JEE Main 2023 April 12 Shift 1 Question Paper with Solutions

Time Allowed : 3 Hours	**Maximum Marks :** 300	**Total Questions :** 90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

MATHEMATICS

SECTION-A

1. The number of five-digit numbers, greater than 40000 and divisible by 5, which can be formed using the digits 0, 1, 3, 5, 7, and 9 without repetition, is equal to:

(1) 120

(2) 132

(3) 72

(4) 96

Correct Answer: (1) 120

Solution: Step 1: Determine the possibilities for the unit place. Since the number must be divisible by 5, the unit digit must be either 0 or 5.

Step 2: Consider the case when the unit digit is 0. The first digit must be greater than 4 (i.e., 5, 7, or 9). After choosing the first digit, there are 4 remaining digits for the second, third, and fourth places. Thus, the number of such five-digit numbers is:

$$3 \times 4 \times 3 \times 2 = 72$$

Step 3: Consider the case when the unit digit is 5. The first digit must be greater than 4 (i.e., 7 or 9). After choosing the first digit, there are 4 remaining digits for the second, third, and fourth places. Thus, the number of such five-digit numbers is:

$$2 \times 4 \times 3 \times 2 = 48$$

Step 4: Total number of such five-digit numbers Adding the results from Step 2 and Step 3:

$$72 + 48 = 120$$

Quick Tip

To solve this type of problem, break it into cases based on the divisibility condition (here, the number must end in 0 or 5) and calculate the possibilities for each case separately.

2. Let α, β be the roots of the quadratic equation $x^2 + \sqrt{6}x + 3 = 0$. Then, $\frac{\alpha^{23}+\beta^{23}+\alpha^{14}+\beta^{14}}{\alpha^{15}+\beta^{15}+\alpha^{10}+\beta^{10}}$ is equal to:

(1) 729

(2) 72

(3) 81

(4) 9

Correct Answer: (3) 81

Solution: Step 1: Find the roots α and β. The given quadratic equation is $x^2 + \sqrt{6}x + 3 = 0$. Using the quadratic formula:

$$\alpha, \beta = \frac{-\sqrt{6} \pm \sqrt{\sqrt{6}^2 - 4(1)(3)}}{2(1)} = \frac{-\sqrt{6} \pm \sqrt{6-12}}{2} = \frac{-\sqrt{6} \pm \sqrt{-6}}{2} = \frac{-\sqrt{6} \pm \sqrt{6}i}{2}$$

Thus,

$$\alpha, \beta = \frac{-\sqrt{6} \pm \sqrt{6}i}{2} = \sqrt{3}e^{\pm\frac{3\pi}{4}i}$$

Step 2: Calculate the required expression. We are asked to evaluate:

$$\frac{\alpha^{23} + \beta^{23} + \alpha^{14} + \beta^{14}}{\alpha^{15} + \beta^{15} + \alpha^{10} + \beta^{10}}$$

Using Euler's formula, we rewrite the powers of α and β:

$$\left(\sqrt{3}\right)^{23}\left(2\cos\left(\frac{69\pi}{4}\right)\right)+\left(\sqrt{3}\right)^{14}\left(2\cos\left(\frac{42\pi}{4}\right)\right)+\left(\sqrt{3}\right)^{15}\left(2\cos\left(\frac{45\pi}{4}\right)\right)+\left(\sqrt{3}\right)^{10}\left(2\cos\left(\frac{30\pi}{4}\right)\right)$$

After simplifying, we get the final value as:

$$\left(\sqrt{3}\right)^{8} = 81$$

Quick Tip

For powers of complex numbers in polar form, Euler's formula $re^{i\theta}$ helps simplify large powers. The magnitude of the result is raised to the power, while the angle is multiplied by the exponent.

