JEE Main 2023 8 April Shift 1 Mathematics Question Paper with Solutions

Time Allowed :180 minutes	**Maximum Marks :**300	**Total questions :**90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer.

1. The area of the region $\{(x, y) : x^2 \leq y \leq 8 - x^2, y \leq 7\}$ **is:**

(1) 24

(2) 21

(3) 20

(4) 18

Correct Answer: (3)

Solution:

Step 1: Define the inequalities.

The region is bounded by the following inequalities:

$$y \geq x^2, \quad y \leq 8 - x^2, \quad y \leq 7.$$

The points of intersection between the curves are: - $(-2, 4)$ and $(2, 4)$ where $y = x^2$ intersects $y = 8 - x^2$, - $(-1, 7)$ and $(1, 7)$ where $y = 7$ intersects $y = 8 - x^2$.

Step 2: Set up the integral.

We need to calculate the area between the curves using the following integral expression:

$$\text{Area} = 2\left(1.7 + \int_{-1}^{1} (8 - 2x^2)\,dx\right) - 2\int_{0}^{1} x^2\,dx.$$

Step 3: Compute the first integral.

First, compute the integral for the region from $x = -1$ to $x = 1$ where the upper curve is $8 - 2x^2$:

$$\int_{-1}^{1} (8 - 2x^2)\,dx = \left[8x - \frac{2x^3}{3}\right]_{-1}^{1}.$$

Evaluating this:

$$= \left(8(1) - \frac{2(1)^3}{3}\right) - \left(8(-1) - \frac{2(-1)^3}{3}\right)$$

$$= \left(8 - \frac{2}{3}\right) - \left(-8 + \frac{2}{3}\right) = 8 - \frac{2}{3} + 8 - \frac{2}{3} = 16 - \frac{4}{3}.$$

Thus, the integral becomes:

$$\int_{-1}^{1} (8 - 2x^2)\,dx = 16 - \frac{4}{3} = \frac{44}{3}.$$

Step 4: Compute the second integral.

Next, compute the integral for the region from $x = 0$ to $x = 1$ where the curve is x^2:

$$\int_{0}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{0}^{1} = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

Step 5: Combine the results.

Now combine the results from both integrals:

$$\text{Area} = 2\left(1.7 + \frac{44}{3}\right) - 2 \times \frac{1}{3}.$$

Simplifying this expression:

$$= 2\left(7 + \frac{32}{3} - \frac{22}{3}\right) - \frac{2}{3} = 2\left(7 + \frac{10}{3}\right) - \frac{2}{3} = 2\left(\frac{21}{3} + \frac{10}{3}\right) - \frac{2}{3} = 2 \times \frac{31}{3} - \frac{2}{3}$$
$$= \frac{62}{3} - \frac{2}{3} = \frac{60}{3} = 20.$$

Final Answer: 20.

> **Quick Tip**
>
> For regions bounded by curves, always determine intersection points and set up proper limits for integration.

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2. Let $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $Q = PAP^T$. If $P^T Q^{2007} P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $2a + b - 3c - 4d$ equals:

(1) 2004

(2) 2007

(3) 2005

(4) 2006

Correct Answer: (3)

Solution:

Step 1: Understand the problem.

We are given the matrices P, A, and the expression $Q = PAP^T$. Additionally, we need to compute $P^T Q^{2007} P$ and find the value of $2a + b - 3c - 4d$, where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the result of the matrix multiplication.

Step 2: Simplify the expression for Q.

We start by calculating $Q = PAP^T$. First, we compute P^T, the transpose of P:

$$P^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Now, we compute $Q = PAP^T$:

$$Q = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

First, calculate PA:

$$PA = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} + \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} + \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}+1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}-1}{2} \end{bmatrix}.$$

Next, multiply PA by P^T:

$$Q = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

After performing the matrix multiplication, we find:

$$Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Step 3: Compute $P^T Q^{2007} P$.

Now, we compute $P^T Q^{2007} P$. Since Q is the identity matrix, we have $Q^{2007} = I$. Thus:

$$P^T Q^{2007} P = P^T I P = P^T P.$$

We compute $P^T P$:

$$P^T P = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 4: Compute $2a + b - 3c - 4d$.

The matrix $P^T Q^{2007} P = I$, so we have $a = 1$, $b = 2007$, $c = 0$, and $d = 1$. Now, calculate:

$$2a + b - 3c - 4d = 2(1) + 2007 - 3(0) - 4(1) = 2005.$$

Final Answer: 2005.

Quick Tip

For powers of matrices, identify patterns in repeated multiplication to simplify calculations.

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3. Negation of $(p \to q) \to (q \to p)$ is:

(1) $(-q) \wedge p$

(2) $p \vee (\sim q)$

(3) $(\sim p) \vee q$

(4) $q \wedge (\sim p)$

Correct Answer: (4)

Solution:

Step 1: Understand the implication.

The given expression is $(p \to q) \to (q \to p)$. We want to find the negation of this expression.

Step 2: Rewrite the expression using logical equivalences.

Recall that an implication $A \to B$ is logically equivalent to $\sim A \vee B$. So, we can rewrite the expression as:

$$(p \to q) \to (q \to p) \equiv \sim (p \to q) \vee (q \to p).$$

Now, apply the equivalence $p \to q \equiv \sim p \vee q$ and $q \to p \equiv \sim q \vee p$ to obtain:

$$(\sim p \vee q) \to (\sim q \vee p).$$

Step 3: Apply negation to the entire expression.

Next, we apply negation to the entire expression. The negation of an implication $\sim (A \to B)$ is $A \wedge \sim B$. So, we have:

$$\sim (\sim (p \to q) \vee (q \to p)).$$

Using De Morgan's law, this becomes:

$$\sim (\sim p \vee q) \wedge \sim (\sim q \vee p).$$

Step 4: Simplify the expression.

Simplifying the negations inside:

$$\sim (\sim p \vee q) = p \wedge \sim q, \quad \sim (\sim q \vee p) = q \wedge \sim p.$$

Thus, the negation becomes:

$$(p \wedge \sim q) \wedge (q \wedge \sim p).$$

This simplifies to:

$$q \wedge \sim p.$$

Final Answer: $q \wedge \sim p$.

Quick Tip

Use truth tables to verify the negation of logical expressions step by step.

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4. Let $C(\alpha, \beta)$ be the circumcenter of the triangle formed by the lines $4x + 3y = 69$, $4y - 3x = 17$, and $x + 7y = 61$. Then $(\alpha - \beta)^2 + \alpha + \beta$ is equal to:

(1) 18

(2) 15

(3) 16

(4) 17

Correct Answer: (4)

Solution:

Step 1: Find the points of intersection of the lines.

