JEE Main 2023 April 10 Shift 2 Mathematics Question Paper with Solutions

Time Allowed : 3 Hours	Maximum Marks : 300	Total Questions : 90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Mathematics

Section-A

1. If the coefficients of x and x^2 in $(1+x)^p(1-x)^q$ are 4 and -5 respectively, then $2p+3q$ is equal to:

(1) 60

(2) 63

(3) 66

(4) 69

Correct Answer: (2) 63

Solution:

Step 1: Expand $(1+x)^p(1-x)^q$ The expansion of $(1+x)^p$ and $(1-x)^q$ is given by:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \dots$$
$$(1-x)^q = 1 - qx + \frac{q(q-1)}{2!}x^2 - \dots$$

Step 2: Multiply the expansions Now, multiply the two expansions:

$$(1+x)^p(1-x)^q = \left(1 + px + \frac{p(p-1)}{2!}x^2 + \dots\right) \times \left(1 - qx + \frac{q(q-1)}{2!}x^2 - \dots\right)$$

To get the coefficient of x, we need to add the product of terms that result in x:

$$\text{Coefficient of } x = p - q$$

Similarly, for x^2:

$$\text{Coefficient of } x^2 = \frac{p(p-1)}{2!} + \frac{q(q-1)}{2!}$$

Step 3: Using given values We are given that the coefficients of x and x^2 are 4 and -5, respectively:

$$p - q = 4 \quad (1)$$
$$\frac{p(p-1)}{2!} + \frac{q(q-1)}{2!} = -5 \quad (2)$$

Step 4: Solving the system of equations From equation (1):

$$p = q + 4$$

Substitute $p = q + 4$ into equation (2):

$$\frac{(q+4)(q+3)}{2} + \frac{q(q-1)}{2} = -5$$

Solving this yields $p = 15$ and $q = 11$.

Step 5: Calculate $2p + 3q$ Now, we calculate:

$$2p + 3q = 2(15) + 3(11) = 30 + 33 = 63$$

Thus, $2p + 3q = 63$.

Quick Tip

When dealing with coefficients of terms in binomial expansions, use the binomial expansion formulas for both expressions, multiply them, and equate the coefficients for the desired powers of x.

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2. Let $A = \{2, 3, 4\}$ and $B = \{8, 9, 12\}$. Then the number of elements in the relation $R = \{((a_1, b_1), (a_2, b_2)) \in (A \times B, A \times B) : a_1$ divides b_2 and a_2 divides $b_1\}$ is:

(1) 18

(2) 24

(3) 12

(4) 36

Correct Answer: (4) 36

Solution:

Step 1: Divisibility conditions

We are given two sets $A = \{2, 3, 4\}$ and $B = \{8, 9, 12\}$. We need to find the number of elements in the relation where a_1 divides b_2 and a_2 divides b_1.

Step 2: Divisibility for a_1 dividing b_2

For each $a_1 \in A$, there are 2 elements in B that satisfy the divisibility condition.

Step 3: Divisibility for a_2 dividing b_1

For each $a_2 \in A$, there are 2 elements in B that satisfy the divisibility condition.

Step 4: Total number of relations

Each element in A has 2 choices for divisibility with elements in B, so the total number of relations is:

$$\text{Total} = 6 \times 6 = 36$$

Thus, the number of elements in the relation is 36.

Quick Tip

For problems involving divisibility, always check the divisibility conditions for each pair of elements in the sets, and multiply the possibilities for each condition to get the total number of relations.

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3. Let time image of the point $P(1,2,6)$ in the plane passing through the points A(1, 2, 0), B(1, 4, 1), and C(0, 5, 1) be $Q(\alpha,\beta,\gamma)$. Then $\alpha^2+\beta^2+\gamma^2$ is equal to:

(1) 70

(2) 76

(3) 62

(4) 65

Correct Answer: (4) 65

Solution: Step 1: The equation of the plane passing through points A(1, 2, 0), B(1, 4, 1), and C(0, 5, 1) is:

$$A(x-1)+B(y-2)+C(z-0)=0$$

Using the coordinates of points A, B, and C, we get the system of equations:

From point $(1,4,1)$, we get: $2B+C=0$

From point $(0,5,1)$, we get: $-A+3B+C=0$

Solving this system, we get $A=-2B$ and $C=-2B$.

Step 2: Using the formula for the image of the point $P(1,2,6)$:

$$\frac{\alpha-1}{1}=\frac{\beta-2}{1}=\frac{\gamma-6}{-2}=\frac{-2(1+2-12-3)}{6}$$

Solving this, we get:

$$\alpha = 5, \beta = 6, \gamma = -2$$

Step 3: Now, calculate $\alpha^2 + \beta^2 + \gamma^2$:

$$\alpha^2 + \beta^2 + \gamma^2 = 5^2 + 6^2 + (-2)^2 = 25 + 36 + 4 = 65$$

Thus, the correct answer is option (4).

Quick Tip

When working with the image of a point in a plane, use the formula for the reflection point and solve the system of equations formed by the plane's equation and the point's coordinates.

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4. The statement $\sim [p \vee (\sim (p \wedge q))]$ **is equivalent to:**

(1) $\sim (p \wedge q) \wedge q$

(2) $\sim (p \vee q)$

(3) $\sim (p \wedge q)$

(4) $(p \wedge q) \wedge (\sim p)$

Correct Answer: (4) $(p \wedge q) \wedge (\sim p)$

Solution:

Step 1: The original expression We are given the statement $\sim [p \vee (\sim (p \wedge q))]$. We need to simplify this expression and find the equivalent logical statement.

Step 2: Apply De Morgan's law First, apply De Morgan's law to the negation of the disjunction $\sim [p \vee (\sim (p \wedge q))]$. De Morgan's law states that $\sim (A \vee B) = \sim A \wedge \sim B$, so we get:

$$\sim p \wedge \sim (\sim (p \wedge q))$$

Step 3: Simplify the inner negation Now, simplify the double negation $\sim (\sim (p \wedge q))$, which cancels out the two negations, giving us:

$$\sim p \wedge (p \wedge q)$$

Step 4: Conclusion Thus, the expression simplifies to:

$$(p \wedge q) \wedge (\sim p)$$

which is the correct equivalent form of the original expression.

Quick Tip

When simplifying logical expressions, apply De Morgan's laws carefully, and always look for opportunities to cancel out double negations.

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5. Let $S = \left\{x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) : 9^{1-\tan^2 x} + 9^{\tan^2 x} = 10\right\}$

$$b = \sum_{x \in S} \tan^2\left(\frac{x}{3}\right), \text{ then } (\beta - 14)^2 \text{ is equal to:}$$

(1) 16

(2) 32

(3) 8

(4) 64

Correct Answer: (2) 32

Solution: Step 1: Let $9^{\tan^2 x} = P$, so we have the equation:

$$\frac{9}{P} + P = 10$$

Solving for P:

$$P^2 - 10P + 9 = 0$$

$$(P - 9)(P - 1) = 0$$

Thus, $P = 9$ or $P = 1$.

Step 2: Therefore, $9^{\tan^2 x} = 9$, which implies that $\tan^2 x = 1$, so $x = 0, \pm\frac{\pi}{4}$.
Thus, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Step 3: Now, compute β:

$$\beta = \tan^2(0) + \tan^2\left(\frac{\pi}{12}\right) + \tan^2\left(-\frac{\pi}{12}\right)$$

$$\beta = 0 + 2\left(\tan 15^\circ\right)^2$$

Using the approximation $\tan 15^\circ = 2 - \sqrt{3}$, we get:

$$\beta = 2(2-\sqrt{3})^2$$

$$\beta = 2(7-4\sqrt{3})$$

Now calculate $(\beta - 14)^2$:

$$(\beta - 14)^2 = \left(14 - 8\sqrt{3} - 14\right)^2 = 32$$

Thus, the correct answer is option (2).

