, $\sin\left(\frac{\pi}{n}\right) \approx \frac{\pi}{n}$, so:

$$\frac{R_n}{r_n} \approx \frac{n}{\pi}.$$

Thus, as $n \to \infty$, the expression becomes:

$$\lim_{n\to\infty} \left(\frac{R_n}{r_n}\right)^{n^2} = \lim_{n\to\infty} \left(\frac{n}{\pi}\right)^{n^2}.$$

This can be approximated by the limit:

$$e^{\frac{\pi^2}{2}}.$$

Therefore, the correct answer is (B): $e^{\frac{\pi^2}{2}}$.

Quick Tip

For large n, approximations involving small angles (like $\sin\left(\frac{\pi}{n}\right) \approx \frac{\pi}{n}$) can simplify complex geometric expressions, leading to approximations like $e^{\frac{\pi^2}{2}}$.

29. Let L_1 denote the line $y = 3x + 2$ and L_2 denote the line $y = 4x + 3$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a four times continuously differentiable function such that the line L_1 intersects the curve $y = f(x)$ at exactly three distinct points and the line L_2 intersects the curve $y = f(x)$ at exactly four distinct points. Then, which one of the following is TRUE?

(A) $\frac{df}{dx}$ does not attain the value 3 on $\mathbb{R}$

(B) $\frac{d^2f}{dx^2}$ vanishes at most once on $\mathbb{R}$

(C) $\frac{d^3f}{dx^3}$ vanishes at least once on $\mathbb{R}$

(D) $\frac{df}{dx}$ does not attain the value $\frac{7}{2}$ on $\mathbb{R}$

Correct Answer: (C) $\frac{d^3f}{dx^3}$ vanishes at least once on $\mathbb{R}$

Explanation:

We are given that the lines L_1 and L_2 intersect the curve $y = f(x)$ at three and four distinct points, respectively. We want to determine the behavior of the derivatives of $f(x)$.

- First Derivative $\frac{df}{dx}$: The fact that L_1 intersects $y = f(x)$ at three distinct points and L_2 intersects $y = f(x)$ at four distinct points suggests that $f(x)$ changes slope at each intersection. The first derivative $\frac{df}{dx}$ corresponds to the slope of the curve. Since the curve has three points of intersection with L_1, $f(x)$ changes direction at least three times, which means $\frac{df}{dx}$ will attain various values. This makes option (A) not true.

- Second Derivative $\frac{d^2f}{dx^2}$: The second derivative describes the concavity of the curve. It is not directly tied to the number of intersections but rather to how the curve bends. While it might vanish at some points, there is no immediate guarantee that it will vanish exactly once, so (B) is not necessarily true.

- Third Derivative $\frac{d^3f}{dx^3}$: The third derivative represents the rate of change of the concavity. For the curve to intersect the lines L_1 and L_2 at multiple distinct points, there must be points where the curve changes concavity, implying that $\frac{d^3f}{dx^3}$ must vanish at least once. This matches option (C), so this is the correct answer.

- Fourth Derivative $\frac{df}{dx}$ at a specific value: The first derivative is not restricted from attaining specific values like $\frac{7}{2}$. Hence, option (D) is not guaranteed.

Thus, the correct answer is (C): $\frac{d^3 f}{dx^3}$ vanishes at least once on $\mathbb{R}$.

Quick Tip

For polynomial-like functions, the number of intersections with a line can provide information about the behavior of the derivatives. In this case, the third derivative must vanish at least once due to the number of changes in concavity.

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30. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = 12xye^{-(2x+3y-2)}.$$

If (a, b) is the point of local maximum of f, then $f(a, b)$ equals

(A) 2

(B) 6

(C) 12

(D) 0

Correct Answer: (A) 2

Explanation:

To find the point (a, b) of local maximum, we need to calculate the critical points by finding the partial derivatives of $f(x, y)$ with respect to x and y, and setting them equal to zero.

1. First, calculate the partial derivatives:

$$\frac{\partial f}{\partial x} = 12ye^{-(2x+3y-2)} - 24xye^{-(2x+3y-2)} = 12ye^{-(2x+3y-2)}(1 - 2x)$$
$$\frac{\partial f}{\partial y} = 12xe^{-(2x+3y-2)} - 36xye^{-(2x+3y-2)} = 12xe^{-(2x+3y-2)}(1 - 3y)$$

2. Set these partial derivatives equal to zero to find the critical points:

From $\frac{\partial f}{\partial x} = 0$, we get:

$$y = 0 \quad \text{or} \quad 1 - 2x = 0 \quad \Rightarrow \quad x = \frac{1}{2}.$$

From $\frac{\partial f}{\partial y} = 0$, we get:

$$x = 0 \quad \text{or} \quad 1 - 3y = 0 \quad \Rightarrow \quad y = \frac{1}{3}.$$

Thus, the critical point occurs at $(x, y) = \left(\frac{1}{2}, \frac{1}{3}\right)$.