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3. Let $< a_n >$ be a sequence such that $a_1 + a_2 + \cdots + a_n = \frac{n^2+3n}{(n+1)(n+2)}$. If

$$\sum_{k=1}^{10} \frac{1}{a_k} = p_1 p_2 p_3 \dots p_m$$

where $p_1, p_2, \ldots, p_m$ are the first m prime numbers, then m is equal to:

(1) 7

(2) 6

(3) 5

(4) 8

Correct Answer: (2) 6

Solution: The given sequence is $a_n = S_n - S_{n-1}$, where $S_n = \frac{n^2+3n}{(n+1)(n+2)}$. Thus,

$$a_n = \frac{n^2 + 3n}{(n+1)(n+2)} - \frac{(n-1)^2 + 3(n-1)}{n(n+1)}$$

Simplifying this gives:

$$a_n = \frac{4}{n(n+1)(n+2)}$$

Step 2: Compute $\sum_{k=1}^{10} \frac{1}{a_k}$. We compute the sum:

$$\sum_{k=1}^{10} \frac{1}{a_k} = \sum_{k=1}^{10} \frac{k(k+1)(k+2)}{4} = \frac{7}{4} \sum_{k=1}^{10} (k(k+1)(k+2))$$

Calculating this sum gives:

$$\sum_{k=1}^{10} \frac{1}{a_k} = 7 \times 11 \times 13 \times 2.3 \times 5.7 \times 11.13$$

Thus, $m = 6$.

Quick Tip

To solve summations involving fractions, break down the terms into manageable parts and look for simplifications that reveal familiar patterns.

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4. Let the lines $l_1 : \frac{x+5}{3} = \frac{y+4}{1} = \frac{z-\alpha}{-2}$ and $l_2 : 3x + 2y + z - 2 = 0,\ x - 3y + 2z - 13 = 0$ be coplanar. If the point P(a, b, c) on l_1 is nearest to the point Q(-4, -3, 2), then $|a| + |b| + |c|$ is equal to:

(1) 12

(2) 14

(3) 10

(4) 8

Correct Answer: (3) 10

Solution: Step 1: Consider the equation of the lines l_1 and l_2. The equation of line l_1 is given by

$$\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-\alpha}{-2}$$

Let the parameter be λ. Then the parametric form of l_1 is:

$$x = 3\lambda - 5, \quad y = \lambda - 4, \quad z = -2\lambda + \alpha$$

For l_2, we have two equations:

$$3x + 2y + z - 2 = 0 \quad \text{and} \quad x - 3y + 2z - 13 = 0$$

Substitute the parametric equations of l_1 into these equations.

Step 2: Solve for α and λ. From the first equation of l_2:

$$3(3\lambda - 5) + 2(\lambda - 4) + (-2\lambda + \alpha) - 2 = 0$$

This simplifies to:

$$9\lambda - 15 + 2\lambda - 8 - 2\lambda + \alpha - 2 = 0$$

$$9\lambda - 25 + \alpha = 0 \quad \Rightarrow \quad \alpha = 25 - 9\lambda$$

Now substitute $\alpha = 25 - 9\lambda$ into the second equation of l_2:

$$3(3\lambda - 5) - 3(\lambda - 4) + 2(-2\lambda + 25 - 9\lambda) - 13 = 0$$

Simplifying gives:

$$9\lambda - 15 - 3\lambda + 12 - 4\lambda + 50 - 18\lambda - 13 = 0$$

$$-16\lambda + 34 = 0 \quad \Rightarrow \quad \lambda = \frac{34}{16} = \frac{9}{4}$$

Step 3: Find the coordinates of point P. Substitute $\lambda = \frac{9}{4}$ into the parametric equations of l_1:

$$x = 3\left(\frac{9}{4}\right) - 5 = \frac{27}{4} - 5 = \frac{7}{4}, \quad y = \frac{9}{4} - 4 = \frac{9}{4} - \frac{16}{4} = -\frac{7}{4}, \quad z = -2\left(\frac{9}{4}\right) + \alpha = -\frac{18}{4} + \alpha$$

Substitute $\alpha = 25 - 9\lambda = 25 - 9 \times \frac{9}{4} = \frac{100}{4} - \frac{81}{4} = \frac{19}{4}$:

$$z = -\frac{18}{4} + \frac{19}{4} = \frac{1}{4}$$

Thus, the coordinates of P are $\left(\frac{7}{4}, -\frac{7}{4}, \frac{1}{4}\right)$.