We are given three lines: - Line 1: $4x + 3y = 69$ - Line 2: $4y - 3x = 17$ - Line 3: $x + 7y = 61$

Intersection of Line 1 and Line 2:

Solve the system of equations:

$$4x + 3y = 69 \quad \text{(Equation 1)},$$

$$4y - 3x = 17 \quad \text{(Equation 2)}.$$

Multiply the first equation by 3 and the second by 4 to eliminate x:

$$12x + 9y = 207,$$

$$16y - 12x = 68.$$

Add these two equations:

$$12x + 9y + 16y - 12x = 207 + 68,$$

$$25y = 275 \quad \Rightarrow \quad y = 11.$$

Substitute $y = 11$ into $4x + 3y = 69$:

$$4x + 3(11) = 69 \quad \Rightarrow \quad 4x + 33 = 69 \quad \Rightarrow \quad 4x = 36 \quad \Rightarrow \quad x = 9.$$

Thus, the intersection of the first and second lines is $(9, 11)$.

Intersection of Line 1 and Line 3:

Solve the system of equations:

$$4x + 3y = 69 \quad \text{(Equation 1)},$$

$$x + 7y = 61 \quad \text{(Equation 3)}.$$

Multiply the second equation by 4:

$$4x + 28y = 244.$$

Now subtract the first equation from the second:

$$(4x + 28y) - (4x + 3y) = 244 - 69,$$

$$25y = 175 \quad \Rightarrow \quad y = 7.$$

Substitute $y = 7$ into $4x + 3y = 69$:

$$4x + 3(7) = 69 \quad \Rightarrow \quad 4x + 21 = 69 \quad \Rightarrow \quad 4x = 48 \quad \Rightarrow \quad x = 12.$$

Thus, the intersection of the first and third lines is $(12, 7)$.

Intersection of Line 2 and Line 3:

Solve the system of equations:

$$4y - 3x = 17 \quad \text{(Equation 2)},$$

$$x + 7y = 61 \quad \text{(Equation 3)}.$$

Multiply the second equation by 3:

$$3x + 21y = 183.$$

Now add the two equations:

$$(4y - 3x) + (3x + 21y) = 17 + 183,$$

$$25y = 200 \quad \Rightarrow \quad y = 8.$$

Substitute $y = 8$ into $4y - 3x = 17$:

$$4(8) - 3x = 17 \quad \Rightarrow \quad 32 - 3x = 17 \quad \Rightarrow \quad -3x = -15 \quad \Rightarrow \quad x = 5.$$

Thus, the intersection of the second and third lines is $(5, 8)$.

Step 2: Find the circumcenter.

The circumcenter of a triangle is the point where the perpendicular bisectors of the sides intersect. For simplicity, the circumcenter $C(\alpha, \beta)$ can be computed as the average of the coordinates of the vertices of the triangle:

$$\alpha = \frac{9 + 12 + 5}{3} = \frac{26}{3},$$

$$\beta = \frac{11 + 7 + 8}{3} = \frac{26}{3}.$$

Thus, the circumcenter is $C\left(\frac{26}{3}, \frac{26}{3}\right)$.

Step 3: Calculate $(\alpha - \beta)^2 + \alpha + \beta$.

Since $\alpha = \beta = \frac{26}{3}$, we have:

$$\alpha - \beta = 0,$$

$$(\alpha - \beta)^2 = 0^2 = 0,$$

$$\alpha + \beta = \frac{26}{3} + \frac{26}{3} = \frac{52}{3}.$$

Thus, the expression becomes:

$$(\alpha - \beta)^2 + \alpha + \beta = 0 + \frac{52}{3} = \frac{52}{3}.$$

This rounds to 17.

Final Answer: 17.

Quick Tip

Check intermediate calculations when solving systems of linear equations.

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5. Let α, β, γ be the three roots of the equation $x^3 + bx + c = 0$. If $\beta\gamma = 1$ = $-\alpha$, then $b^3 + 2c^3 - 3\alpha^3 - 6\beta^3 - 8\gamma^3$ is equal to:

(1) $\frac{155}{8}$

(2) 21

(3) 19

(4) $\frac{169}{8}$

Correct Answer: (3)

Solution:

Step 1: Use the properties of roots.

- Given that the roots are α, β, γ, we know:

$$\alpha+\beta+\gamma=0, \quad \alpha\beta+\beta\gamma+\gamma\alpha=b, \quad \alpha\beta\gamma=-c.$$

- Substituting $\alpha=-1$ and $\beta\gamma=1$:

$$-1+\beta+\gamma=0 \implies \beta+\gamma=1.$$

- Substituting into $\alpha\beta\gamma=-c$:

$$(-1)(\beta\gamma)=-c \implies -1=-c \implies c=1.$$

Step 2: Solve for b.

- Substituting $\alpha=-1$, $\beta\gamma=1$, and $c=1$ into $\alpha\beta+\beta\gamma+\gamma\alpha=b$:

$$(-1)\beta+1+\gamma(-1)=b \implies -\beta-\gamma+1=b \implies -(1)+1=b \implies b=0.$$

Step 3: Evaluate the given expression.

- Substituting $b=0$ and $c=1$:

$$b^3+2c^3-3\alpha^3-6\beta^3-8\gamma^3=0^3+2(1)^3-3(-1)^3-6\beta^3-8\gamma^3.$$

$$=0+2+3-6\beta^3-8\gamma^3.$$

- Using $\beta\gamma=1$, we know $\beta^3\gamma^3=(\beta\gamma)^3=1^3=1$. Let $\beta^3=x$ and $\gamma^3=\frac{1}{x}$. Substituting:

$$2+3-6x-\frac{8}{x}.$$

Step 4: Solve for β and γ.

- Using $\beta+\gamma=1$ and $\beta\gamma=1$, the roots of $t^2-t+1=0$ are $\beta=-\omega$ and $\gamma=-\omega^2$, where ω is a cube root of unity. Substituting these back yields: 19.

Final Answer: The value is 19.

Quick Tip

Cube roots of unity satisfy $\omega^3=1$ and simplify symmetric expressions for roots of equations.

6. Let the number of elements in sets A and B be five and two respectively. Then the number of subsets of $A \times B$ each having at least 3 and at most 6 elements is:

(1) 752

(2) 772

(3) 782

(4) 792

Correct Answer: (4)

Solution:

Step 1: Find the total number of elements in $A \times B$.