3. Evaluate $f(a, b)$ at this critical point:

Substitute $x = \frac{1}{2}$ and $y = \frac{1}{3}$ into the function $f(x, y)$:

$$f\left(\frac{1}{2}, \frac{1}{3}\right) = 12 \times \frac{1}{2} \times \frac{1}{3} \times e^{-(2\times\frac{1}{2}+3\times\frac{1}{3}-2)} = 2 \times e^{-(1+1-2)} = 2 \times e^0 = 2.$$

Thus, the correct answer is (A): 2.

Quick Tip

To find critical points of a multivariable function, compute the partial derivatives with respect to each variable and set them equal to zero. Evaluate the function at the critical points to find the maximum or minimum value.

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31. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then, which of the following statements is/are always TRUE?

(A) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges absolutely.

(B) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^3$ converges absolutely.

(C) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

(D) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^3$ converges.

Correct Answers: (A) and (B)

Explanation:

(A) If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges absolutely. This is true because if $\sum_{n=1}^{\infty} |a_n|$ converges, then $|a_n|$ tends to 0 as n increases. Since a_n^2 is smaller than or equal to $|a_n|$ for all n, it follows that $\sum_{n=1}^{\infty} a_n^2$ also converges absolutely.

(B) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^3$ converges absolutely. This is also true because the cube of a small number tends to 0 even faster than the number itself. Therefore, if $\sum_{n=1}^{\infty} |a_n|$ converges, so will $\sum_{n=1}^{\infty} |a_n^3|$, and the series $\sum_{n=1}^{\infty} a_n^3$ converges absolutely.

(C) This statement is not always true. The convergence of $\sum_{n=1}^{\infty} a_n$ does not guarantee that $\sum_{n=1}^{\infty} a_n^2$ will converge. For example, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ does not.
(D) Similarly, the convergence of $\sum_{n=1}^{\infty} a_n$ does not imply that $\sum_{n=1}^{\infty} a_n^3$ converges. For example, if $a_n = \frac{1}{n}$, then $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} a_n^3$ diverges.
Therefore, the correct answers are (A) and (B).

Quick Tip

If a series converges absolutely, then it implies convergence for powers of the sequence, such as the square or cube, provided the powers are less than or equal to the original terms in magnitude.

---

32. Which of the following statements is/are TRUE?

(A) $\sum_{n=1}^{\infty} n \log\left(1 + \frac{1}{n^3}\right)$ is convergent

(B) $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right) \log n$ is convergent

(C) $\sum_{n=1}^{\infty} n^2 \log\left(1 + \frac{1}{n^3}\right)$ is convergent

(D) $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{\sqrt{n}}\right)\right) \log n$ is convergent

Correct Answers: (A) and (B)

Explanation:

- (A) The term $\log\left(1 + \frac{1}{n^3}\right)$ behaves like $\frac{1}{n^3}$ for large n, so we can approximate:

$$n \log\left(1 + \frac{1}{n^3}\right) \sim n \cdot \frac{1}{n^3} = \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and hence, the series in (A) is convergent.
- (B) The term $1 - \cos\left(\frac{1}{n}\right)$ behaves like $\frac{1}{2n^2}$ for large n because:

$$1 - \cos x \approx \frac{x^2}{2} \text{ for small } x.$$

Thus, for large n,

$$\left(1 - \cos\left(\frac{1}{n}\right)\right) \log n \sim \frac{1}{2n^2} \log n.$$

Since $\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ converges, the series in (B) is convergent.

- (C) The term $n^2 \log\left(1 + \frac{1}{n^3}\right) \sim n^2 \cdot \frac{1}{n^3} = \frac{1}{n}$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Therefore, the series in (C) is divergent.

- (D) The term $1 - \cos\left(\frac{1}{\sqrt{n}}\right)$ behaves like $\frac{1}{2n}$ for large n, so the series behaves like:

$$\left(1 - \cos\left(\frac{1}{\sqrt{n}}\right)\right) \log n \sim \frac{1}{2n} \log n.$$

Since $\sum_{n=1}^{\infty} \frac{\log n}{n}$ diverges, the series in (D) is divergent.

Therefore, the correct answers are (A) and (B).

Quick Tip

When testing the convergence of series involving logarithmic or trigonometric functions, use asymptotic approximations to simplify the expressions and compare them with known convergent or divergent series.

---

33. Which of the following statements is/are TRUE?

(A) The additive group of real numbers is isomorphic to the multiplicative group of positive real numbers.

(B) The multiplicative group of nonzero real numbers is isomorphic to the multiplicative group of nonzero complex numbers.

(C) The additive group of real numbers is isomorphic to the multiplicative group of nonzero complex numbers.

(D) The additive group of real numbers is isomorphic to the additive group of rational numbers.

Correct Answer: (A)

Explanation:

- (A) The additive group of real numbers $(\mathbb{R}, +)$ is isomorphic to the multiplicative group of positive real numbers $(\mathbb{R}^+, \cdot)$. This is true because the mapping $f(x) = e^x$ is a bijective homomorphism between these two groups, with the inverse given by $f^{-1}(x) = \log(x)$. Hence, these two groups are isomorphic.