Step 4: Calculate $|a|+|b|+|c|$. For point P, $a=\frac{7}{4}, b=-\frac{7}{4}, c=\frac{1}{4}$. Hence,

$$|a|+|b|+|c|=\left|\frac{7}{4}\right|+\left|-\frac{7}{4}\right|+\left|\frac{1}{4}\right|=\frac{7}{4}+\frac{7}{4}+\frac{1}{4}=10$$

Thus, $|a|+|b|+|c|=10$.

Quick Tip

For problems involving distances from a point to a line, use the formula for the distance between a point and a line in 3D space:

$$d=\frac{|\vec{PQ}\times\vec{d}|}{|\vec{d}|}$$

where $\vec{PQ}$ is the vector from the point to the line and $\vec{d}$ is the direction ratio of the line.

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5. Let $P\left(\frac{2\sqrt{3}}{7}, \frac{6}{\sqrt{7}}\right), Q, R,$ and S be four points on the ellipse $9x^2+4y^2=36$. Let PQ and RS be mutually perpendicular and pass through the origin. If

$$\frac{1}{(PQ)^2}+\frac{1}{(RS)^2}=\frac{p}{q}$$

where p and q are coprime, then $p+q$ is equal to:

(1) 143

(2) 137

(3) 157

(4) 147

Correct Answer: (3) 157

Solution: Step 1: Apply the given conditions to the ellipse equation. The equation of the ellipse is $9x^2+4y^2=36$. Dividing through by 36, we get:

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

Step 2: Relating the points $P, Q, R,$ and S. Let the vector $\overrightarrow{OP}=(\frac{2\sqrt{3}}{7}, \frac{6}{\sqrt{7}})$ represent point P. We also know that the points PQ and RS are mutually perpendicular and pass through the origin.

Step 3: Determine the perpendicularity condition. Let $R(2\cos\theta, 3\sin\theta)$ such that

$OP \perp OR$. So,

$$3\sin\theta \times \frac{6}{\sqrt{7}} = -1 \quad \text{and} \quad 2\cos\theta \times \frac{2\sqrt{3}}{\sqrt{7}} = -1$$

Solving this, we get:

$$\tan\theta = \frac{-2}{3\sqrt{3}}$$

Step 4: Calculate $\frac{1}{(PQ)^2} + \frac{1}{(RS)^2}$. Now, we compute:

$$\frac{1}{(PQ)^2} + \frac{1}{(RS)^2} = \frac{1}{4}\left(\frac{1}{(OP)^2} + \frac{1}{(OR)^2}\right)$$

Substituting the values for OP and OR, we get:

$$\frac{1}{4}\left(\frac{7}{48} + \frac{31}{144}\right) = \frac{1}{4}\left(\frac{7}{48} + \frac{31}{144}\right) = \frac{13}{144}$$

Thus, $p = 157$ and $q = 144$, and $p + q = 157 + 144 = 157$.

Quick Tip

In problems involving perpendicular vectors, always check the dot product condition, $\overrightarrow{A} \cdot \overrightarrow{B} = 0$, which implies that the vectors are perpendicular.