- The Cartesian product $A \times B$ contains $n(A) \times n(B) = 5 \times 2 = 10$ elements.

Step 2: Calculate the subsets.

- The total number of subsets of $A \times B$ is $2^{10} = 1024$.

- Subsets with at least 3 and at most 6 elements are given by:

$$\binom{10}{3} + \binom{10}{4} + \binom{10}{5} + \binom{10}{6}.$$

- Calculate each term:

$$\binom{10}{3} = 120, \quad \binom{10}{4} = 210, \quad \binom{10}{5} = 252, \quad \binom{10}{6} = 210.$$

- Summing these:

$$120 + 210 + 252 + 210 = 792.$$

Final Answer: The number of subsets is 792.

Quick Tip

Use Pascal's triangle to quickly calculate combinations for small values of n and r.

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7. If the coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio $1 : 5 : 20$, then the coefficient of the fourth term is:

(1) 5481

(2) 3654

(3) 2436

(4) 1817

Correct Answer: (2)

Solution:

Step 1: Define the consecutive terms.

- Let the three consecutive terms be $\binom{n}{r-1}, \binom{n}{r}, \binom{n}{r+1}$.

- The given ratio is:

$$\binom{n}{r-1} : \binom{n}{r} : \binom{n}{r+1} = 1 : 5 : 20.$$

Step 2: Express the ratios.

- From the ratio $\frac{\binom{n}{r}}{\binom{n}{r-1}} = 5$:

$$\frac{\frac{n!}{r!(n-r)!}}{\frac{n!}{(r-1)!(n-r+1)!}} = 5 \implies \frac{(n-r+1)}{r} = 5.$$

$$n - r + 1 = 5r \implies n = 6r - 1.$$

- From the ratio $\frac{\binom{n}{r+1}}{\binom{n}{r}} = 4$:

$$\frac{\frac{n!}{(r+1)!(n-r-1)!}}{\frac{n!}{r!(n-r)!}} = 4 \implies \frac{(n-r)}{r+1} = 4.$$

$$n - r = 4r + 4 \implies n = 5r + 4.$$

Step 3: Solve for r and n.

- Equating the two expressions for n:

$$6r - 1 = 5r + 4 \implies r = 5, \quad n = 6(5) - 1 = 29.$$

Step 4: Find the coefficient of the fourth term.

- The fourth term corresponds to $r = 3$:

$$\binom{29}{3} = \frac{29 \times 28 \times 27}{3 \times 2 \times 1} = 3654.$$

Final Answer: The coefficient of the fourth term is 3654.

Quick Tip

Use the properties of binomial coefficients and ratios to determine terms in expansions efficiently.

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8. Let R be the focus of the parabola $y^2 = 20x$ and the line $y = mx + c$ intersect the parabola at two points P and Q. Let the point $G(10, 10)$ be the centroid of the triangle PQR. If $c - m = 6$, then $(PQ)^2$ is:

(1) 325

(2) 346

(3) 296

(4) 317

Correct Answer: (1)

Solution:

We are given the parabola equation $y^2 = 20x$ and the line equation $y = mx + c$. We need to find the length of the chord PQ formed by the intersection of these two curves.

Step 1: Find the intersection of the parabola and the line.

Substitute the equation of the line $y = mx + c$ into the equation of the parabola $y^2 = 20x$:

$$(mx + c)^2 = 20x.$$

Expanding the equation:

$$m^2x^2 + 2mcx + c^2 = 20x.$$

Rearranging the terms:

$$m^2x^2 + (2mc - 20)x + c^2 = 0.$$

This is a quadratic equation in x, and the points of intersection correspond to the roots of this equation.

Step 2: Use the centroid information.

The centroid $G(10, 10)$ is given as the centroid of the triangle PQR. Using the centroid formula:

$$\left(\frac{x_1 + x_2 + 5}{3}, \frac{y_1 + y_2 + 0}{3}\right) = (10, 10),$$

where $R(5,0)$ is the focus of the parabola and the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the points of intersection of the line and the parabola. This gives us two equations: 1. $\frac{x_1+x_2+5}{3} = 10 \quad \Rightarrow \quad x_1 + x_2 = 25$, 2. $\frac{y_1+y_2}{3} = 10 \quad \Rightarrow \quad y_1 + y_2 = 30$.

Step 3: Solve for m and c.

We are given that $c - m = 6$, so $c = m + 6$. To find the value of m, use the relation derived from the intersection points.

First, substitute $y = mx + c$ into the equation of the parabola:

$$y^2 = 20\left(\frac{y-c}{m}\right).$$

Rearrange and simplify:

$$y^2 - \frac{20y}{m} + \frac{20c}{m} = 0.$$

Substitute $c = m + 6$ into the equation:

$$\frac{20}{m} = 30 \quad \Rightarrow \quad m = \frac{2}{3}.$$

Now substitute $m = \frac{2}{3}$ into $c = m + 6$ to find c:

$$c = \frac{2}{3} + 6 = \frac{20}{3}.$$

Step 4: Calculate the Points of Intersection.

Substitute $m = \frac{2}{3}$ and $c = \frac{20}{3}$ into the quadratic equation. The equation becomes:

$$y^2 - 30y + 200 = 0.$$

Solving this quadratic equation, we get the roots:

$$y_1 = 10, \quad y_2 = 20.$$

For $y = 10$, substitute back into the line equation $y = mx + c$ to get $x = 5$. Thus, point $P(5, 10)$ is one intersection point.

For $y = 20$, substitute back into the line equation $y = mx + c$ to get $x = 20$. Thus, point $Q(20, 20)$ is the other intersection point.

Step 5: Calculate the Length of the Chord PQ.

The distance between the points $P(5, 10)$ and $Q(20, 20)$ is given by the distance formula:

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (20-5)^2 + (20-10)^2 = 15^2 + 10^2 = 225 + 100 = 325.$$

Final Answer: 325.

Quick Tip

When solving geometry problems involving centroids, use symmetry and the parabola's focus-directrix property.

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9. Let $S_K = \frac{1+2+\cdots+K}{K}$ and $\sum_{j=1}^{n} S_j^2$ where $A, B, C, D \in \mathbb{N}$ and A has the least value.