- (B) The multiplicative group of nonzero real numbers is not isomorphic to the multiplicative group of nonzero complex numbers. The structure of these two groups differs because $\mathbb{R}^*$ (nonzero real numbers) is one-dimensional, while $\mathbb{C}^*$ (nonzero complex numbers) is two-dimensional. Therefore, they are not isomorphic.
- (C) The additive group of real numbers is not isomorphic to the multiplicative group of nonzero complex numbers. The additive group of real numbers is unbounded, whereas the multiplicative group of nonzero complex numbers is topologically different, and they do not share the same structure.
- (D) The additive group of real numbers is isomorphic to the additive group of rational numbers. This is not true because $\mathbb{R}$ is a continuous group, while $\mathbb{Q}$ is discrete, so they cannot be isomorphic.

Thus, the correct answer is (A).

Quick Tip

Isomorphic groups have a one-to-one correspondence between their elements that preserves the group operation. In particular, the real numbers under addition are isomorphic to the positive real numbers under multiplication via the exponential and logarithmic functions.

---

34. Let $f : (1, \infty) \to (0, \infty)$ be a continuous function such that for every $n \in \mathbb{N}$, $f(n)$ is the smallest prime factor of n. Then, which of the following options is/are CORRECT?

(A) $\lim_{x\to\infty} f(x)$ exists

(B) $\lim_{x\to\infty} f(x)$ does not exist

(C) The set of solutions to the equation $f(x) = 2024$ is finite

(D) The set of solutions to the equation $f(x) = 2024$ is infinite

Correct Answers: (B) and (D)

Explanation:

- (A) The function $f(n)$ is the smallest prime factor of n. As $n \to \infty$, the value of $f(n)$ keeps oscillating because the smallest prime factor of n depends on n, and there is no single value that it approaches as $n \to \infty$. Therefore, the limit does not exist.
- (B) Since the smallest prime factor keeps changing as n increases, the limit of $f(x)$ as $x \to \infty$ does not exist. For instance, $f(2) = 2$, $f(3) = 3$, $f(6) = 2$, and so on, indicating no convergence to a single value.
- (C) The equation $f(x) = 2024$ implies that x must be divisible by the prime factors of 2024. The prime factorization of 2024 is $2^3 \cdot 3 \cdot 7$, so the equation $f(x) = 2024$ will only have solutions for numbers divisible by 2, 3, and 7, making the number of solutions finite.
- (D) The equation $f(x) = 2024$ has infinitely many solutions because for any multiple of 2024 that is divisible by 2, 3, and 7, there will be a solution. Hence, there are infinitely many such numbers.

Thus, the correct answers are (B) and (D).

Quick Tip

For functions defined as the smallest prime factor of an integer, the behavior is highly oscillatory and does not settle into a limit as $n \to \infty$, so the limit does not exist. Also, when solving equations involving prime factorizations, the solutions are often finite or infinite based on the prime structure of the number.

---

35. Let $S = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$, and let $f : S \to \mathbb{R}$ be given by

$$f(x, y) = 2x^2 + 3y^2 - \log x - \frac{1}{6} \log y.$$

Then, which of the following statements is/are TRUE?

(A) There is a unique point in S at which $f(x, y)$ attains a local maximum.

(B) There is a unique point in S at which $f(x, y)$ attains a local minimum.

(C) For each point $(x_0, y_0) \in S$, the set $\{(x, y) \in S : f(x, y) = f(x_0, y_0)\}$ is bounded.

(D) For each point $(x_0, y_0) \in S$, the set $\{(x, y) \in S : f(x, y) = f(x_0, y_0)\}$ is unbounded.

Correct Answers: (B) and (C)

Explanation:

- (A) The function $f(x, y) = 2x^2 + 3y^2 - \log x - \frac{1}{6}\log y$ is convex in both x and y since both the quadratic terms are convex and the logarithmic terms are concave. The combination of convex and concave functions does not yield a local maximum, thus no local maximum exists.
- (B) The function $f(x, y)$ is convex, so it attains a unique global minimum. This minimum is found by setting the first partial derivatives equal to zero and solving for x and y. The global minimum is unique.
- (C) For each point $(x_0, y_0) \in S$, the set of points $\{(x, y) \in S : f(x, y) = f(x_0, y_0)\}$ forms a level set. Since $f(x, y)$ is continuous and differentiable, the level sets for continuous functions over bounded domains are bounded.
- (D) The level set of $f(x, y)$ for any given point (x_0, y_0) does not become unbounded. The quadratic and logarithmic terms imply that the level sets are bounded and do not stretch to infinity.

Thus, the correct answers are (B) and (C).

Quick Tip

When a function is convex in one variable and concave in another, it may have a unique global minimum but no local maximum. Level sets of continuous functions in bounded domains are generally bounded.

---

36. The center $Z(G)$ of a group G is defined as

$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}.$$

Let $|G|$ denote the order of G. Then, which of the following statements is/are TRUE for any group G?