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6. Let a, b, c be three distinct real numbers, none equal to one. If the vectors $a\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + b\hat{j} + \hat{k}$, $a\hat{i} + \hat{j} + c\hat{k}$ are coplanar, then $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is equal to:

(1) 1

(2) -1

(3) -2

(4) 2

Correct Answer: (1) 1

Solution: Step 1: Use the condition for coplanarity of vectors. The vectors $a\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + b\hat{j} + \hat{k}$, $a\hat{i} + \hat{j} + c\hat{k}$ are coplanar if and only if the determinant of the matrix formed by the vectors is zero. This gives the following equation:

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

Step 2: Expand the determinant. Expanding the determinant, we get:

$$a\begin{vmatrix} b & 1 \\ 1 & c \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & c \end{vmatrix} + 1\begin{vmatrix} 1 & b \\ 1 & 1 \end{vmatrix} = 0$$

$$a(bc-1) - (c-1) + (1-b) = 0$$

$$a(bc-1) - c + 1 + 1 - b = 0$$

$$a(bc-1) - b - c + 2 = 0 \quad \cdots (1)$$

Step 3: Express the condition in terms of fractions. Now, consider the given equation $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$. Rewriting it:

$$-1 + \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

This simplifies to:

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

Thus, the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is 1.

Quick Tip

For problems involving coplanarity of vectors, use the condition that the determinant of the matrix formed by the vectors is zero. This can help solve for relationships between the components.

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7. If the local maximum value of the function

$$f(x) = \left(\frac{\sqrt{3e}}{2\sin x}\right)^{\sin^2 x}, \quad x \in \left(0, \frac{\pi}{2}\right)$$

is $\frac{k}{e}$, then

$$\left(\frac{k}{e}\right)^8 + \frac{k^8}{e^5} + k^8$$

is equal to:

(1) $e^5 + e^6 + e^{11}$

(2) $e^3 + e^5 + e^{11}$

(3) $e^3 + e^6 + e^{11}$

(4) $e^3 + e^6 + e^{10}$

Correct Answer: (3) $e^3 + e^6 + e^{11}$

Solution: Step 1: Let $y = \left(\frac{\sqrt{3e}}{2\sin x}\right)^{\sin^2 x}$**.** Taking the natural logarithm of both sides, we get:

$$\ln y = \sin^2 x \ln\left(\frac{\sqrt{3e}}{2\sin x}\right)$$

Step 2: Differentiate y **with respect to** x**.** The derivative is:

$$\frac{1}{y}\frac{dy}{dx} = \ln\left(\frac{\sqrt{3e}}{2\sin x}\right) \cdot \left(2\sin x\cos x + \sin^2 x\right)$$

Now, setting $\frac{dy}{dx} = 0$, we get:

$$\ln\left(\frac{\sqrt{3e}}{2\sin x}\right) \cdot (2\sin x\cos x - \sin x\cos x) = 0$$

Step 3: Solve for x**.** This simplifies to:

$$\ln\left(\frac{3e}{4\sin^2 x}\right) = 1$$

which gives:

$$\frac{3e}{4\sin^2 x} = e \quad \Rightarrow \quad \sin^2 x = \frac{3}{4}$$

Thus,

$$\sin x = \frac{\sqrt{3}}{2} \quad (\text{for } x \in (0, \frac{\pi}{2}))$$

Step 4: Calculate the local maximum value. The local maximum value is:

$$\left(\frac{\sqrt{3e}}{\sqrt{3}}\right)^{3/4} = e^{3/8}$$

Hence, the maximum value is $\frac{k}{e}$, where $k = e^3$.

Step 5: Calculate the required expression. Now, we compute:

$$\left(\frac{k}{e}\right)^8 + \frac{k^8}{e^5} + k^8 = e^3 + e^6 + e^{11}$$

Quick Tip

When solving for maximum or minimum values, take the derivative of the function and solve for critical points by setting the derivative equal to zero. This will help in finding the local maxima or minima.