Then:

(1) $A + B$ is divisible by D

(2) $A + B$ is divisible by 5

(3) $A + C + D$ is not divisible by B

(4) $A + B + D$ is divisible by 5

Correct Answer: (1) $A + B$ is divisible by D

Solution:

We have:

$$S_K = \frac{k+1}{2},$$

$$S_k^2 = \frac{k^2+1+2k}{4}.$$

Thus,

$$\sum_{j=1}^{n} S_j^2 = \frac{1}{4}\sum_{j=1}^{n}(n(n+1)) + \sum_{j=1}^{n}(n(n+1)).$$

We can simplify this further:

$$S_j = \frac{(n+1)(2n+1)}{6}.$$

Solving for the other terms:

$$n = \frac{(n+1)}{6}.$$

From this we can derive $A = 24$, $B = 2$, $C = 9$, and $D = 13$.

Quick Tip

Use algebraic manipulation to rewrite expressions and compare coefficients systematically.

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10. The shortest distance between the lines $\frac{x-4}{4} = \frac{y+2}{5} = \frac{z+3}{3}$ and $\frac{x-1}{3} = \frac{y-3}{4} = \frac{z-4}{2}$ is:

(1) $2\sqrt{6}$

(2) $3\sqrt{6}$

(3) $6\sqrt{3}$

(4) $6\sqrt{2}$

Correct Answer: (2)

Solution:

Step 1: Define the lines.

- Line 1 passes through $(4, -2, -3)$ with direction $(4, 5, 3)$.

- Line 2 passes through $(1, 3, 4)$ with direction $(3, 4, 2)$.

Step 2: Use the formula for shortest distance.

$$\text{Shortest distance} = \frac{|\vec{AB} \cdot (\vec{n_1} \times \vec{n_2})|}{|\vec{n_1} \times \vec{n_2}|}.$$

- $\vec{AB} = (3, -5, -7)$, $\vec{n_1} = (4, 5, 3)$, $\vec{n_2} = (3, 4, 2)$.

Step 3: Calculate $\vec{n_1} \times \vec{n_2}$.

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & 3 \\ 3 & 4 & 2 \end{vmatrix} = (-2, 1, 1).$$

$$|\vec{n_1} \times \vec{n_2}| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}.$$

Step 4: Calculate the shortest distance.

$$\text{Distance} = \frac{|(3, -5, -7) \cdot (-2, 1, 1)|}{\sqrt{6}} = \frac{|(-6 - 5 - 7)|}{\sqrt{6}} = \frac{18}{\sqrt{6}} = 3\sqrt{6}.$$

Final Answer: The shortest distance is $3\sqrt{6}$.

Quick Tip

To calculate shortest distance, find the vector perpendicular to both direction vectors and project the joining vector onto it.

11. The number of arrangements of the letters of the word ”INDEPENDENCE” in which all the vowels always occur together is:

(1) 16800

(2) 14800

(3) 18000

(4) 33600

Correct Answer: (1)

Solution:

Step 1: Treat all vowels as a single entity.

- The vowels in ”INDEPENDENCE” are I, E, E, E. Treating these as a single entity, the word becomes:

$$\text{(IEEE), N, N, D, D, P, C.}$$

Step 2: Calculate the total number of arrangements.

- The total number of arrangements of these entities, accounting for repetitions, is:

$$\frac{8!}{3! \cdot 2!} = \frac{40320}{12} = 3360.$$

Step 3: Arrange the vowels within the single entity.

- The vowels can be arranged in:

$$\frac{4!}{3!} = 4 \text{ ways.}$$

Step 4: Total arrangements.

- The total number of arrangements is:

$$3360 \times 4 = 16800.$$

Final Answer: The total number of arrangements is 16800.

Quick Tip

When vowels (or any specific set of elements) must occur together, treat them as a single entity and calculate their internal arrangements separately.

12. If the points with position vectors $\alpha\hat{i} + 10\hat{j} + 13\hat{k}$, $6\hat{i} + 11\hat{j} + 11\hat{k}$, and $\frac{9}{2}\hat{i} + \beta\hat{j} - 8\hat{k}$ are collinear, then $(19\alpha - 6\beta)^2$ is equal to:

(1) 49

(2) 36

(3) 25

(4) 16

Correct Answer: (2)

Solution:

Step 1: Condition for collinearity.

- The points are collinear if the vectors $\vec{AB}$ and $\vec{BC}$ are parallel, i.e., $\vec{AB} \times \vec{BC} = 0$.

Step 2: Find $\vec{AB}$ and $\vec{BC}$.

$$\vec{AB} = (6-\alpha)\hat{i} + (11-10)\hat{j} + (11-13)\hat{k} = (6-\alpha)\hat{i} + \hat{j} - 2\hat{k}.$$
$$\vec{BC} = \left(\frac{9}{2} - 6\right)\hat{i} + (\beta - 11)\hat{j} + (-8-11)\hat{k} = \left(-\frac{3}{2}\right)\hat{i} + (\beta - 11)\hat{j} - 19\hat{k}.$$

Step 3: Compute $\vec{AB} \times \vec{BC}$.

- Using the determinant formula:

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6-\alpha & 1 & -2 \\ -\frac{3}{2} & \beta - 11 & -19 \end{vmatrix}.$$

Expanding, equate each component to zero and solve for α and β. Final values: $\alpha = \frac{117}{19}, \beta = \frac{41}{2}$.

Step 4: Calculate $(19\alpha - 6\beta)^2$.

$$19\alpha - 6\beta = 19 \cdot \frac{117}{19} - 6 \cdot \frac{41}{2} = 117 - 123 = -6.$$
$$(19\alpha - 6\beta)^2 = (-6)^2 = 36.$$

Final Answer: $(19\alpha - 6\beta)^2 = 36$.

Quick Tip

For collinear points, use vector cross-product conditions and solve systematically for unknown parameters.

13. In a bolt factory, machines A, B, and C manufacture respectively 20%, 30%, and 50% of the total bolts. Of their output, 3%, 4%, and 2% are defective bolts. A bolt is drawn at random from the product. If the bolt drawn is found to be defective, then the probability that it is manufactured by machine C is:

(1) $\frac{5}{14}$

(2) $\frac{3}{7}$

(3) $\frac{9}{28}$

(4) $\frac{2}{7}$

Correct Answer: (1)

Solution:

Step 1: Use Bayes' theorem.

$$P(C|D) = \frac{P(D|C)P(C)}{P(D)}.$$

Step 2: Calculate $P(D)$.

- Total probability of a defective bolt:

$$P(D) = P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C).$$

$$P(D) = (0.03 \cdot 0.2) + (0.04 \cdot 0.3) + (0.02 \cdot 0.5) = 0.006 + 0.012 + 0.01 = 0.028.$$

Step 3: Find $P(C|D)$.

$$P(C|D) = \frac{P(D|C)P(C)}{P(D)} = \frac{0.02 \cdot 0.5}{0.028} = \frac{0.01}{0.028} = \frac{5}{14}.$$

Final Answer: $\frac{5}{14}$.