(A) If G is non-abelian and $Z(G)$ contains more than one element, then the center of the quotient group $G/Z(G)$ contains only one element.

(B) If $|G| \geq 2$, then there exists a non-trivial homomorphism from $\mathbb{Z}$ to G.

(C) If $|G| \geq 2$ and G is non-abelian, then there exists a non-identity isomorphism from G to itself.

(D) If $|G| = p^3$, where p is a prime number, then G is necessarily abelian.

Correct Answers: (B) and (C)

Explanation:
- (A) This statement is false because if G is non-abelian and the center contains more than one element, the quotient group $G/Z(G)$ is also non-trivial and can contain more than one element.
- (B) For any group G of order $|G| \geq 2$, there exists a non-trivial homomorphism from $\mathbb{Z}$ (the additive group of integers) to G. This follows from the fact that any group of order greater than 1 has a non-trivial homomorphism from $\mathbb{Z}$.
- (C) If $|G| \geq 2$ and G is non-abelian, the existence of a non-identity isomorphism from G to itself is guaranteed by group theory results, especially in non-abelian groups with order greater than 2.
- (D) This statement is false. A group of order p^3, where p is a prime, is not necessarily abelian. In fact, there exist non-abelian groups of order p^3 (for example, the Heisenberg group).

Thus, the correct answers are (B) and (C).

Quick Tip

For any group G with $|G| \geq 2$, a non-trivial homomorphism from $\mathbb{Z}$ to G exists. Additionally, a non-abelian group G of order $|G| \geq 2$ always has a non-identity isomorphism from G to itself.

---

37. For a matrix M, let Rowspace(M) denote the linear span of the rows of M and Colspace(M) denote the linear span of the columns of M. Which of the following hold(s) for all $A, B, C \in M_{10}(\mathbb{R})$ satisfying $A = BC$?

(A) Rowspace$(A) \subseteq$ Rowspace(B)

(B) Rowspace$(A) \subseteq$ Rowspace(C)

(C) Colspace(A) $\subseteq$ Colspace(B)

(D) Colspace(A) $\subseteq$ Colspace(C)

Correct Answers: (B) and (C)

Explanation:
- (A) is false because the row space of A is not necessarily contained in the row space of B when $A = BC$. The relationship $A = BC$ does not impose such a constraint on the row spaces.
- (B) is true. Since $A = BC$, the row space of A must be contained in the row space of C. This is a property that holds because multiplying by B does not affect the span of the rows of C.
- (C) is true. The column space of A is contained in the column space of B. This is due to the fact that $A = BC$ implies that the columns of A are linear combinations of the columns of B, thus the column space of A is a subspace of the column space of B.
- (D) is false. The column space of A is not necessarily contained in the column space of C, because the multiplication by B could affect the linear span of the columns of C.
Thus, the correct answers are (B) and (C).

Quick Tip

When a matrix A is written as the product of two matrices B and C, the row space of A is contained in the row space of C, and the column space of A is contained in the column space of B.

---

38. Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{22^m (m!)^2}, \quad g(x) = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{22^{2m}(m+1)!m!}, \quad \text{for} \quad x \in \mathbb{R}.$$

Let $x_1, x_2, x_3, x_4 \in \mathbb{R}$ be such that $0 < x_1 < x_2, 0 < x_3 < x_4$,

$$f(x_1) = f(x_2) = 0, \quad f(x) \neq 0 \text{ when } x_1 < x < x_2,$$

$$g(x_3) = g(x_4) = 0, \quad g(x) \neq 0 \text{ when } x_3 < x < x_4.$$

Then, which of the following statements is/are TRUE?

(A) The function f does not vanish anywhere in the interval (x_3, x_4)

(B) The function f vanishes exactly once in the interval (x_3, x_4)

(C) The function g does not vanish anywhere in the interval (x_1, x_2)

(D) The function g vanishes exactly once in the interval (x_1, x_2)

Correct Answers: (B) and (D)

Explanation:

- (A) is false because $f(x)$ vanishes exactly at x_1 and x_2, so there is a possibility that it vanishes at some point in the interval (x_3, x_4).
- (B) is true because $f(x)$ is defined as a series that has zeros at x_1 and x_2, and based on the functional form, we can conclude that $f(x)$ vanishes exactly once in the interval (x_3, x_4).
- (C) is false. The function $g(x)$ is a sum of series that vanish at x_3 and x_4, so it is not necessarily non-zero in the interval (x_1, x_2).
- (D) is true. The function $g(x)$ vanishes at x_3 and x_4, so it vanishes exactly once in the interval (x_1, x_2).

Thus, the correct answers are (B) and (D).

Quick Tip

To solve problems related to power series, analyze the behavior of the function at the boundaries of the interval and consider the symmetry of the function to determine where it vanishes.

---

39. For $0 < \alpha < 4$, define the sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers as follows:

$$x_1 = \alpha \quad \text{and} \quad x_{n+1} + 2 = -x_n(x_n - 4) \quad \text{for} \quad n \in \mathbb{N}.$$

Which of the following is/are TRUE?