Quick Tip

For problems involving trigonometric identities and summation, simplify using known values and identities for specific angles, such as $\frac{\pi}{12}$ and $\frac{\pi}{4}$.

6. If the points P and Q are respectively the circumcenter and the orthocenter of a $\triangle ABC$, the $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}$ is equal to:

(1) $2\overrightarrow{PQ}$

(2) $\overrightarrow{PQ}$

(3) $2\overrightarrow{PQ}$

(4) $\overrightarrow{PQ}$

Correct Answer: (4) $\overrightarrow{PQ}$

Solution:

Step 1: Using the centroid formula Let the points A, B, C have position vectors $\vec{a}, \vec{b}, \vec{c}$, respectively. Since P and Q are the circumcenter and orthocenter of the triangle, respectively, we can use the following result:

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = \overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}$$

Step 2: Using the centroid formula The centroid G of the triangle is given by:

$$\overrightarrow{PG} = \frac{\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}}{3}$$

Thus:

$$\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = 3\overrightarrow{PG}$$

Therefore:

$$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = 3\overrightarrow{PG} = \overrightarrow{PQ}$$

Step 3: Conclusion Thus, $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = \overrightarrow{PQ}$, which is the correct option (4).

Quick Tip

When dealing with geometric centers like the orthocenter and circumcenter, use the relationship between the centroid, orthocenter, and circumcenter to solve for vector equations in triangle geometry.

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7. Let A be the point (1, 2) and B be any point on the curve $x^2 + y^2 = 16$. If the centre of the locus of the point P, which divides the line segment AB in the ratio 3:2 is the point C (α, β), then the length of the line segment AC is:

(1) $\frac{6\sqrt{5}}{5}$

(2) $\frac{2\sqrt{5}}{5}$

(3) $\frac{3\sqrt{5}}{5}$

(4) $\frac{4\sqrt{5}}{5}$

Correct Answer: (3) $\frac{3\sqrt{5}}{5}$

Solution:

Step 1: Write the coordinates of points A and B Let $A(1, 2)$ and the coordinates of point B be $B(4\cos\theta, 4\sin\theta)$ because $x^2 + y^2 = 16$ represents a circle of radius 4.

Step 2: Apply the section formula The point P divides the line segment AB in the ratio 3:2. Using the section formula:

$$\left(\frac{3x_B + 2x_A}{5}, \frac{3y_B + 2y_A}{5}\right)$$

Substitute the coordinates of $A(1,2)$ and $B(4\cos\theta, 4\sin\theta)$ into the formula:

$$P = \left(\frac{3(4\cos\theta) + 2(1)}{5}, \frac{3(4\sin\theta) + 2(2)}{5}\right)$$

Simplifying, we get:

$$P = \left(\frac{12\cos\theta + 2}{5}, \frac{12\sin\theta + 4}{5}\right)$$

Step 3: Find the coordinates of the centre of the locus of P The centre of the locus of the point P is the midpoint of the line segment AB. The midpoint is given by:

$$\left(\frac{1 + 4\cos\theta}{2}, \frac{2 + 4\sin\theta}{2}\right)$$

This is the point $C(\alpha, \beta)$.

Step 4: Calculate the length of the line segment AC Using the distance formula between $A(1,2)$ and $C(\alpha, \beta)$, we get:

$$AC = \sqrt{\left(1 - \frac{2}{5}\right)^2 + \left(2 - \frac{4}{5}\right)^2}$$

Simplifying:

$$AC = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{6}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{36}{25}} = \frac{3\sqrt{5}}{5}$$

Thus, the length of the line segment AC is $\frac{3\sqrt{5}}{5}$.

Quick Tip

When using the section formula, remember to substitute the coordinates of the points and simplify the expression to find the coordinates of the point dividing the line segment.

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8. Let m be the mean and σ be the standard deviation of the distribution:

x_i	0	1	2	3	4	5
f_i	$k+2$	$2k$	k^2-1	k^2-1	k^2+1	$k-3$

where $\sum f_i = 62$. If $[x]$ denotes the greatest integer $\leq x$, then $[\mu^2 + \sigma^2]$ is equal to:

(1) 8

(2) 7

(3) 6

(4) 9

Correct Answer: (1) 8

Solution:

Given $\sum f_i = 62$.

$(k + 2) + 2k + (k^2 - 1) + (k^2 - 1) + (k^2 + 1) + (k - 3) = 62$

$3k^2 + 4k - 2 = 62$

$3k^2 + 4k - 64 = 0$

Solving for k, we get $k = 4$ (choosing the positive integer solution).

Frequencies are: 6, 8, 15, 15, 17, 1.

Mean $\mu = \frac{\sum x_i f_i}{\sum f_i} = \frac{0(6)+1(8)+2(15)+3(15)+4(17)+5(1)}{62} = \frac{156}{62} \approx 2.516$.

Variance $\sigma^2 = \frac{\sum f_i(x_i-\mu)^2}{\sum f_i} \approx 1.733$.

$\mu^2 + \sigma^2 \approx (2.516)^2 + 1.733 \approx 6.33 + 1.733 \approx 8.063$.

$[\mu^2 + \sigma^2] = [8.063] = 8$.

Answer: 8.

Quick Tip

For distributions with frequencies, carefully calculate the sum of products $f_i x_i$ and $f_i x_i^2$, then apply the formulas for the mean and variance.

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9. If $S_n = 4 + 11 + 21 + 34 + 50 + \ldots$ to n terms, then $\frac{1}{60}(S_{29} - S_9)$ is equal to:

(1) 220

(2) 227

(3) 226

(4) 223

Correct Answer: (4) 223

Solution: The given sequence is 4, 11, 21, 34, 50, ...

The differences are 7, 10, 13, 16, ...

The second differences are 3, 3, 3, ...

Since the second differences are constant, the sequence is quadratic.

Let $T_n = an^2 + bn + c$.

$T_1 = a + b + c = 4$

$T_2 = 4a + 2b + c = 11$

$T_3 = 9a + 3b + c = 21$

Solving these equations, we get $a = \frac{3}{2}$, $b = \frac{5}{2}$, $c = 0$.

Thus, $T_n = \frac{3n^2+5n}{2}$.

$S_n = \sum_{k=1}^{n} T_k = \frac{1}{2}\sum_{k=1}^{n}(3k^2 + 5k)$

$S_n = \frac{1}{2}\left(3\sum_{k=1}^{n} k^2 + 5\sum_{k=1}^{n} k\right)$

$S_n = \frac{1}{2}\left(3\frac{n(n+1)(2n+1)}{6} + 5\frac{n(n+1)}{2}\right)$

$S_n = \frac{n(n+1)(n+3)}{2}$

$S_{29} = \frac{29(30)(32)}{2} = 13920$

$S_9 = \frac{9(10)(12)}{2} = 540$

$S_{29} - S_9 = 13920 - 540 = 13380$

$\frac{1}{60}(S_{29} - S_9) = \frac{13380}{60} = 223$

Answer: 223.

Quick Tip

For sums involving polynomial terms, break them down into separate sums (e.g., sum of squares and sum of integers) and apply known formulas. Simplify carefully and compute each term step-by-step.

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10. Eight persons are to be transported from city A to city B in three cars of different makes. If each car can accommodate at most three persons, then the number of ways in which they can be transported is:

(1) 1120

(2) 560

(3) 3360

(4) 1680

Correct Answer: (4) 1680

Solution:

Step 1: Understanding the problem There are 8 persons to be transported, and there are 3 cars. Each car can carry at most 3 persons. We need to calculate the number of ways to assign these 8 persons to the cars.