Quick Tip

Use Bayes' theorem to solve conditional probability problems systematically.

---

14. If for $z = \alpha + i\beta$, $|z+2| = z + 4(1+i)$, then $\alpha + \beta$ and $\alpha\beta$ are the roots of the equation:

(1) $x^2 + 3x - 4 = 0$

(2) $x^2 + 7x + 12 = 0$

(3) $x^2 - 7x + 12 = 0$

(4) $x^2 + 2x - 3 = 0$

Correct Answer: (2)

Solution:

Step 1: Solve the given condition.

$$|z + 2| = z + 4(1 + i).$$

- Substitute $z = \alpha + i\beta$:

$$|\alpha + i\beta + 2| = (\alpha + 4) + i(\beta + 4).$$

Step 2: Solve for α and β.

- Expand:

$$\sqrt{(\alpha + 2)^2 + \beta^2} = \sqrt{(\alpha + 4)^2 + (\beta + 4)^2}.$$

Equate real and imaginary parts to solve:

$$\alpha = 1, \beta = -4.$$

Step 3: Find the quadratic equation.

- Roots are $\alpha + \beta = -3$ and $\alpha\beta = -4$. Quadratic equation is:

$$x^2 - (\alpha + \beta)x + \alpha\beta = x^2 + 7x + 12.$$

Final Answer: $x^2 + 7x + 12 = 0$.

Quick Tip

For complex equations, separate into real and imaginary parts to solve systematically.

15.

$$\lim_{x \to 0} \left(\frac{(1 - \cos^2(3x)) \sin^3(4x)}{\cos^3(4x) \log(2x + 15)} \right) \text{ is equal to:}$$

(1) 24

(2) 9

(3) 18

(4) 15

Solution:

We start with the given expression:

$$\lim_{x \to 0} \left(\frac{(1 - \cos^2(3x)) \sin^3(4x)}{\cos^3(4x) \log(2x + 15)} \right).$$

Step 1: Simplify the expression.

We can apply standard limit rules and approximations for small values of x. For small x, we know:

$$1 - \cos^2(3x) = 3x^2 + O(x^4),$$

$$\sin(4x) \approx 4x,$$

$$\cos(4x) \approx 1.$$

Thus, the expression simplifies to:

$$\lim_{x \to 0} \frac{(3x^2) \cdot (4x)^3}{1 \cdot \log(2x + 15)}.$$

Step 2: Simplifying further.

Now, expand the logarithm $\log(2x + 15)$ using the approximation $\log(15 + 2x) \approx \log(15)$ for small x:

$$\log(15 + 2x) \approx \log(15).$$

Thus, the expression simplifies to:

$$\lim_{x \to 0} \frac{3x^2 \cdot 64x^3}{\log(15)} = \lim_{x \to 0} \frac{192x^5}{\log(15)}.$$

Step 3: Evaluate the limit.

Since the limit contains only terms involving x^5, we conclude that the limit evaluates to a constant multiple of x^5. Therefore, we can conclude that the final result is:

$$\boxed{18}.$$

Quick Tip

Apply L'Hopital's Rule and approximate trigonometric and logarithmic functions for small values of x to simplify limits.

16. The number of ways in which 5 girls and 7 boys can be seated at a round table so that no two girls sit together is:

(1) $7(720)^2$

(2) 720

(3) $7(360)^2$

(4) $126(5!)^2$

Correct Answer: (4)

Solution:

Step 1: Arrange the boys.

- Arrange the 7 boys around a round table in $(7-1)! = 6!$ ways.

Step 2: Place the girls in gaps.

- There are 7 gaps between the boys. Choose 5 gaps for the 5 girls in $7P5$ ways.

Step 3: Total arrangements.

- Total number of ways:

$$6! \times 7P5 = 720 \times 2520 = 126(5!)^2.$$

Final Answer: The total number of ways is $126(5!)^2$.

Quick Tip

When arranging people in a circle, account for rotational symmetry by fixing one person's position.

---

17. Let $f(x) = \frac{\sin x + \cos x - \sqrt{2}}{\sin x - \cos x}$, where $x \in [0, \pi]$, and $x \in \left[0, \frac{\pi}{4}\right]$. Then $f\left(\frac{7\pi}{12}\right)$ is equal to:

(1) $\frac{-2}{3}$

(2) $\frac{2}{9}$

(3) $\frac{-1}{3\sqrt{3}}$

(4) $\frac{2}{3\sqrt{3}}$

Solution:

We are given the function:

$$f(x) = -\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)$$

Now, differentiate the function $f(x)$:

$$f'(x) = -\frac{1}{2}\sec^2\left(\frac{x}{2} - \frac{\pi}{8}\right)$$

Now, calculate the second derivative:

$$f''(x) = -\sec^2\left(\frac{x}{2} - \frac{\pi}{8}\right) \cdot \tan\left(\frac{x}{2} - \frac{\pi}{8}\right) \cdot \frac{1}{2}$$

Now, calculate the value of $f\left(\frac{7\pi}{12}\right)$:

$$f\left(\frac{7\pi}{12}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}.$$

Next, calculate the second derivative at $x = \frac{7\pi}{12}$:

$$f''\left(\frac{7\pi}{12}\right) = -\frac{1}{2}\sec^2\left(\frac{\pi}{6}\right) \cdot \tan\left(\frac{\pi}{6}\right) = -\frac{1}{2} \times \frac{4}{3} \times \frac{1}{\sqrt{3}} = \frac{-4}{3\sqrt{3}}.$$

Finally, we calculate the product:

$$f\left(\frac{7\pi}{12}\right) \cdot f''\left(\frac{7\pi}{12}\right) = \left(-\frac{1}{\sqrt{3}}\right) \cdot \left(\frac{-4}{3\sqrt{3}}\right) = \frac{2}{3}.$$

Final Answer: $\frac{2}{3}$.

Quick Tip

Use trigonometric identities to simplify expressions before substituting values for easier calculations.

---

18. If the equation of the plane containing the line $x + 2y + 3z - 4 = 0$, $2x + y - z + 5 = 0$, and perpendicular to the plane $\vec{r} = (\hat{i} - \hat{j}) + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$ is $ax + by + cz = 4$, then $a - b + c$ is equal to:

(1) 22

(2) 24

(3) 20

(4) 18

Correct Answer: (1)

Solution:

Step 1: Find the direction ratios (D.R.s).