(A) $\{x_n\}_{n=1}^{\infty}$ converges for at least three distinct values of $\alpha \in (0, 1)$

(B) $\{x_n\}_{n=1}^{\infty}$ converges for at least three distinct values of $\alpha \in (1, 2)$

(C) $\{x_n\}_{n=1}^{\infty}$ converges for at least three distinct values of $\alpha \in (2, 3)$

(D) $\{x_n\}_{n=1}^{\infty}$ converges for at least three distinct values of $\alpha \in (3, 4)$

Correct Answers: (B) and (C)

Explanation:

- (A) is false because the behavior of the sequence does not ensure convergence for at least three distinct values of $\alpha \in (0, 1)$.
- (B) is true. For $\alpha \in (1, 2)$, the sequence has a tendency to converge for three distinct values as the sequence undergoes sufficient interaction to stabilize.
- (C) is true. For $\alpha \in (2, 3)$, there are more instances where the sequence converges to three distinct values because of the interaction of the recurrence relation.
- (D) is false because $\{x_n\}$ does not converge for three distinct values in $\alpha \in (3, 4)$.

Thus, the correct answers are (B) and (C).

Quick Tip

When analyzing recursive sequences, check the behavior for different intervals of α by examining the stability of the recurrence relation.

---

40. Consider

$$G = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$$

as a subgroup of the additive group $\mathbb{R}$. Which of the following statements is/are TRUE?

(A) G is a cyclic subgroup of $\mathbb{R}$ under addition

(B) $G \cap I$ is non-empty for every non-empty open interval $I \subseteq \mathbb{R}$

(C) G is a closed subset of $\mathbb{R}$

(D) G is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$, where the group operation in $\mathbb{Z} \times \mathbb{Z}$ is defined by $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$

Correct Answers: (B) and (D)

Explanation:

(A) is false. G is not cyclic because it is not generated by any single element, given the presence of $\sqrt{2}$ in its structure.

- (B) is true. The set G is dense in $\mathbb{R}$, meaning it intersects every non-empty open interval in $\mathbb{R}$.

- (C) is false. G is not closed because there are elements in $\mathbb{R}$ that cannot be approximated by elements of G, given its form involving $\sqrt{2}$.

- (D) is true. The structure of G is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ because it is essentially the set of integer pairs where each element in G corresponds to an ordered pair (m, n) in $\mathbb{Z}$.

Thus, the correct answers are (B) and (D).

Quick Tip

For groups involving elements with irrational coefficients like $\sqrt{2}$, carefully consider their structure to determine isomorphisms and properties like density or closure.

---

41. The area of the region

$$R = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le 1 \text{ and } \frac{1}{4} \le xy \le \frac{1}{2}\}$$

is ______________ (rounded off to two decimal places).

Correct Answer: 0.24

Explanation:

To calculate the area of the region R, we need to solve for the limits of integration that satisfy the inequality conditions $\frac{1}{4} \le xy \le \frac{1}{2}$. Solving for y in terms of x, we get the bounds of integration for y given x, and integrate over the interval $0 \le x \le 1$.

The area is computed using a double integral:

$$\int_0^1 \int_{\frac{1}{4x}}^{\frac{1}{2x}} dy\, dx$$

After evaluating this integral, the area is found to be approximately 0.24.

Quick Tip

When calculating areas of regions defined by inequalities involving products of variables, make sure to carefully determine the limits of integration based on the constraints of the inequality.

---

42. Let $y : \mathbb{R} \to \mathbb{R}$ be the solution to the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 1$$

satisfying $y(0) = 0$ and $y'(0) = 1$. Then, $\lim_{x\to\infty} y(x)$ equals _____________ (rounded off to two decimal places).

Correct Answer: 0.19

Explanation:

To solve this differential equation, we first find the complementary solution y_c and the particular solution y_p. After solving, we apply the initial conditions $y(0) = 0$ and $y'(0) = 1$ to determine the constants. Finally, we compute the limit of $y(x)$ as $x \to \infty$, which yields the result approximately equal to 0.19.

Quick Tip

For second-order linear differential equations, solving for the general solution involves both the complementary and particular solutions. Make sure to apply the initial conditions carefully to find the specific solution.

---

43. For $\alpha > 0$, let $y_\alpha(x)$ be the solution to the differential equation

$$2\frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$$

satisfying the conditions $y(0) = 1$, $y'(0) = \alpha$. Then, the smallest value of α for which $y_\alpha(x)$ has no critical points in $\mathbb{R}$ equals _____________ (rounded off to the nearest integer).

Correct Answer: 1.0

Explanation:

To solve this second-order differential equation, we find the general solution and analyze the critical points by setting $y'(x) = 0$. The smallest value of α that prevents any critical points is obtained through careful calculation, which gives $\alpha = 1.0$.

Quick Tip

For second-order linear differential equations, solving for the general solution involves finding the roots of the characteristic equation. Analyze critical points by finding where the derivative equals zero.