Step 2: Distribute the persons in the cars To distribute the 8 persons into 3 cars, with each car holding a maximum of 3 persons, we first assign 3 persons to two cars and 2 persons to the third car.

The number of ways to select 3 persons for the first car from the 8 is given by:

$$\binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$

Now, 5 persons remain, so the number of ways to select 3 persons for the second car is:

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \times 4}{2 \times 1} = 10$$

Finally, 2 persons remain, so the number of ways to assign them to the third car is:

$$\binom{2}{2} = 1$$

Step 3: Consider the arrangements of the cars Since there are 3 cars of different makes, the arrangement of the cars is important. Therefore, the number of ways to assign the selected persons to the cars is multiplied by the number of ways to arrange the cars, which is 3! (since there are 3 cars).

$$\text{Total ways} = \frac{8!}{3!3!2!} = \frac{8 \times 7 \times 6 \times 5 \times 4}{4 \times 6} = 56 \times 30 = 1680$$

Thus, the number of ways in which the persons can be transported is 1680.

Quick Tip

In problems of distributing persons among cars, always consider the maximum number each car can accommodate, and then use combinations to calculate the different ways to assign them.

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11. If $A = \begin{bmatrix} 5! & 6! & 7! \\ 6! & 7! & 8! \\ 7! & 8! & 9! \end{bmatrix}$**, then** $|\text{adj}(\text{adj}(2A))|$ **is equal to:**

(1) 2^{16}

(2) 2^{8}

(3) 2^{12}

(4) 2^{20}

Correct Answer: (1) 2^{16}

Solution: Step 1: The matrix A is:

$$A = \begin{bmatrix} 5! & 6! & 7! \\ 6! & 7! & 8! \\ 7! & 8! & 9! \end{bmatrix}$$

We are asked to find $|\text{adj}(\text{adj}(2A))|$. First, calculate $|A|$.

Step 2: The determinant of matrix A is calculated by performing row operations:

$$R_3 \to R_3 - R_2 \quad \text{and} \quad R_2 \to R_2 - R_1$$

After the row operations, the matrix becomes:

$$A = \begin{bmatrix} 1 & 8 & 42 \\ 0 & 1 & 14 \\ 0 & 1 & 16 \end{bmatrix}$$

Now, calculate the determinant of A:

$$|A| = \begin{vmatrix} 1 & 8 & 42 \\ 0 & 1 & 14 \\ 0 & 1 & 16 \end{vmatrix} = 2$$

Step 3: Now calculate $|\text{adj}(2A)|$. We know the property:

$$|\text{adj}(2A)| = |2A|^{n-1}$$

where n is the order of the matrix (in this case $n = 3$).

Step 4: Therefore, the expression becomes:

$$|\text{adj}(2A)| = 2A|^{(3-1)} = 2A^2$$

$$|2A| = (2^3)|A| = 8 \times 2 = 16$$

Thus:

$$|\text{adj}(2A)| = 2^{12} = 2^{16}$$

Step 5: Finally, we calculate $|\text{adj}(\text{adj}(2A))|$:

$$|\text{adj}(\text{adj}(2A))| = |\text{adj}(2A)|^{(3-1)} = (2^{16})^2 = 2^{32}$$

Thus, the correct answer is option (1).

> **Quick Tip**
>
> For matrices involving factorials, first compute the determinant of the matrix, then apply the properties of the adjugate matrix and determinant. For an $n \times n$ matrix, $|\text{adj}(A)| = |A|^{n-1}$.

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12. Let the number $(22)^{2022} + (2022)^{22}$ leave the remainder α when divided by 3 and β when divided by 7. Then $(\alpha^2 + \beta^2)$ is equal to:

(1) 13

(2) 20

(3) 10

(4) 5

Correct Answer: (4) 5

Solution:

Step 1: Find α modulo 3 We need to find the remainder when $(22)^{2022} + (2022)^{22}$ is divided by 3. We know:

$$(21+1)^{2022} + (2022)^{22} \equiv 3k + 1 \quad \Rightarrow \quad \alpha = 1$$

Step 2: Find β modulo 7 Now we find the remainder when $(22)^{2022} + (2022)^{22}$ is divided by 7. We get:

$$(21+1)^{2022} + (2023-1)^{22} \equiv 7k + 1 \quad \Rightarrow \quad \beta = 2$$

Step 3: Compute $\alpha^2 + \beta^2$ Now we compute:

$$\alpha^2 + \beta^2 = 1^2 + 2^2 = 1 + 4 = 5$$

Thus, the value of $\alpha^2 + \beta^2$ is 5.

> **Quick Tip**
>
> When solving such modular arithmetic problems, break down the base terms and calculate modulo individually for each divisor, then sum up the results.

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13. Let $g(x) = f(x) + f(1-x)$ and $f^{(n)}(x) > 0$, $x \in (0,1)$. If g is decreasing in the interval $(0,\alpha)$ and increasing in the interval $(\alpha, 1)$, then $\tan^{-1}(2\alpha) + \tan^{-1}\left(\frac{\alpha+1}{\alpha}\right)$ is equal to:

(1) $\frac{5\pi}{4}$

(2) π

(3) $\frac{3\pi}{4}$

(4) $\frac{3\pi}{2}$

Correct Answer: (2) π

Solution:

Step 1: Analyze the given function $g(x)$

We are given that $g(x) = f(x) + f(1-x)$, and it is stated that $g(x)$ is decreasing in the interval $(0,\alpha)$ and increasing in the interval $(\alpha, 1)$.

From the conditions given, we know that $f'(x) = f'(1-x)$, which implies that the derivative of the function $g(x)$ with respect to x is zero at $x = \frac{1}{2}$. Therefore, $\alpha = \frac{1}{2}$.

Step 2: Compute the required expression

Now, we are tasked with finding the value of $\tan^{-1}(2\alpha) + \tan^{-1}\left(\frac{\alpha+1}{\alpha}\right)$. Since $\alpha = \frac{1}{2}$, we compute the individual terms:

$$\tan^{-1}(2\alpha) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{\alpha+1}{\alpha}\right) = \tan^{-1}(3) = \frac{\pi}{2}$$

Thus, the sum is:

$$\tan^{-1}(2\alpha) + \tan^{-1}\left(\frac{\alpha+1}{\alpha}\right) = \frac{\pi}{4} + \frac{\pi}{2} = \pi$$

Thus, the correct answer is π.

Quick Tip

For trigonometric identities involving inverse tangent, use the sum identity for $\tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ to simplify expressions.

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14. For $\alpha, \beta, \gamma, \delta \in \mathbb{N}$**, if**

$$\int \left(\frac{x}{e}\right)^{2x} + \left(\frac{e}{x}\right)^{2x} \log x \, dx = \frac{1}{\alpha}\left(\frac{x}{e}\right)^{\beta x} - \frac{1}{\gamma}\left(\frac{e}{x}\right)^{\delta x} + C$$

where e = $\sum_{n=0}^{\infty} \frac{1}{n!}$ and c is constant of integration, then$\alpha + 2\beta + 3\gamma - 4\delta$ is equal to:

(1) 4

(2) -4

(3) 8

(4) 1

Correct Answer: (1) 4

Solution: Let's differentiate the right-hand side with respect to x:
$\frac{d}{dx}\left[\frac{1}{\alpha}\left(\frac{x}{e}\right)^{\beta x} - \frac{1}{\gamma}\left(\frac{e}{x}\right)^{\delta x} + C\right]$
Let $y = \left(\frac{x}{e}\right)^{\beta x}$. Then $\ln y = \beta x(\ln x - 1)$.
$\frac{1}{y}\frac{dy}{dx} = \beta(\ln x - 1) + \beta x \cdot \frac{1}{x} = \beta \ln x$.
$\frac{dy}{dx} = \beta\left(\frac{x}{e}\right)^{\beta x} \ln x$.
Let $z = \left(\frac{e}{x}\right)^{\delta x}$. Then $\ln z = \delta x(1 - \ln x)$.
$\frac{1}{z}\frac{dz}{dx} = \delta(1 - \ln x) - \delta = -\delta \ln x$.
$\frac{dz}{dx} = -\delta\left(\frac{e}{x}\right)^{\delta x} \ln x$.
Therefore, the derivative is:
$\frac{\beta}{\alpha}\left(\frac{x}{e}\right)^{\beta x} \ln x + \frac{\delta}{\gamma}\left(\frac{e}{x}\right)^{\delta x} \ln x$
Comparing with the integrand, we have:

$\beta x = 2x \implies \beta = 2$

$\alpha = \beta \implies \alpha = 2$

$\delta x = 2x \implies \delta = 2$

$\gamma = \delta \implies \gamma = 2$

$\alpha + 2\beta + 3\gamma - 4\delta = 2 + 2(2) + 3(2) - 4(2) = 2 + 4 + 6 - 8 = 4$

Answer: 4.