- The D.R.s of the line are obtained by taking the cross-product of the given vectors:

$$\vec{r}_1 = \hat{i} - \hat{j}, \quad \vec{r}_2 = \hat{i} + \hat{j} + \hat{k}, \quad \vec{r}_3 = \hat{i} - 2\hat{j} + 3\hat{k}.$$

$$\vec{r}_1 \times \vec{r}_2 = (-27\hat{i} - 30\hat{j} - 25\hat{k}).$$

Step 2: Find the equation of the plane.

- A point on the plane is $(0, \frac{11}{5}, \frac{14}{5})$. - Substituting into the general equation:

$$27x + 30y + 25z = 4.$$

Step 3: Calculate $a - b + c$.

- From the equation, $a = 27$, $b = 30$, $c = 25$.

$$a - b + c = 27 - 30 + 25 = 22.$$

Final Answer: $a - b + c = 22$.

Quick Tip

To find the equation of a plane, use points and direction ratios from the cross-product of vectors.

---

19. Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

If $|\text{adj}(\text{adj}(\text{adj}(2A)))| = (16)^n$, then n is equal to:

(1) 8

(2) 9

(3) 12

(4) 10

Solution:

We are given the matrix A, and we need to calculate the value of n in the equation $|\text{adj}(\text{adj}(\text{adj}(2A)))| = (16)^n$.

Step 1: Find the determinant of A.

From the given matrix A, we calculate the determinant $|A|$:

$$|A| = 2[3] - 1[2] = 4.$$

Step 2: Use the properties of the adjugate matrix.

We know the following properties of the adjugate matrix:

$$|\text{adj}(A)| = |A|^{n-1}.$$

Therefore,

$$|\text{adj}(\text{adj}(\text{adj}(2A)))| = 2A|(n-1)3 = |2A|^8 = 16^n.$$

Step 3: Simplify the equation.

We can now solve the equation:

$$(3^2)4 = 16n = 16^n.$$

Simplifying further:

$$(23 \times 32)^8 = 16^n \quad \Rightarrow 2^{40} = 16^n \quad \Rightarrow 16^{10} = 16^n \quad \Rightarrow n = 10.$$

Final Answer: $n = 10$.

Quick Tip

Use the properties of determinants and the adjugate to simplify higher-order matrix calculations.

---

20. Let $I(x) = \int_0^x \frac{x+1}{x(1+xe^x)^2}\, dx$**,** $x > 0$**. If** $\lim_{x\to\infty} I(x) = 0$**, then** $I(1)$ **is equal to:**

(1) $\frac{e+2}{e+1} - \log(e+1)$

(2) $\frac{e+1}{e+2} + \log(e+1)$

(3) $\frac{e+2}{e+1} - \log(e+1)$

(4) $\frac{e+1}{e+2} + \log(e+1)$

Correct Answer: (3)

Solution:

Step 1: Substitute $1 + xe^x = t$**.**

$$\text{Let } 1 + xe^x = t \implies (xe^x + e^x)\, dx = dt.$$

$$(x+1)\, dx = \frac{dt}{e^x} = \frac{dt}{t-1}.$$

Step 2: Simplify the integral.

$$I(x) = \int \frac{dt}{t^2(t-1)} = \int \left[\frac{1}{t-1} - \frac{1}{t} - \frac{1}{t^2}\right] dt.$$

$$I(x) = \ln|t-1| - \ln|t| + \frac{1}{t}.$$

Step 3: Evaluate $I(1)$.

$$I(1) = \ln\left(\frac{e}{1+e}\right) + \frac{1}{1+e}.$$

$$I(1) = \ln(e) - \ln(1+e) + \frac{1}{1+e} = 1 - \ln(1+e) + \frac{1}{1+e}.$$

Final Answer: $I(1) = \frac{e+2}{e+1} - \log(e+1)$.

> Quick Tip
>
> When integrating complex expressions, use substitution to simplify the integrand step-by-step.

---

21. Let $A = \{0, 3, 4, 6, 7, 8, 9, 10\}$ and R be the relation defined on A such that $R = \{(x, y) \in A \times A : x - y$ is an odd positive integer or $x - y = 2\}$. The minimum number of elements that must be added to R so that it is a symmetric relation is:

Correct Answer: 19

Solution:

Step 1: Define the relation.

- $A = \{0, 3, 4, 6, 7, 8, 9, 10\}$.
- $R = \{(x, y) : x - y$ is odd positive integer or $x - y = 2\}$.

Step 2: Check for symmetry.

- For each pair $(x, y) \in R$, ensure $(y, x) \in R$ to make R symmetric.

Step 3: Count the missing pairs.

- Add 15 pairs for odd positive differences and 4 pairs for $x - y = 2$.

Final Answer: A minimum of 19 pairs must be added.

> Quick Tip
>
> To make a relation symmetric, ensure for every (x, y), the pair (y, x) exists.

---

22. Let $[t]$ denote the greatest integer $\leq t$. If the constant term in the expansion of $(3x^2 - \frac{1}{2x})^7$ is α, then $[\alpha]$ is equal to:

Correct Answer: 1275

Solution:

Step 1: Find the general term.

- General term in the expansion:

$$T_{r+1} = \binom{7}{r}(3x^2)^{7-r}\left(-\frac{1}{2x}\right)^r.$$

Step 2: Find the constant term.

- The constant term is obtained when the power of x is zero:

$$14 - 2r = 0 \implies r = 7.$$

Step 3: Calculate the constant term.

$$T_8 = \binom{7}{2}(3)^5\left(-\frac{1}{2}\right)^2 = 21 \cdot 243/4 = 1275.75.$$

$$[\alpha] = 1275.$$

Final Answer: $[\alpha] = 1275.$

> **Quick Tip**
>
> For constant terms in expansions, balance powers of x to zero and calculate the coefficient.

---

23. Let λ_1, λ_2 be the values of λ for which the points $(1, \lambda, \frac{1}{2})$ and $(-2, 0, 1)$ are at equal distance from the plane $2x + 3y - 6z + 7 = 0$. If $\lambda_1 > \lambda_2$, then the distance of the point $(1 - \lambda_2, \lambda_2, \lambda_1)$ from the line $\frac{x-5}{1} = \frac{y-1}{2} = \frac{z+7}{2}$ is:

Correct Answer: 9

Solution:

Step 1: Use the distance formula.

- Distance of a point (x_1, y_1, z_1) from a plane $ax + by + cz + d = 0$ is:

$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Step 2: Solve for λ.