8. Let D be the domain of the function $f(x) = \sin^{-1}\left(\log_{3x}\left(\frac{6+2\log_3 x}{-5x}\right)\right)$. If the range of the function $g : D \to \mathbb{R}$ defined by $g(x) = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer function, is (α, β), then $\alpha^2 + \frac{5}{\beta}$ is equal to:

(1) 46

(2) 135

(3) 136

(4) 45

Correct Answer: (2) 135

Solution: Step 1: Solve for the domain of $f(x)$. We are given:

$$6 + 2\log_3 x > 0 \quad \text{and} \quad x > 0 \quad \text{and} \quad x \neq \frac{1}{3}$$

This gives:

$$x \in \left(0, \frac{1}{27}\right) \quad \cdots \quad (1)$$

Step 2: Apply the range conditions. Next, we consider:

$$-1 \leq \log_3\left(\frac{6 + 2\log_3 x}{-5x}\right) \leq 1$$

This leads to:

$$\left(\frac{3}{4\sin^2 x}\right) = 1 \quad \Rightarrow \quad \sin^2 x = \frac{3}{4}$$

Thus,

$$\sin x = \frac{\sqrt{3}}{2} \quad (\text{for } x \in (0, \frac{\pi}{2}))$$

Step 3: Solve for α and β. From the previous steps, we calculate:

$$\left(\frac{3}{4\sin^2 x}\right) = 1 \quad \Rightarrow \quad \sin x = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \alpha = 3$$

and

$$\beta = \frac{1}{27}$$

Step 4: Compute the value of $\alpha^2 + \frac{5}{\beta}$. Now we compute:

$$\alpha^2 + \frac{5}{\beta} = 3^2 + \frac{5}{\frac{1}{27}} = 9 + 135 = 135$$

Quick Tip

When dealing with the greatest integer function, be careful with the domain and range. Make sure to take into account the constraints imposed by the logarithmic and trigonometric functions.

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9. Let $y = y(x),\ y > 0$, be a solution curve of the differential equation $(1 + x^2)\,dy = y(x - y)\,dx$. If $y(0) = 1$ and $y(2\sqrt{2}) = \beta$, then:

(1) $e^{3\beta-1} = e^{(3+2\sqrt{2})}$

(2) $e^{\beta-1} = e^{-2(5+\sqrt{2})}$

(3) $e^{\beta-1} = e^{-2(3+\sqrt{2})}$

(4) $e^{3\beta-1} = e^{(5+\sqrt{2})}$

Correct Answer: (1) $e^{3\beta-1} = e^{(3+2\sqrt{2})}$

Solution: Step 1: Rearrange the given differential equation. The given equation is:

$$(1 + x^2)\,\frac{dy}{dx} = y(x - y)$$

We can express it as:

$$\frac{dy}{dx} = \frac{y(x - y)}{1 + x^2}$$

Step 2: Transform into a solvable form. Rewrite the equation as:

$$\frac{dy}{dx} + y\left(\frac{-x}{1 + x^2}\right) = \left(\frac{-1}{1 + x^2}\right) y^2$$

Multiply through by $\frac{1}{y}$:

$$\frac{1}{y}\frac{dy}{dx} + \frac{-x}{(1 + x^2)y} = \frac{-1}{(1 + x^2)}$$

Step 3: Simplify the integrals. Let $\frac{1}{y} = t$, then we have the differential equation:

$$\frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

Integrating both sides:

$$\int \frac{1}{1 + x^2}\,dx = \int \frac{1}{y}\,dt$$

Thus, we obtain the general solution:

$$\sqrt{1 + x^2} = y\ln(e(x + \sqrt{1 + x^2}))$$

Step 4: Apply the boundary conditions. We know that $y(0) = 1$, so we can use this to find the value of the constant. Substitute $x = 0$ and $y = 1$ into the equation:

$$1 = \sqrt{1+0^2}\ln(e(0+\sqrt{1+0^2}))$$

This simplifies to $1 = \ln(e(1)) = 1$, confirming the constant is correct.

Step 5: Find the value of β. Now, for $y(2\sqrt{2}) = \beta$, we substitute $x = 2\sqrt{2}$ into the solution to find β:

$$\beta = \frac{3}{\ln(e(3+2\sqrt{2}))}$$

Thus, we obtain $3 = \ln(e(3+2\sqrt{2}))$, which leads to:

$$e^{3\beta-1} = e^{(3+2\sqrt{2})}$$

Quick Tip

When solving first-order differential equations, use the method of integrating factors. This helps in simplifying the equation to an easily solvable form.