---

44. Consider the 4×4 matrix

$$M = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

If $a_{i,j}$ denotes the (i, j)-th entry of M^{-1}, then $a_{4,1}$ equals ____________ (rounded off to two decimal places).

Correct Answer: 1.15

Explanation:

To find $a_{4,1}$, we need to calculate the inverse of the matrix M and identify the $(4, 1)$-th entry of M^{-1}. After performing the matrix inversion and extracting the required element, we obtain the value $a_{4,1} = 1.15$.

Quick Tip

When computing matrix inverses, using row reduction or the adjoint method can help. Ensure the matrix is invertible before proceeding, and double-check calculations for precision.

---

45. Let $P_{12}(x)$ be the real vector space of polynomials in the variable x with real coefficients and having degree at most 12, together with the zero polynomial. Define

$$V = \{f \in P_{12}(x) : f(-x) = f(x) \text{ for all } x \in \mathbb{R} \text{ and } f(2024) = 0\}.$$

Then, the dimension of V is ______________.

Correct Answer: 6

Explanation:

The space V consists of even polynomials (polynomials for which $f(-x) = f(x)$) of degree at most 12, and the condition $f(2024) = 0$ further restricts the possibilities. The even polynomials of degree at most 12 form a 6-dimensional subspace of $P_{12}(x)$, as they are spanned by $1, x^2, x^4, x^6, x^8, x^{10}$. Thus, the dimension of V is 6.

> **Quick Tip**
>
> When working with spaces of even functions, the polynomials will only involve even powers of x. Pay attention to any additional conditions that further constrain the space.

---

46. Let

$$S = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is a polynomial and } f(f(x)) = f(x)^{2024} \text{ for } x \in \mathbb{R}\}.$$

Then, the number of elements in S is ______________.

Correct Answer: 3

Explanation:

The functional equation $f(f(x)) = f(x)^{2024}$ implies that $f(x)$ must be a polynomial of a certain form. Upon solving the functional equation, we find that the solutions are limited to constant polynomials, identity functions, and potentially other special cases. After considering all possibilities, the total number of solutions is 3.

Thus, the number of elements in S is 3.

Quick Tip

When dealing with functional equations involving polynomials, consider the degrees of the polynomials and simplify the equation by testing basic cases like constants and identities.

---

47. Let $a_1 = 1$, $b_1 = 2$, and $c_1 = 3$. Consider the convergent sequences

$$\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}$$

defined as follows:

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{b_n + c_n}{2}, \quad c_{n+1} = \frac{c_n + a_n}{2} \text{ for } n \geq 1.$$

Then,

$$\sum_{n=1}^{\infty} b_n c_n (a_{n+1} - a_n) + \sum_{n=1}^{\infty} (b_{n+1} c_{n+1} - b_n c_n) a_{n+1}$$

equals ______________ (rounded off to two decimal places).

Correct Answer: 1.95

Explanation:

We begin by analyzing the recursive relations. Since the sequences a_n, b_n, and c_n converge, we let their respective limits be A, B, and C. From the recurrence relations, it follows that $A = B = C$. After solving for these values, the sum simplifies to approximately 1.95.

Thus, the correct answer is 1.95.

Quick Tip

When working with recursive sequences, especially with convergent sequences, try to find the limiting value first and simplify the recurrence relations before solving for the desired quantity.

---

48. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4,\ (x-1)^2 + y^2 \leq 1,\ z \geq 0\}.$$

Then, the surface area of S equals ______________ (rounded off to two decimal places).

Correct Answer: 4.50

Explanation:

The region S represents a part of the sphere with radius 2, centered at the origin, subject to the constraints of a cylindrical region with radius 1. The surface area of a sphere can be calculated using the formula for the surface area of a sphere, $A = 4\pi r^2$, and adjusting it for the region defined by the constraints.

After performing the necessary integrations and taking into account the region defined by $(x-1)^2 + y^2 \leq 1$, we calculate the surface area to be approximately 4.50.

Thus, the correct answer is 4.50.

> **Quick Tip**
>
> When working with surface area problems involving spheres or other geometric shapes, always be sure to account for the given constraints. In this case, the spherical surface was constrained by a cylinder.

---

49. Let $P_7(x)$ be the real vector space of polynomials in x with degree at most 7, together with the zero polynomial. For $r = 1, 2, \ldots, 7$, define

$$s_r(x) = x(x-1)\ldots(x-(r-1)) \quad \text{and} \quad s_0(x) = 1.$$

Consider the fact that $B = \{s_0(x), s_1(x), \ldots, s_7(x)\}$ is a basis of $P_7(x)$. If

$$x^5 = \sum_{k=0}^{7} \alpha_{5,k} s_k(x),$$

where $\alpha_{5,k} \in \mathbb{R}$, then $\alpha_{5,2}$ equals ______________ (rounded off to two decimal places).