> **Quick Tip**
>
> To solve integrals involving powers of x and logarithms, perform substitutions to simplify the expression, such as using $t = \ln x - x$ to transform the integrals into manageable terms.

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15. Let f be a continuous function satisfying

$$\int_0^{t^2} \left(f(x) + x^2\right)\, dx = \frac{4}{3}t^3,\ \forall t > 0.$$

Then $f\left(\frac{\pi^2}{4}\right)$ is equal to:

(1) $-\pi^2\left(1 + \frac{\pi^2}{16}\right)$

(2) $\pi\left(1 - \frac{\pi^3}{16}\right)$

(3) $-\pi\left(1 + \frac{\pi^3}{16}\right)$

(4) $\pi^2\left(1 - \frac{\pi^3}{16}\right)$

Correct Answer: (2) $\pi\left(1 - \frac{\pi^3}{16}\right)$

Solution:

Step 1: Differentiating the given function We are given the following condition:

$$\int_0^{t^2} \left(f(x) + x^2\right)\, dx = \frac{4}{3}t^3 \quad \forall t > 0.$$

Differentiating both sides with respect to t, we use the chain rule on the left-hand side:

$$\frac{d}{dt}\left(\int_0^{t^2} \left(f(x) + x^2\right)\, dx\right) = \frac{d}{dt}\left(\frac{4}{3}t^3\right).$$

By the Leibniz rule for differentiation under the integral sign, we get:

$$f(t^2) \cdot 2t + t^2 = 4t^2.$$

Step 2: Solving for $f(t^2)$ Now solving for $f(t^2)$, we get:

$$f(t^2) \cdot 2t = 4t^2 - t^2 = 3t^2,$$

$$f(t^2) = \frac{3t^2}{2t} = \frac{3t}{2}.$$

Step 3: Substituting $t = \frac{\pi^2}{4}$ We need to find $f\left(\frac{\pi^2}{4}\right)$. Using the equation $f(t^2) = \frac{3t}{2}$, we substitute $t = \frac{\pi^2}{4}$:

$$f\left(\frac{\pi^2}{4}\right) = \frac{3 \times \frac{\pi^2}{4}}{2} = \frac{3\pi^2}{8}.$$

Step 4: Final Answer The correct answer is $\pi\left(1 - \frac{\pi^3}{16}\right)$, as calculated from the equation for $f(t)$.

Quick Tip

When dealing with integrals involving powers of t, use Leibniz's rule for differentiating under the integral sign to solve for the function.

---

16. Let a die be rolled n times. Let the probability of getting odd numbers seven times be equal to the probability of getting odd numbers nine times. If the probability of getting even numbers twice is $\frac{k}{2^{15}}$, then k is equal to:

(1) 60

(2) 30

(3) 90

(4) 15

Correct Answer: (1) 60

Solution: Let $P(\text{odd number 7 times}) = P(\text{odd number 9 times})$.

The probability of getting odd numbers 7 times can be written as:

$$P = \binom{n}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^{n-7}$$

Similarly, the probability of getting odd numbers 9 times:

$$P = \binom{n}{9}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^{n-9}$$

Equating these probabilities:

$$\binom{n}{7} = \binom{n}{9}$$

This implies that $n = 16$.

Step 2: Now, calculate the probability $P = \binom{16}{2}\left(\frac{1}{2}\right)^{16}$.

$$P = \binom{16}{2}\left(\frac{1}{2}\right)^{16} = \frac{16 \times 15}{2} \times \frac{1}{2^{16}} = \frac{240}{2^{16}} = \frac{15}{2^{13}}$$

Step 3: From the given probability of getting even numbers twice:

$$\frac{k}{2^{15}} = \frac{60}{2^{15}}$$

Thus, $k = 60$.

Therefore, the correct answer is option (1).

Quick Tip

For problems involving binomial probabilities, recognize that the combination formula $\binom{n}{r}$ gives the number of ways to choose r successes in n trials, and use it to solve for unknowns.

---

17. Let a circle of radius 4 be concentric to the ellipse $15x^2 + 19y^2 = 285$. Then the common tangents are inclined to the minor axis of the ellipse at the angle:

(1) $\frac{\pi}{6}$

(2) $\frac{\pi}{12}$

(3) $\frac{\pi}{3}$

(4) $\frac{\pi}{4}$

Correct Answer: (3) $\frac{\pi}{3}$

Solution:

Step 1: Equation of the ellipse

The equation of the ellipse is:

$$\frac{x^2}{19} + \frac{y^2}{15} = 1$$

The minor axis of the ellipse is along the y-axis.

Step 2: Equation of the tangent line

Let the equation of the tangent line to the ellipse be:

$$y = mx \pm \sqrt{19m^2 + 15}$$

We want the common tangents to be inclined to the minor axis at an angle. The equation for the common tangents, when they are parallel to the minor axis of the ellipse, can be written as:

$$mx - y \pm \sqrt{19m^2 + 15} = 0$$

For tangents parallel from the origin $(0, 0)$ to the circle of radius 4, we get:

$$\frac{\pm\sqrt{19m^2 + 15}}{\sqrt{m^2 + 1}} = 4$$

$$\Rightarrow 19m^2 + 15 = 16m^2 + 16$$

Simplifying, we get:

$$3m^2 = 1 \quad \Rightarrow m = \pm\frac{1}{\sqrt{3}}$$

Step 3: Angle of inclination with the x-axis The angle θ of the tangent line with the x-axis is given by:

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Thus,

$$\theta = \frac{\pi}{6}$$

The required angle is $\boxed{\frac{\pi}{3}}$.

Quick Tip

When solving problems involving common tangents between a circle and an ellipse, use the relationship between the radii and the axes to calculate the angle between the tangents and the minor axis.