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10. Among the two statements

(S1): $(p \Rightarrow q) \wedge (q \wedge (\sim q))$ **is a contradiction and**

(S2): $(p \wedge q) \vee (\sim p) \wedge (\sim q)$ **is a tautology,**

(1) Only (S2) is true

(2) Only (S1) is true

(3) Both are false

(4) Both are true

Correct Answer: (4) Both are true

Solution: Step 1: Analyzing the truth table for S1. Given $S1 : (p \Rightarrow q) \wedge (p \wedge (\sim q))$, we need to create the truth table:

p	q	$p \Rightarrow q$	$p \wedge (\sim q)$	$S1$
T	T	T	F	F
T	F	F	F	F
F	T	T	F	F
F	F	T	T	F

The statement $S1$ is false for all values of p and q, indicating that it is a contradiction.

Step 2: Analyzing the truth table for S2. Given $S2 : (p \wedge q) \vee (\sim p) \wedge (\sim q)$, we need to create the truth table:

p	q	$p \wedge q$	$(\sim p) \wedge (\sim q)$	$S2$
T	T	T	F	T
T	F	F	F	F
F	T	F	F	F
F	F	F	T	T

The statement $S2$ is true for all possible values of p and q, indicating that it is a tautology.

Conclusion: Thus, $S1$ is a contradiction and $S2$ is a tautology, so the correct answer is (4), both are true.

Quick Tip

To analyze logical statements like contradictions and tautologies, construct truth tables for the components of the statements. A statement that is false for all possible truth values is a contradiction, while one that is true for all truth values is a tautology.

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11. Let $\lambda \in \mathbb{Z}, \vec{a} = \lambda\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$. Let $\vec{c}$ be a vector such that

$$(\vec{a} + \vec{b} + \vec{c}) \times \vec{c} = 0, \quad \vec{a} \cdot \vec{c} = -17 \quad \textbf{and} \quad \vec{b} \cdot \vec{c} = -20.$$

Then $\left|\vec{c} \times (\lambda\hat{i} + \hat{j} + \hat{k})\right|^2$ is equal to:

(1) 62

(2) 46

(3) 53

(4) 49

Correct Answer: (2) 46

Solution: Step 1: Solve for $\vec{c}$. We are given:

$$(\vec{a} + \vec{b} + \vec{c}) \times \vec{c} = 0 \quad \Rightarrow \quad (\vec{a} + \vec{b}) \times \vec{c} = 0$$

So,

$$\vec{c} = \alpha(\vec{a} + \vec{b}) = \alpha(\lambda + 3)\hat{i} + \alpha\hat{j} + \alpha(\lambda - 1)\hat{k}$$

Next, using the condition $\vec{b} \cdot \vec{c} = -20$, we get:

$$3\alpha(\lambda + 3) + (-1)\alpha + 2\alpha(\lambda - 1) = -20$$

which simplifies to:

$$\alpha(5\lambda + 6) = -20$$

Solving for α:

$$\alpha = -\frac{20}{5\lambda + 6}$$

Step 2: Solve for $\vec{c} \times (\lambda\hat{i} + \hat{j} + \hat{k})$. Now, compute:

$$\vec{c} \times (\lambda\hat{i} + \hat{j} + \hat{k})$$

Using the cross-product formula and simplifying, we get:

$$\left|\vec{c} \times (\lambda\hat{i} + \hat{j} + \hat{k})\right|^2 = 46$$

Quick Tip

For problems involving vectors and dot/cross products, express the vectors in component form and use properties of dot and cross products to simplify calculations.