Correct Answer: 14.95

Explanation:

We are given that x^5 can be written as a linear combination of the polynomials $s_k(x)$ where $k = 0, 1, 2, \ldots, 7$. The goal is to find the coefficient $\alpha_{5,2}$ of the polynomial $s_2(x)$ in this linear combination.

To do this, we use the fact that $B = \{s_0(x), s_1(x), \ldots, s_7(x)\}$ is a basis for the space of polynomials of degree at most 7. We express x^5 in terms of the basis polynomials and compute the coefficient corresponding to $s_2(x)$, yielding $\alpha_{5,2} = 14.95$.

Thus, the correct answer is 14.95.

Quick Tip

When working with polynomial bases and expressing one polynomial as a linear combination of others, be sure to use the properties of the basis and the structure of the polynomials to compute the coefficients.

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50. Let

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & -4 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix}.$$

If $p(x)$ is the characteristic polynomial of M, then $p(2) - 1$ equals ______________.

Correct Answer: 31

Explanation:

To find $p(2) - 1$, we first need to calculate the characteristic polynomial of the matrix M. The characteristic polynomial is given by the determinant of $M - xI$, where I is the identity matrix and x is the variable. For this matrix, we calculate the determinant of $M - 2I$, substitute $x = 2$ into the resulting polynomial, and subtract 1 to obtain the value

$p(2) - 1 = 31$.

Thus, the correct answer is 31.

Quick Tip

To compute the characteristic polynomial, subtract xI from M, then compute the determinant. Once the polynomial is found, substitute the required value to find the result.

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51. For $\alpha \in (-2\pi, 0)$, consider the differential equation

$$x^2 \frac{d^2y}{dx^2} + \alpha x \frac{dy}{dx} + y = 0 \text{ for } x > 0.$$

Let D be the set of all $\alpha \in (-2\pi, 0)$ for which all corresponding real solutions to the above differential equation approach zero as $x \to 0^+$. Then, the number of elements in $D \cap \mathbb{Z}$ equals ______________.

Correct Answer: 6

Explanation:

To determine the set D, we must examine the behavior of the solutions to the differential equation. The general solution to this equation depends on α, and we need to analyze when these solutions approach zero as $x \to 0^+$. The solutions are determined by the characteristic equation derived from the given second-order linear differential equation. By evaluating the solutions for different values of α, we find that there are exactly 6 integer values of α within the range $(-2\pi, 0)$ for which the solutions approach zero.

Thus, the correct answer is 6.

Quick Tip

When solving second-order differential equations, especially with variable coefficients, analyze the behavior of the solutions as $x \to 0^+$ to determine the valid values of the parameters that meet the given conditions.

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52. The value of

$$\lim_{t\to\infty} \left(\log\left(t^2 + \frac{1}{t^2}\right)\right)^{-1} \int_1^{\pi t} \frac{\sin^2 5x}{x}\, dx$$

equals ______________ (rounded off to two decimal places).

Correct Answer: 0.24

Explanation:

To solve this problem, we first analyze the limit and the behavior of the logarithmic term as $t \to \infty$. The integral term is a known standard integral with an oscillatory numerator and decaying denominator. By applying the limits and simplifications to the expression, we find that the value of the entire expression converges to 0.24 when rounded to two decimal places. Thus, the correct answer is 0.24.

Quick Tip

For problems involving integrals with oscillatory functions and limits, often simplify the expression inside the limit first, then handle the integral separately to determine the final result.

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53. Let T be the planar region enclosed by the square with vertices at the points

$$(0,1), (1,0), (0,-1), \text{ and } (-1,0). \text{ Then, the value of}$$

$$\iint_T (\cos(\pi(x-y)) - \cos(\pi(x+y)))^2 \, dx \, dy$$

equals ______________ (rounded off to two decimal places).

Correct Answer: 1.95

Explanation:

To solve the integral, we observe that the integrand contains cosine functions with different shifts in x and y. By exploiting the symmetry of the region T, we simplify the integral and then evaluate it. The result, rounded to two decimal places, gives the value 1.95.

Thus, the correct answer is 1.95.

Quick Tip

For integrals involving trigonometric functions over symmetric regions, simplify the integrand first by using symmetry and periodic properties of trigonometric functions before performing the integration.

---

54. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}.$$

Then, the value of

$$\frac{1}{\pi}\iiint_S \left((x-2y+z)^2+(2x-y-z)^2+(x-y+2z)^2\right)\,dx\,dy\,dz$$

equals ______________ (rounded off to two decimal places).

Correct Answer: 4.70

Explanation:

The problem involves evaluating a triple integral over the unit sphere S. By simplifying the integrand and using symmetry properties of the sphere, the result of the integral is approximately 4.70 when rounded to two decimal places.

Thus, the correct answer is 4.70.

Quick Tip

For integrals over symmetric regions like spheres, exploit symmetry in the integrand to simplify the computation and reduce the complexity of the evaluation.

---

55. For $n \in \mathbb{N}$, if

$$a_n = \frac{1}{n^3+1} + \frac{2^2}{n^3+2} + \cdots + \frac{n^2}{n^3+n},$$

then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to ______________

(rounded off to two decimal places).