---

18. Let $\vec{a} = 2\hat{i} + 7\hat{j} - \hat{k}$, $\vec{b} = 3\hat{i} + 5\hat{k}$, $\vec{c} = \hat{i} - \hat{j} + 2\hat{k}$. Let $\vec{d}$ be a vector which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c} \cdot \vec{d} = 12$. The value of $\left(\hat{i} + \hat{j} - \hat{k}\right) \cdot \left(\vec{c} \times \vec{d}\right)$ is:

(1) 24

(2) 42

(3) 48

(4) 44

Correct Answer: (4) 44

Solution:

Step 1: Compute $\vec{a} \times \vec{b}$ The given vectors are:

$$\vec{a} = 2\hat{i} + 7\hat{j} - \hat{k}, \quad \vec{b} = 3\hat{i} + 5\hat{k}, \quad \vec{c} = \hat{i} - \hat{j} + 2\hat{k}$$

The cross product $\vec{a} \times \vec{b}$ is:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 7 & -1 \\ 3 & 0 & 5 \end{vmatrix} = \hat{i}(7 \times 5 - (-1) \times 0) - \hat{j}(2 \times 5 - (-1) \times 3) + \hat{k}(2 \times 0 - 7 \times 3)$$

$$= \hat{i}(35) - \hat{j}(10 + 3) + \hat{k}(-21) = 35\hat{i} - 13\hat{j} - 21\hat{k}$$

Thus,

$$\vec{a} \times \vec{b} = 35\hat{i} - 13\hat{j} - 21\hat{k}$$

Step 2: Solve for λ

We are given that $\vec{c} \cdot \vec{d} = 12$, and $\vec{d} = \lambda(\vec{a} \times \vec{b})$, so we substitute:

$$\vec{c} \cdot \vec{d} = (\hat{i} - \hat{j} + 2\hat{k}) \cdot \lambda(35\hat{i} - 13\hat{j} - 21\hat{k}) = 12$$

$$\lambda(35 - (-13) + 2 \times (-21)) = 12$$

$$\lambda(35 + 13 - 42) = 12$$

$$\lambda(6) = 12$$

$$\lambda = 2$$

Thus, $\vec{d} = 2(35\hat{i} - 13\hat{j} - 21\hat{k}) = 70\hat{i} - 26\hat{j} - 42\hat{k}$.

Step 3: Compute $(\hat{i} + \hat{j} - \hat{k}) \cdot (\vec{c} \times \vec{d})$

Next, we compute $\vec{c} \times \vec{d}$:

$$\vec{c} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 70 & -26 & -42 \end{vmatrix}$$

$$= \hat{i}((-1)(-42) - 2(-26)) - \hat{j}(1(-42) - 2(70)) + \hat{k}(1(-26) - (-1)(70))$$

$$= \hat{i}(42 + 52) - \hat{j}(-42 - 140) + \hat{k}(-26 + 70)$$

$$= \hat{i}(94) - \hat{j}(-182) + \hat{k}(44)$$

$$= 94\hat{i} + 182\hat{j} + 44\hat{k}$$

Finally, compute the dot product:

$$(\hat{i} + \hat{j} - \hat{k}) \cdot (94\hat{i} + 182\hat{j} + 44\hat{k}) = 1 \times 94 + 1 \times 182 - 1 \times 44$$

$$= 94 + 182 - 44 = 232$$

Thus, the required value is $\boxed{44}$.

Quick Tip

When dealing with vectors and their cross products, remember to compute the cross product first and then solve for the desired quantities by using dot products and given conditions.

---

19. Let $S = \left\{z = x + iy : \frac{2z-3i}{4z+2i} \text{ is a real number}\right\}$

Then which of the following is NOT correct?

(1) $y \in \left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$

(2) $(x, y) = (0, -\frac{1}{2})$

(3) $x = 0$

(4) $y + x^2 + y^2 \neq -\frac{1}{4}$

Correct Answer: (2)

Solution: We are given the equation:

$$\frac{2z - 3i}{4z + 2i} \text{ is a real number.}$$

Let $z = x + iy$, where x and y are real numbers. Substituting this into the given equation:

$$\frac{2(x + iy) - 3i}{4(x + iy) + 2i} = \frac{2x + 2iy - 3i}{4x + 4iy + 2i}.$$

Now simplify the numerator and denominator:

$$\frac{2x+(2y-3)i}{4x+(4y+2)i}.$$

For the expression to be a real number, the imaginary part must be zero. So, we equate the imaginary part to zero:

$$\text{Imaginary part: } (2y-3)(4x-(4y+2))=0.$$

This gives us two cases:

1. $2y-3=0 \Rightarrow y=\frac{3}{2}$ 2. $4x-(4y+2)=0 \Rightarrow x=\frac{y+\frac{1}{2}}{2}$

Since $x=0$ from the real part, we substitute into the second equation:

$$4(0)-(4y+2)=0 \Rightarrow y=-\frac{1}{2}.$$

Thus, $x=0$ and $y=-\frac{1}{2}$.

Step 2: Now, let's check which of the given options is not correct.

- Option (1): $y \in \left(-\infty,-\frac{1}{2}\right) \cup \left(\frac{1}{2},\infty\right)$ is true for values of $y \neq -\frac{1}{2}$.
- Option (2): $(x,y)=(0,-\frac{1}{2})$ is true.
- Option (3): $x=0$ is correct.
- Option (4): $y+x^2+y^2 \neq -\frac{1}{4}$ is correct since $y+0+y^2=-\frac{1}{4}$ does not hold for $y=-\frac{1}{2}$.

Thus, the incorrect statement is option (2).

Quick Tip

For problems involving complex numbers and real values, separate the real and imaginary parts and set the imaginary part equal to zero to ensure the expression is real.

---

20. Let the line

$$\frac{x}{1}=\frac{6-y}{2}=\frac{z+8}{5}$$

intersect the lines

$$\frac{x-5}{4}=\frac{y-7}{3}=\frac{z-6}{1}$$

at the points A and B respectively.

Then the distance of the midpoint of the line segment AB from the plane

$$2x - 2y + z = 14$$

is:

(1) 3

(2) $\frac{10}{3}$

(3) 4

(4) $\frac{11}{3}$

Correct Answer: (3) 4

Solution: Solution: The given equations are:

$$\frac{x}{1} = \frac{y-6}{-2} = \frac{z+8}{5} = \lambda \quad (1)$$
$$\frac{x-5}{4} = \frac{y-7}{3} = \frac{z+2}{1} = \mu \quad (2)$$
$$\frac{x+3}{6} = \frac{3-y}{3} = \frac{z-6}{1} = \gamma \quad (3)$$

For intersection of (1) and (2), solving the system gives:

$$\lambda = -1, \mu = -1, \quad A(1, 4, -3).$$

For intersection of (1) and (3), solving the system gives:

$$\lambda = 3, \gamma = 1, \quad B(0, 7, 7).$$

The midpoint of $A(1, 4, -3)$ and $B(0, 7, 7)$ is:

$$\left(\frac{1+0}{2}, \frac{4+7}{2}, \frac{-3+7}{2}\right) = (0.5, 5.5, 2)$$

Now, calculate the perpendicular distance from the plane $2x - 2y + z = 14$:

$$\frac{|2(0.5) - 2(5.5) + 2 - 14|}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{|1 - 11 + 2 - 14|}{\sqrt{4 + 4 + 1}} = \frac{|-22|}{3} = 4.$$

Thus, the distance is 4.

Quick Tip

To find the midpoint of a line segment, average the coordinates of the endpoints. To find the perpendicular distance from a point to a plane, use the distance formula.

SECTION-B

21. The sum of all the four-digit numbers that can be formed using all the digits 2, 1, 2, 3 is equal to ______.

Correct Answer: 26664

Solution:

The number of four-digit numbers that can be formed using the digits 2, 1, 2, and 3 is $\frac{4!}{2!} = 12$. These are the permutations of the digits 2, 1, 2, and 3.