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12. The sum of the coefficients of the first 50 terms in the binomial expansion of $(1 - x)^{100}$, is equal to

(1) $-{}^{101}C_{50}$

(2) ${}^{99}C_{49}$

(3) $-{}^{99}C_{49}$

(4) ${}^{101}C_{50}$

Correct Answer: (3) $-{}^{99}C_{49}$

Solution: The binomial expansion of $(1 - x)^{100}$ is:

$$(1 - x)^{100} = C_0 - C_1 x + C_2 x^2 - C_3 x^3 + \cdots + C_{99} x^{99} + C_{100} x^{100}$$

where $C_n = \binom{100}{n}$. Thus, the general form of the expansion is:

$$(1 - x)^{100} = \sum_{n=0}^{100} (-1)^n \binom{100}{n} x^n$$

We are asked to find the sum of the coefficients of the first 50 terms. These are the terms where $x^0, x^1, \dots, x^{49}$ appear.
The sum of the coefficients of the first 50 terms corresponds to the sum:

$$C_0 + C_1 + C_2 + \cdots + C_{49}$$

This is the sum of the coefficients of the first 50 terms of the expansion. To calculate this sum, we substitute $x = 1$ into the expansion of $(1-x)^{100}$, which gives the sum of all the coefficients:

$$(1-1)^{100} = \sum_{n=0}^{100} (-1)^n \binom{100}{n} 1^n = 0$$

This gives us the sum of all the coefficients of the expansion:

$$\sum_{n=0}^{100} (-1)^n \binom{100}{n} = 0$$

Now, to find the sum of the first 50 terms, we separate the sum into two parts: one for the first 50 terms and one for the remaining terms:

$$(C_0 + C_1 + C_2 + \cdots + C_{49}) + (C_{50} + C_{51} + \cdots + C_{100}) = 0$$

We now note that the sum of the coefficients of the first 50 terms is related to the sum of the coefficients of the last 50 terms. If we subtract the sum of the coefficients of the first 50 terms from the sum of all the coefficients, we get:

$$2\,(C_0 + C_1 + C_2 + \cdots + C_9) + C_{50} = 0$$

Thus, the sum of the first 50 terms is:

$$C_0 - C_1 + C_2 - \cdots - C_{49} = \frac{1}{2}\binom{100}{50}$$

which simplifies to $-99\binom{100}{49}$.

Quick Tip

When summing the coefficients of specific terms in a binomial expansion, set $x = 1$ to compute the sum of all coefficients. Then, apply symmetry or break the sum into parts based on the range of terms you need.

13. The area of the region enclosed by the curve $y = x^3$ and its tangent at the point $(-1, -1)$ is:

(1) $\frac{27}{4}$

(2) $\frac{19}{4}$

(3) $\frac{23}{4}$

(4) $\frac{31}{4}$

Correct Answer: (1) $\frac{27}{4}$

Solution: Step 1: Find the equation of the tangent. The curve is $y = x^3$. The derivative of $y = x^3$ is:

$$\frac{dy}{dx} = 3x^2$$

At the point $(-1, -1)$, the slope of the tangent is:

$$m = 3(-1)^2 = 3$$

Thus, the equation of the tangent line is:

$$y + 1 = 3(x + 1) \quad \Rightarrow \quad y = 3x + 2$$

Step 2: Find the point of intersection of the tangent and the curve. The point of intersection occurs where:

$$x^3 = 3x + 2$$

This simplifies to:

$$x^3 - 3x - 2 = 0$$

Factoring the cubic equation:

$$(x + 1)(x^2 - x - 2) = 0$$

This gives the roots $x = -1, 2, -1$, and the point of intersection is $(2, 8)$.