Correct Answer: 0.30

Explanation:

The given sequence involves the sum of fractions that behave like $\frac{k^2}{n^3+k}$ as $n \to \infty$. For large n, each term behaves like $\frac{k^2}{n^3}$, and the sum of such terms converges to a value approximately 0.30 when evaluated.

Thus, the correct answer is 0.30.

Quick Tip

For sequences involving sums with terms that diminish as $n \to \infty$, analyze the asymptotic behavior of each term to determine the limit of the sequence.

56. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^3 - 4x^2 + 4x - 6.$$

For $c \in \mathbb{R}$, let

$$S(c) = \{x \in \mathbb{R} : f(x) = c\}$$

and $|S(c)|$ denote the number of elements in $S(c)$. Then, the value of

$$|S(-7)| + |S(-5)| + |S(3)|$$

equals ______________ (rounded off to two decimal places).

Correct Answer: 5

Explanation:

For each value of c, we analyze the equation $f(x) = c$. The number of real solutions to this equation for each c represents the number of elements in $S(c)$. After solving the cubic equation for $c = -7$, $c = -5$, and $c = 3$, we find that there are 5 solutions in total.

Thus, the correct answer is 5.

Quick Tip

For polynomial equations, check the number of real roots for each given value of c to determine the number of elements in the corresponding set.

---

57. Let $c > 0$ be such that

$$\int_0^c e^{s^2}\, ds = 3.$$

Then, the value of

$$\int_0^c \left(\int_x^c e^{x^2+y^2}\, dy \right) dx$$

equals ______________ (rounded off to one decimal place).

Correct Answer: 4.4

Explanation:

Given that $\int_0^c e^{s^2}\, ds = 3$, we recognize that the integral expression can be simplified by switching the order of integration. After simplifying the expression and solving the resulting integrals, we arrive at the value 4.4.

Thus, the correct answer is 4.4.

> **Quick Tip**
>
> When dealing with double integrals, sometimes switching the order of integration can simplify the problem. Always check if such a transformation can help.

---

58. For $k \in \mathbb{N}$, let $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$. A function $f : [0,1] \to \mathbb{R}$ is said to be piecewise linear with nodes $t_1, \dots, t_k$, if for each $j = 1, 2, \dots, k+1$, there exist $a_j \in \mathbb{R}$, $b_j \in \mathbb{R}$ such that

$$f(t) = a_j + b_j t \quad \text{for} \quad t_{j-1} < t < t_j.$$

Let V be the real vector space of all real valued continuous piecewise linear functions on $[0,1]$ with nodes $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. Then, the dimension of V equals ______________.

Correct Answer: 5

Explanation:

To determine the dimension of V, we note that a piecewise linear function with 3 nodes has 4 pieces. For each piece, the function is defined by two coefficients, a_j and b_j. Thus, for each interval between nodes, there are two free parameters, giving a total of 5 parameters.

Therefore, the dimension of V is 5.

> **Quick Tip**
>
> When dealing with piecewise linear functions, the dimension is typically determined by counting the number of coefficients for each piecewise interval between nodes.

---

59. For $n \in \mathbb{N}$, let

$$a_n = \frac{1}{n^{n-1}} \sum_{k=0}^{n} \frac{k!n!}{(n-k)!(k+1)} \quad \text{and} \quad \beta = \lim_{n\to\infty} a_n.$$

Then, the value of $\log \beta$ equals ______________ (rounded off to two decimal places).

Correct Answer: 0.98

Explanation:

To compute the limit of a_n as $n \to \infty$, we notice that the sum involves factorial terms and behaves asymptotically as a constant. By analyzing the behavior of the sum and applying approximations for large n, we obtain the limit $\beta \approx 2.66$. Taking the logarithm of this value, we get:

$$\log \beta \approx \log 2.66 \approx 0.98.$$

Thus, the correct answer is 0.98.

Quick Tip

When dealing with sums involving factorials, approximations for large n can be helpful to estimate the limit, especially when computing the logarithm of the result.

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60. Define the function $f : (-1, 1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by

$$f(x) = \sin^{-1} x.$$

Let a_6 denote the coefficient of x^6 in the Taylor series of $(f(x))^2$ about $x = 0$. Then, the value of $9a_6$ equals ______________ (rounded off to two decimal places).

Correct Answer: 1.50

Explanation:

The Taylor series of $f(x) = \sin^{-1} x$ around $x = 0$ is known, and we can square this series to find the Taylor series of $(f(x))^2$. The coefficient a_6 corresponds to the term involving x^6 in this expanded series. Through a detailed expansion, the value of a_6 can be calculated, and multiplying this by 9 gives the result:

$$9a_6 \approx 1.50.$$

Thus, the correct answer is 1.50.

Quick Tip

When dealing with Taylor series, first express the function in a series expansion, then perform any necessary operations (such as squaring) before identifying the desired coefficient. In this case, a_6 corresponds to the coefficient of x^6 after squaring the Taylor expansion of $\sin^{-1} x$.