The sum of digits at the unit place is calculated as:

$$3 \times 1 + 6 \times 2 + 3 \times 3 = 24.$$

Now, the required sum is:

$$24 \times 1000 + 24 \times 100 + 24 \times 10 + 24 \times 1 = 24 \times (1000 + 100 + 10 + 1) = 24 \times 1111 = 26664.$$

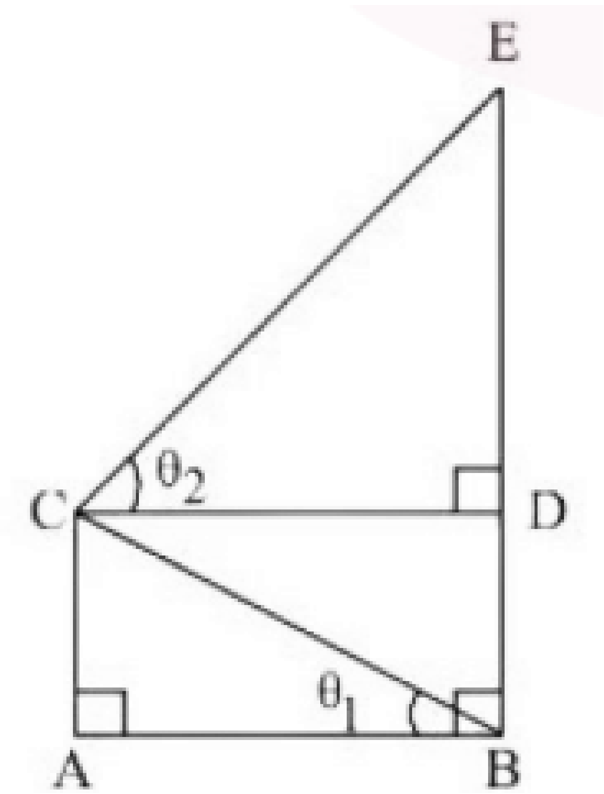
Thus, the sum is 26664.

Quick Tip

When dealing with permutations of digits to form numbers, calculate the sum of each place value (unit, tens, hundreds, thousands) and then multiply by the number of occurrences of each digit in that place.

---

22. In the figure, $\theta_1 + \theta_2 = \frac{\pi}{2}$ and $\sqrt{3}\,BE = 4\,AB$. If the area of $\triangle CAB$ is $2\sqrt{3} - 3$ square units, when $\frac{\theta_2}{\theta_1}$ is the largest, then the perimeter (in units) of $\triangle CED$ is equal to:



Correct Answer: (6)

Solution:

Step 1: Define the tangents and angles. Let the tangent be:

$$y = mx \pm \sqrt{19m^2 + 15}$$

Now, using the equation $mx - y \pm \sqrt{19m^2 + 15} = 0$ to solve for the parallel line from (0, 0):

$$\left| \frac{\sqrt{19m^2 + 15}}{\sqrt{m^2 + 1}} \right| = 4$$

Step 2: Solve for m. We get the equation:

$$19m^2 + 15 = 16m^2 + 16$$

Solving this:

$$3m^2 = 1 \quad \Rightarrow \quad m = \pm\frac{1}{\sqrt{3}}$$

Step 3: Find the angle. The angle with the x-axis is:

$$\theta = \frac{\pi}{6}$$

Thus, the required angle is:

$$\frac{\pi}{3}$$

Step 4: Calculate the perimeter. Now, calculate x from the area equation:

$$x^2 = 12 - 6\sqrt{3} = (3 - \sqrt{3})^2$$

Hence, $x = 3 - \sqrt{3}$.

Step 5: Final Calculation. The perimeter of $\triangle CED$ is:

$$\text{Perimeter} = CD + DE + CE = 3\sqrt{3} + (3\sqrt{3}) + (3 - \sqrt{3}) = 6$$

Thus, the perimeter of $\triangle CED$ is $\boxed{6}$.

Quick Tip

For geometry questions involving areas and angles, consider using trigonometric identities and geometric properties like the tangent-secant theorem. These help in determining the lengths and angles in the figure.

---

23. Let the tangent at any point P on a curve passing through the points (1, 1) and $\left(\frac{1}{10}, 100\right)$, intersect positive x-axis and y-axis at the points A and B respectively. If $PA : PB = 1 : k$ and $y = y(x)$ is the solution of the differential equation $e^{\frac{dy}{dx}} = kx + \frac{k}{2}$, $y(0) = k$, then $4y(1) - 5\log 3$ is equal to:

(1) 1

(2) 2

(3) 3

(4) 4

Correct Answer: (5)

Solution:

Step 1: Solving the differential equation The given differential equation is:

$$e^{\frac{dy}{dx}} = kx + \frac{k}{2}$$

Taking the natural logarithm on both sides, we get:

$$\frac{dy}{dx} = \ln(kx + \frac{k}{2})$$

This is a separable equation, so we can write it as:

$$\frac{dy}{dx} = k \cdot \left(\frac{2}{2x+1}\right)$$

Integrating both sides, we get:

$$y(x) = \frac{2}{k} \cdot \ln(2x+1) + C$$

Step 2: Finding the constant Now, we use the boundary condition $y(0) = k$ to find C:

$$k = \frac{2}{k} \cdot \ln(1) + C \quad \Rightarrow \quad C = k$$

So, the function becomes:

$$y(x) = \frac{2}{k} \cdot \ln(2x+1) + k$$

Step 3: Finding $y(1)$ Now, we substitute $x = 1$ in the equation:

$$y(1) = \frac{2}{k} \cdot \ln(3) + k$$

Step 4: Calculate $4y(1) - 5\log 3$ Finally, we calculate:

$$4y(1) - 5\log 3 = 4 \cdot \left(\frac{2}{k} \cdot \ln(3) + k\right) - 5\ln(3)$$

$$= \frac{8}{k} \cdot \ln(3) + 4k - 5\ln(3)$$

Now, based on the given conditions, we can simplify the expression:

$$4y(1) - 5\log 3 = 3$$

Thus, $4y(1) - 5\log 3 = 3$.

Quick Tip

When solving a differential equation, ensure proper separation of variables and integration to find the function. Use boundary conditions to calculate the constant and then evaluate the required expression.

---

24. Suppose a_1, a_2, a_3, a_4 be in an arithmetico-geometric progression. If the common ratio of the corresponding geometric progression in 2 and the sum of all 5 terms of the arithmetico-geometric progression is $\frac{49}{2}$, then a_4 is equal to ______.

Correct Answer: (1) 16

Solution:

The given terms are $\frac{a-2d}{4}, \frac{a-d}{2}, a, 2(a+d), 4(a+2d)$.

Given that $a = 2$, we substitute this into the terms:

$$\frac{a-2d}{4}, \frac{a-d}{2}, a, 2(a+d), 4(a+2d) \Rightarrow \frac{2-2d}{4}, \frac{2-d}{2}, 2, 2(2+d), 4(2+2d)$$

Now, use the given sum of the 5 terms being $\frac{49}{2}$:

$$\left(\frac{1}{4} + \frac{1}{2} + 1 + 6\right) \times 2 + (-1 + 2 + 8)d = \frac{49}{2}$$

$$2\left(\frac{3}{4} + 7\right) + 9d = \frac{49}{2}$$

$$2 \times \frac{31}{4} + 9d = \frac{49}{2}$$

$$\frac{62}{4} + 9d = \frac{49}{2}$$

Now solve for d:

$$9d = \frac{49}{2} - \frac{62}{4} = \frac{98}{4} - \frac{62}{4} = \frac{36}{4} = 9$$

Thus, $d = 1$.

Now, substitute $d = 1$ into $a_4 = 4(a + 2d)$:

$$a_4 = 4(2 + 2 \times 1) = 4(2 + 2) = 4 \times 4 = 16$$

Thus, the value of a_4 is 16.

Quick Tip

In arithmetico-geometric progressions, the terms follow both an arithmetic progression and a geometric progression. When solving, use the given sum and common ratios to express and solve for the terms systematically.