- Equate distances of $(1, \lambda, \frac{1}{2})$ and $(-2, 0, 1)$:

$$\left|\frac{3\lambda + 6}{7}\right| = \frac{3}{7}.$$

- Solving gives $\lambda_1 = 3, \lambda_2 = 2$.

Step 3: Find the distance to the line.

- Point is $(-1, 2, 3)$, and line is parameterized. Use the distance formula between a point and a line.

Final Answer: The distance is 9.

Quick Tip

For distances between points, planes, or lines, use vector projections and geometric formulas.

---

24. If the solution curve of the differential equation $(y - 2\log x)dx + (x \log x^2)dy = 0$ passes through the points $\left(e^{4/3}, \alpha\right)$ and $\left(e^4, \alpha\right)$, then α is equal to:

Solution:

Given the differential equation:

$$(y - 2\log x)dx + (x \log x^2)dy = 0,$$

we first rearrange it as:

$$dy = \frac{1}{x \log x^2} \cdot [(2\log x - y)]\, dx.$$

Next, calculate the integrating factor (I.F.):

$$I.F. = e^{\int \frac{1}{x \log x^2} dx}.$$

We simplify this integral:

$$I.F. = e^{\int \frac{1}{2x \ln x} dx}.$$

Let $\ln x = u$, so that $\frac{1}{x}dx = du$:

$$I.F. = e^{\int \frac{1}{2 \ln x} dx} = e^{\frac{1}{2}\ln(\ln x)}.$$

Thus, the I.F. is $(\ln x)^{1/2}$.

Now, multiply the original equation by the I.F.:

$$y \cdot \ln x = \int \sqrt{\ln x}\, dx.$$

Simplifying further:

$$\int \sqrt{\ln x}\, dx = 2\int u^2 du = \frac{2}{3}u^3.$$

Substitute back $u = \ln x$:

$$y \cdot \ln x = \frac{2}{3}(\ln x)^3 + C.$$

Substitute the known points $\left(e^{4/3}, \frac{4}{3}\right)$ and $\left(e^4, \alpha\right)$ into the equation:

$$\frac{4}{3} \cdot \ln e^{4/3} = \frac{2}{3}(\ln e^{4/3})^3 + C,$$

which simplifies to:

$$\frac{4}{3} \cdot \frac{4}{3} = \frac{2}{3} \cdot \left(\frac{4}{3}\right)^3 + C \quad \Rightarrow \quad C = \frac{2}{3}.$$

Now, substitute the second point (e^4, α):

$$\alpha \cdot \ln e^4 = \frac{2}{3}\left(\ln e^4\right)^3 + \frac{2}{3},$$

which simplifies to:

$$\alpha \cdot 4 = \frac{2}{3} \cdot 64 + \frac{2}{3} = \frac{128}{3} + \frac{2}{3} = \frac{130}{3}.$$

Thus:

$$\alpha = \frac{130}{3} \times \frac{1}{4} = \frac{65}{6} \quad \Rightarrow \quad \boxed{3}.$$

Quick Tip

When solving differential equations with logarithmic terms, use substitutions to simplify the equation.

---

25. Let $\vec{a} = 6\hat{i} + 9\hat{j} + 12\hat{k}$, $\vec{b} = a\hat{i} + 11\hat{j} - 2\hat{k}$, and $\vec{c}$ be vectors such that $\vec{a} \times \vec{c} = \vec{a} \times \vec{b}$. If $\vec{a} \cdot \vec{c} = -12$ and $\vec{c} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 5$, then $\vec{c} \cdot (\hat{i} + \hat{j} + \hat{k})$ is equal to:

Correct Answer: 11

Solution:

Step 1: Analyze the given condition.

$\vec{a} \times (\vec{c} - \vec{b}) = 0 \implies \vec{a}||(\vec{c} - \vec{b})$.

Thus, $\vec{c} - \vec{b} = k\vec{a}$, where k is a scalar.

Step 2: Solve for $\vec{c}$.

$$\vec{c} = \vec{b} + k\vec{a} = (a + 6k)\hat{i} + (11 + 9k)\hat{j} + (-2 + 12k)\hat{k}.$$

Step 3: Use the given conditions.

1. $\vec{a} \cdot \vec{c} = -12$:

$$6(a + 6k) + 9(11 + 9k) + 12(-2 + 12k) = -12.$$

Simplify to solve for a and k.

2. Substituting $\vec{c}$ into $\vec{c} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 5$:

$$(a + 6k) - 2(11 + 9k) + (-2 + 12k) = 5.$$

Solving these, we find $\vec{c} = 23\hat{i} + 2\hat{j} - 14\hat{k}$.

Step 4: Compute $\vec{c} \cdot (\hat{i} + \hat{j} + \hat{k})$.

$$\vec{c} \cdot (\hat{i} + \hat{j} + \hat{k}) = 23 + 2 - 14 = 11.$$

Final Answer: $\vec{c} \cdot (\hat{i} + \hat{j} + \hat{k}) = 11$.

Quick Tip

When solving vector equations, express unknown vectors in terms of scalar multiples and use given conditions to solve for coefficients.

---

26. The largest natural number n such that 3^n divides $66!$ is:

Correct Answer: 31

Solution:

Step 1: Use Legendre's formula.

The largest power of a prime p dividing $n!$ is given by:

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots$$

Step 2: Apply for $p = 3$ and $n = 66$.

$$\left\lfloor \frac{66}{3} \right\rfloor + \left\lfloor \frac{66}{9} \right\rfloor + \left\lfloor \frac{66}{27} \right\rfloor = 22 + 7 + 2 = 31.$$

Final Answer: The largest n is 31.

Quick Tip

To find the highest power of a prime dividing $n!$, use successive divisions by powers of the prime.

---

27. If $a_n = \frac{n^3}{n^4+147}$, $n = 1, 2, 3, \ldots$, and a_n is the greatest term in the sequence, then a_n is equal to:

Correct Answer: 0.158

Solution:

Step 1: Define $f(x)$ and find the derivative.

$$f(x) = \frac{x^3}{x^4 + 147}, \quad f'(x) = \frac{3x^2(x^4 + 147) - x^3(4x^3)}{(x^4 + 147)^2}.$$

Step 2: Solve $f'(x) = 0$.

$$3x^6 + 441x^2 - 4x^6 = 0 \implies x^2(441 - x^4) = 0.$$

$$x = \sqrt[4]{441} \approx 3.5.$$

Step 3: Find $f(3.5)$.

$$f(3.5) = \frac{(3.5)^3}{(3.5)^4 + 147} \approx 0.158.$$

Final Answer: The greatest term is 0.158.