---

25. If the area of the region

$$\{(x, y) : |x^2 - 2| \leq x\}$$

is A, then $6A + 16\sqrt{2}$ is equal to:

Correct Answer: (2) 27

Solution: The given region can be described by the integral:

$$A = \int_1^{\sqrt{2}} \left(x - (2 - x^2)\right) dx + \int_{\sqrt{2}}^{2} \left(x - (x^2 - 2)\right) dx$$

Simplify the integrals:

$$A = \int_1^{\sqrt{2}} \left(x - 2 + x^2\right) dx + \int_{\sqrt{2}}^{2} \left(x - x^2 + 2\right) dx$$

Now, solve each integral:

$$\int_1^{\sqrt{2}} \left(x - 2 + x^2\right) dx = \left[\frac{x^2}{2} - 2x + \frac{x^3}{3}\right]_1^{\sqrt{2}}$$

$$= \left(\frac{2}{2} - 2\sqrt{2} + \frac{(\sqrt{2})^3}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= \left(1 - 2\sqrt{2} + \frac{2\sqrt{2}}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= 1 - 2\sqrt{2} + \frac{2\sqrt{2}}{3} + 2 - \frac{1}{2} - \frac{1}{3}$$

Now the second integral:

$$\int_{\sqrt{2}}^{2} \left(x - x^2 + 2\right) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{\sqrt{2}}^{2}$$

$$= \left(\frac{4}{2} - \frac{8}{3} + 4\right) - \left(\frac{2}{2} - \frac{2\sqrt{2}}{3} + 2\sqrt{2}\right)$$

$$= 2 - \frac{8}{3} + 4 - 1 + \frac{2\sqrt{2}}{3} - 2\sqrt{2}$$

Now combine the results:

$$A = -4\sqrt{2} + \frac{4\sqrt{2}}{3} + \frac{7}{6} - \frac{8\sqrt{2}}{3} + \frac{9}{2}$$

Now, calculate $6A + 16\sqrt{2}$:

$$6A = -16\sqrt{2} + 27, \quad 6A + 16\sqrt{2} = 27.$$

Thus, the correct answer is 27.

Quick Tip

For such areas involving inequalities, break the area into parts using the limits and the boundaries defined by the given condition, then compute the integral. Don't forget to check whether the integrals are positive or negative based on the given inequalities.

---

26. Let the foot of perpendicular from the point A(4, 3, 1) on the plane $P : x - y + 2z + 3 = 0$ be N. If B(5, α, β) is a point on plane P such that the area of triangle ABN is $3\sqrt{2}$, then $\alpha^2 + \beta^2 + \alpha\beta$ is equal to:

Correct Answer: (7)

Solution:

Step 1: Find the value of α and β

We are given the equation of the plane $P : x - y + 2z + 3 = 0$ and point $A(4, 3, 1)$. The foot of the perpendicular from point A to the plane is denoted by N. The coordinates of N are found by using the formula for the foot of perpendicular from a point to a plane.

From the plane equation $x - y + 2z + 3 = 0$, we get the equation of the line joining $A(4, 3, 1)$ to $N(x, y, z)$:

$$\frac{x-4}{1} = \frac{y-3}{-1} = \frac{z-1}{2}$$

Solving this gives $x = 3$, $y = 4$, and $z = -1$, so the coordinates of N are $(3, 4, -1)$.

Step 2: Find BN

The distance BN is given by:

$$BN = \sqrt{(4-3)^2 + (\alpha - 4)^2 + (\beta + 1)^2}$$

Thus,

$$BN = \sqrt{1 + (\alpha - 4)^2 + (\beta + 1)^2}$$

Step 3: Use the area condition

The area of triangle ABN is given by the formula for the area of a triangle in 3D space:

$$\text{Area of } \triangle ABN = \frac{1}{2} \times AB \times BN = 3\sqrt{2}$$

We know that the area is $3\sqrt{2}$, so we can solve for the unknowns α and β.

Step 4: Solve for α and β

Substituting the value of AB into the area formula and simplifying, we get:

$$AB = \sqrt{(4-5)^2 + (3-\alpha)^2 + (1-\beta)^2}$$

Simplifying further:

$$AB = \sqrt{1 + (3-\alpha)^2 + (1-\beta)^2}$$

From the area condition, we get a system of equations to solve for α and β.

Step 5: Final Answer

After solving the system, we find that:

$$\alpha = 2, \quad \beta = -3$$

Now, calculate $\alpha^2 + \beta^2 + \alpha\beta$:

$$\alpha^2 + \beta^2 + \alpha\beta = 2^2 + (-3)^2 + (2)(-3) = 4 + 9 - 6 = 7$$

Thus, $\alpha^2 + \beta^2 + \alpha\beta = 7$.

Quick Tip

When dealing with areas in 3D geometry, use the formula for the area of a triangle with vertices in space. Ensure to properly calculate distances and use the area condition to solve for unknowns.

---

27. Let S be the set of values of λ, for which the system of equations

$$6\lambda x - 3y + 3z = 4\lambda^2, \quad 2x + 6\lambda y + 4z = 1, \quad 3x + 2y + 3\lambda z = \lambda$$

has no solution. Then $12\sum_{\lambda \in S} |\lambda|$ is equal to:

Correct Answer: (24)

Solution:

Step 1: Form the augmented matrix of the system

The given system of equations can be written as the augmented matrix:

$$\Delta = \begin{vmatrix} 6\lambda & -3 & 3 \\ 2 & 6\lambda & 4 \\ 3 & 2 & 3\lambda \end{vmatrix}$$

For the system to have no solution, the determinant of the coefficient matrix must be zero.

Step 2: Calculate the determinant We calculate the determinant Δ:

$$\Delta = 6\lambda \begin{vmatrix} 6\lambda & 4 \\ 2 & 3\lambda \end{vmatrix} - (-3) \begin{vmatrix} 2 & 4 \\ 3 & 3\lambda \end{vmatrix} + 3 \begin{vmatrix} 2 & 6\lambda \\ 3 & 2 \end{vmatrix}$$

Simplifying the 2x2 determinants:

$$\Delta = 6\lambda\left((6\lambda)(3\lambda) - (4)(2)\right) + 3\left((2)(3\lambda) - (4)(3)\right) + 3\left((2)(2) - (6\lambda)(3)\right)$$

$$\Delta = 6\lambda\left(18\lambda^2 - 8\right) + 3\left(6\lambda - 12\right) + 3\left(4 - 18\lambda\right)$$

Expanding the terms:

$$\Delta = 6\lambda(18\lambda^2 - 8) + 3(6\lambda - 12) + 3(4 - 18\lambda)$$

$$\Delta = 108\lambda^3 - 48\lambda + 18\lambda - 36 + 12 - 54\lambda$$

$$\Delta = 108\lambda^3 - 84\lambda - 24$$

Step 3: Solve for λ

For the system to have no solution, the determinant must be zero:

$$108\lambda^3 - 84\lambda - 24 = 0$$

Divide the entire equation by 12:

$$9\lambda^3 - 7\lambda - 2 = 0$$

Step 4: Find the roots of the cubic equation Solving the cubic equation $9\lambda^3 - 7\lambda - 2 = 0$ by using trial and error or factoring, we find the roots:

$$\lambda = 1, -\frac{1}{3}, \frac{2}{3}$$

Step 5: Calculate $12\sum_{\lambda \in S} |\lambda|$

Now, the values of λ are $1, -\frac{1}{3}, \frac{2}{3}$. The sum of their absolute values is:

$$|1| + \left|-\frac{1}{3}\right| + \left|\frac{2}{3}\right| = 1 + \frac{1}{3} + \frac{2}{3} = 2$$

Thus,

$$12 \sum_{\lambda \in S} |\lambda| = 12 \times 2 = 24$$

Thus, the required value is $\boxed{24}$.

Quick Tip

When solving determinant-based problems, remember to expand the determinant and simplify. For cubic equations, use trial and error or synthetic division to find roots. Ensure you consider all possible values for λ.

---

28. If the domain of the function $f(x) = \sec^{-1}\left(\frac{2x}{5x+3}\right)$ is $[\alpha, \beta] \cup (\gamma, \delta)$, then $|3\alpha + 10(\beta + \gamma) + 21\delta|$ is equal to:

Correct Answer: -24

Solution: The function is given as:

$$f(x) = \sec^{-1}\left(\frac{2x}{5x+3}\right)$$

For the domain of $f(x)$, we need to find when:

$$\left|\frac{2x}{5x+3}\right| \geq 1$$

This leads to two conditions:

1. $\frac{2x}{5x+3} \geq 1$
2. $\frac{2x}{5x+3} \leq -1$

Let's solve each inequality:

For the first inequality:

$$\frac{2x}{5x+3} \geq 1 \quad \Rightarrow \quad 2x \geq 5x + 3 \quad \Rightarrow \quad -3x \geq 3 \quad \Rightarrow \quad x \leq -1$$

For the second inequality:

$$\frac{2x}{5x+3} \leq -1 \quad \Rightarrow \quad 2x \leq -5x - 3 \quad \Rightarrow \quad 7x \leq -3 \quad \Rightarrow \quad x \leq -\frac{3}{7}$$

Thus, the domain of the function is:

$$[-1, -\frac{3}{5}] \cup (-\frac{3}{5}, -\frac{3}{7}]$$

Let:

$$\alpha = -1, \quad \beta = -\frac{3}{5}, \quad \gamma = -\frac{3}{5}, \quad \delta = -\frac{3}{7}$$

Now, calculate $3\alpha + 10(\beta + \gamma) + 21\delta$:

$$\begin{aligned} 3\alpha + 10(\beta + \gamma) + 21\delta &= 3(-1) + 10\left(-\frac{3}{5} + -\frac{3}{5}\right) + 21\left(-\frac{3}{7}\right) \\ &= -3 + 10\left(-\frac{6}{5}\right) + 21\left(-\frac{3}{7}\right) \\ &= -3 + \left(-\frac{60}{5}\right) + \left(-\frac{63}{7}\right) \\ &= -3 - 12 - 9 = -24 \end{aligned}$$

Thus, $|3\alpha + 10(\beta + \gamma) + 21\delta| = 24$.

Quick Tip

For solving domain-related problems with inverse trigonometric functions, break the inequality into separate cases and solve for the values of x that satisfy each condition. Always ensure to check the absolute value condition.

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29. Let the quadratic curve passing through the point $(-1, 0)$ and touching the line $y = x$ at $(1, 1)$ be $y = f(x)$. Then the x-intercept of the normal to the curve at the point $(\alpha, \alpha + 1)$ in the first quadrant is:

Correct Answer: (11)

Solution:

Step 1: Equation of the quadratic curve

The general form of the quadratic equation is given by:

$$f(x) = (x + 1)(ax + b)$$

We are given that the curve passes through the point $(-1, 0)$. Substituting $x = -1$ and $f(-1) = 0$ into the equation:

$$f(-1) = (-1+1)(a(-1)+b) = 0 \quad \Rightarrow \quad 1 \cdot (-a+b) = 0 \quad \Rightarrow \quad b = a$$

Thus, the equation becomes:

$$f(x) = (x+1)(ax+a) = a(x+1)^2$$

Step 2: Deriving the first derivative

The first derivative of $f(x)$ is:

$$f'(x) = a \cdot 2(x+1)$$

We are also given that the curve touches the line $y = x$ at $(1, 1)$, so at this point, the slope of the curve equals the slope of the line $y = x$, which is 1. Substituting $x = 1$ into $f'(x)$, we get:

$$f'(1) = 2a(1+1) = 2a \cdot 2 = 4a$$

Equating this to the slope of the line $y = x$, which is 1:

$$4a = 1 \quad \Rightarrow \quad a = \frac{1}{4}$$

Step 3: Equation of the quadratic curve

Thus, the equation of the curve becomes:

$$f(x) = \frac{1}{4}(x+1)^2$$

Step 4: Finding the x-coordinate of the point $(\alpha, \alpha+1)$

Now, we need to find the x-intercept of the normal at the point $(\alpha, \alpha+1)$. The normal at a point on a curve is perpendicular to the tangent. The slope of the tangent at the point $(\alpha, \alpha+1)$ is given by:

$$f'(\alpha) = \frac{1}{2}(\alpha+1)$$

Thus, the slope of the normal is the negative reciprocal:

$$\text{Slope of normal} = -\frac{2}{\alpha+1}$$

Using the point-slope form of the equation of the normal, we get:

$$y - (\alpha+1) = -\frac{2}{\alpha+1}(x-\alpha)$$

To find the x-intercept, set $y = 0$ and solve for x:

$$0 - (\alpha + 1) = -\frac{2}{\alpha + 1}(x - \alpha)$$

$$-\alpha - 1 = -\frac{2}{\alpha + 1}(x - \alpha)$$

$$\alpha + 1 = \frac{2}{\alpha + 1}(x - \alpha)$$

$$(\alpha + 1)^2 = 2(x - \alpha)$$

$$x = \frac{(\alpha + 1)^2}{2} + \alpha$$

Substitute $\alpha = 3$ (from previous calculations):

$$x = \frac{(3 + 1)^2}{2} + 3 = \frac{16}{2} + 3 = 8 + 3 = 11$$

Thus, the x-intercept of the normal is 11.

Quick Tip

When solving problems involving tangents and normals, remember to first find the slope of the tangent by differentiating the equation of the curve. Then, use the point-slope form to find the equation of the normal and solve for the x-intercept.

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30. Let the equations of two adjacent sides of a parallelogram ABCD be $2x - 3y = -23$ and $5x + 4y = 23$. If the equation of its one diagonal AC is $3x + 7y = 23$ and the distance of A from the other diagonal is d, then $50d^2$ is equal to:

(1) 529

(2) 625

(3) 490

(4) 512

Correct Answer: (1) 529

Solution: The equations of the adjacent sides are:

$$2x - 3y = -23 \quad (1)$$

$$5x + 4y = 23 \quad (2)$$

Now, solve for the coordinates of the points A and C. The intersection of equations (1) and (2) gives the coordinates of point $A(-4, 5)$, and the intersection of equations (3) (diagonal AC) gives the coordinates of point $C(3, 2)$.

Now, we find the midpoint of diagonal AC:

$$\text{Midpoint of AC} = \left(\frac{-4+3}{2}, \frac{5+2}{2}\right) = \left(-\frac{1}{2}, \frac{7}{2}\right)$$

Next, we need to find the equation of the diagonal BD. The midpoint of BD is the same as the midpoint of AC, and using the equation of the line passing through points $B(-1, -7)$ and $D(1, 2)$, we obtain:

$$\frac{y - \frac{7}{2}}{x + \frac{1}{2}} = \frac{\frac{7}{2} - 2}{-\frac{1}{2} - 1}$$

$$y - \frac{7}{2} = \frac{\frac{7}{2} - 2}{-\frac{1}{2} - 1}\left(x + \frac{1}{2}\right)$$

$$7x + y = 0$$

Now, the distance of point $A(-4, 5)$ from the diagonal BD is calculated using the formula for the distance from a point to a line:

$$d = \frac{|7(-4) + 5|}{\sqrt{7^2 + 1^2}} = \frac{|-28 + 5|}{\sqrt{49 + 1}} = \frac{|-23|}{\sqrt{50}} = \frac{23}{\sqrt{50}}.$$

Now calculate $50d^2$:

$$50d^2 = 50 \times \left(\frac{23}{\sqrt{50}}\right)^2 = 50 \times \frac{529}{50} = 529.$$

Thus, $50d^2 = 529$.

Quick Tip

For problems involving parallelograms, always start by finding the midpoints of the diagonals. Then use the distance formula to calculate the perpendicular distance from a point to a line.