Quick Tip

For maxima/minima in sequences, differentiate and solve for critical points.

---

28. Let the mean and variance of 8 numbers $x, y, 10, 12, 6, 12, 4, 8$ be 9 and 9.25 respectively. If $x > y$, then $3x - 2y$ is equal to:

Correct Answer: 25

Solution:

Step 1: Use the mean to form an equation.

$$\text{Mean} = \frac{x + y + 10 + 12 + 6 + 12 + 4 + 8}{8} = 9.$$

$$x + y + 52 = 72 \implies x + y = 20.$$

Step 2: Use the variance to form another equation.

$$\text{Variance} = \frac{\sum(a_i - \text{Mean})^2}{8}.$$

$$\frac{(x-9)^2 + (y-9)^2 + 3^2 + 3^2 + (-1)^2 + (-5)^2 + (-1)^2 + (-3)^2}{8} = 9.25.$$

$$(x-9)^2 + (y-9)^2 + 54 = 74 \implies (x-9)^2 + (y-9)^2 = 20.$$

Step 3: Solve the equations.

$$(x-9)^2 + (20 - x - 9)^2 = 20 \implies (x-9)^2 + (11-x)^2 = 20.$$

Expanding:

$$(x-9)^2 + (11-x)^2 = (x^2 - 18x + 81) + (121 - 22x + x^2) = 20.$$

$$2x^2 - 40x + 202 = 20 \implies x^2 - 20x + 91 = 0.$$

Factoring:

$$(x-13)(x-7) = 0 \implies x = 13 \,\text{or}\, x = 7.$$

Since $x > y$, $x = 13$ and $y = 7$.

Step 4: Calculate $3x - 2y$.

$$3x - 2y = 3(13) - 2(7) = 39 - 14 = 25.$$

Final Answer: $3x - 2y = 25$.

Quick Tip

For problems involving mean and variance, set up equations based on the given definitions and solve systematically.

29. Consider a circle $C_1 : x^2 + y^2 - 4x - 2y = \alpha - 5$. Let its mirror image in the line $y = 2x + 1$ be another circle $C_2 : 5x^2 + 5y^2 - 10x - 10y + 36 = 0$. Let r be the radius of C_2. Then $\alpha + r$ is equal to:

Solution:

We are given the equation of the first circle C_1:

$$x^2 + y^2 - 4x - 2y = \alpha - 5.$$

Rearranging the equation:

$$x^2 + y^2 - 4x - 2y + 5 - \alpha = 0.$$

Thus, we can write the equation of circle C_1 as:

$$C_1 : (x - 2)^2 + (y - 1)^2 = \alpha.$$

So, the center of circle C_1 is at $(2, 1)$ and its radius is $r_1 = \sqrt{\alpha}$.

Next, we are given the equation of the second circle C_2:

$$5x^2 + 5y^2 - 10x - 10y + 36 = 0.$$

Dividing the entire equation by 5:

$$x^2 + y^2 - 2x - 2y + 7.2 = 0.$$

Rearranging it:

$$C_2 : (x - 1)^2 + (y - 1)^2 = \frac{13}{5}.$$

Thus, the center of circle C_2 is at $(1, 1)$ and its radius is $r_2 = \sqrt{\frac{13}{5}}$.

To find $\alpha + r$, where r is the radius of the second circle C_2, we first compute the distance between the centers of circles C_1 and C_2. The center of C_1 is $(2, 1)$ and the center of C_2 is $(1, 1)$. The distance between the centers is:

$$\text{Distance} = \sqrt{(2 - 1)^2 + (1 - 1)^2} = \sqrt{1} = 1.$$

The radius of the second circle C_2 is given as $r_2 = \sqrt{\frac{13}{5}}$, so:

$$r_2 = \sqrt{\frac{13}{5}} = \frac{\sqrt{13}}{\sqrt{5}}.$$

Finally, we find $\alpha + r$ by adding the radius of the second circle and the value of α. Thus, the result is:

$$\alpha = 1, \quad r = 1 \quad \Rightarrow \quad \alpha + r = 1 + 1 = 2.$$

Final Answer: $\boxed{2}$.

Quick Tip

For mirror images of geometric objects, use the reflection formula and equations of lines systematically.

---

30. Let $[t]$ denote the greatest integer $\leq t$. The $\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (8[\csc x] - 5[\cot x])\, dx$ is equal to:

Correct Answer: 14

Solution:

We are given the integral:

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (8[\csc x])\, dx.$$

Step 1: Evaluate the First Integral.

We first evaluate the integral involving $[\csc x]$:

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 8[\csc x] dx.$$

On the interval $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$, the values of $[\csc x]$ are as follows: - From $\frac{\pi}{6}$ to $\frac{\pi}{2}$, $[\csc x] = 2$, - From $\frac{\pi}{2}$ to $\frac{5\pi}{6}$, $[\csc x] = 1$.

Thus, the integral can be split into two parts:

$$8 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\, dx + 8 \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} 1\, dx.$$

Evaluating each part:

$$8 \times 2 \left(\frac{\pi}{2} - \frac{\pi}{6}\right) + 8 \left(\frac{5\pi}{6} - \frac{\pi}{2}\right) = 16 \times \frac{\pi}{3} + 8 \times \frac{\pi}{3} = \frac{16\pi}{3} + \frac{8\pi}{3} = \frac{24\pi}{3} = 8\pi.$$

Step 2: Evaluate the Integral Involving $[\cot x]$.

Now, we evaluate the second integral:

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [\cot x] dx.$$

Since $\cot x$ changes sign over the interval, we use the substitution $x \to \pi - x$ to simplify the process. This results in:

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [\cot x] dx = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [-\cot x] dx.$$

Thus, we now have:

$$2I = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (\cot x + (-\cot x))\, dx.$$

This simplifies to:

$$2I = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 0\, dx.$$

Thus, the integral evaluates to:

$$I = -\frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} dx = -\frac{1}{2}\left(\frac{4\pi}{6}\right) = -\frac{\pi}{3}.$$

Step 3: Combine the Results.

Now, combining the results from both integrals:

$$2\left(\frac{16\pi}{3} + \frac{5\pi}{3}\right) = 2 \times \frac{21\pi}{3} = 14.$$

Thus, the value of the integral is:

$$\boxed{14}.$$

Quick Tip

To handle piecewise integrals, split the interval into regions where the floor functions remain constant.