Step 3: Compute the area enclosed by the curve and the tangent. The area between the curve $y = x^3$ and the line $y = 3x + 2$ is given by the integral:

$$\text{Area} = \int_{-1}^{2} \left((3x + 2) - x^3\right) dx$$

Evaluating the integral:

$$\text{Area} = \int_{-1}^{2} (3x + 2 - x^3) dx = \left[\frac{3x^2}{2} + 2x - \frac{x^4}{4}\right]_{-1}^{2}$$

Substitute the limits:

$$= \left(\frac{3(2)^2}{2} + 2(2) - \frac{(2)^4}{4}\right) - \left(\frac{3(-1)^2}{2} + 2(-1) - \frac{(-1)^4}{4}\right)$$
$$= (6 + 4 - 4) - \left(\frac{3}{2} - 2 - \frac{1}{4}\right)$$
$$= 6 - \left(\frac{3}{2} - 2 - \frac{1}{4}\right)$$
$$= 6 - \left(\frac{6}{4} - \frac{9}{4}\right)$$
$$= 6 - \left(-\frac{3}{4}\right)$$
$$= 6 + \frac{3}{4} = \frac{27}{4}$$

Quick Tip

When calculating the area between a curve and its tangent, set up the integral of the difference between the two functions over the given range. The limits of integration are determined by the points of intersection.

---

14. Let $A = \begin{bmatrix} 1 & \frac{1}{51} \\ 0 & 1 \end{bmatrix}$. If $B = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} A \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$, then the sum of all the elements of the matrix

$$\sum_{n=1}^{50} B^n$$

is equal to:

(1) 100

(2) 50

(3) 75

(4) 125

Correct Answer: (1) 100

Solution: Step 1: Find B. We are given:

$$B = CAD$$

where

$$C = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

First, compute DC:

$$DC = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, $B = CAD = I$, where I is the identity matrix.

Step 2: Compute the sum of elements of B^n. Since $B = I$, the matrix B^n is also I for all values of n. Therefore, the sum of all elements in B^n is:

$$\sum_{n=1}^{50} B^n = 50 \times \text{sum of elements of } I = 50 \times 2 = 100$$

Quick Tip

When dealing with powers of a matrix, if the matrix is the identity matrix, then any power of it is also the identity matrix. The sum of the elements of the identity matrix is simply the number of its dimensions.

---

15. Let the plane $P : 4x - y + z = 10$ be rotated by an angle $\frac{\pi}{2}$ about its line of intersection with the plane $x + y - z = 4$. If α is the distance of the point $(2, 3, -4)$ from the new position of the plane P, then 35α is:

(1) 90

(2) 85

(3) 105

(4) 126

Correct Answer: (4) 126

Solution: Step 1: Equation of the new position of the plane after rotation. Let the equation in the new position of the plane be:

$$(4x - y + z - 10) + \lambda(x + y - z - 4) = 0$$

Simplify the equation:

$$4(4 + \lambda) - 1(-1 + \lambda) + 1(1 - \lambda) = 0$$

$$\Rightarrow \lambda = -9$$

Thus, the equation in the new position is:

$$-5x - 10y + 10z + 26 = 0$$

$$\Rightarrow \alpha = \frac{54}{15}$$

Step 2: Calculate 35α. Now, we calculate 35α:

$$35\alpha = 35 \times \frac{54}{15} = 126$$

Quick Tip

For problems involving rotation of planes, the angle and the direction of rotation can be used to find the equation of the new plane. In this case, the method of finding the value of λ allows us to write the new plane equation.

---

16. If $\frac{1}{n+1}\,{}^nC_n + \frac{1}{n}\,{}^nC_{n-1}$ + ... + $\frac{1}{2}\,{}^nC_1$ + ${}^nC_0$ = $\frac{1023}{10}$ then n is equal to:

(1) 6

(2) 9

(3) 8

(4) 7

Correct Answer: (2) 9

Solution: We are given the equation:

$$\frac{1}{n+1}\binom{n}{n} + \frac{1}{n}\binom{n}{n-1} + \cdots + \frac{1}{2}\binom{n}{1} + \binom{n}{0} = \frac{1023}{10}$$

First, recall that the sum of binomial coefficients is:

$$\sum_{r=0}^{n}\binom{n}{r} = 2^n$$

Thus, we can express the sum as:

$$\frac{1}{n+1}\binom{n}{n} + \frac{1}{n}\binom{n}{n-1} + \cdots + \frac{1}{2}\binom{n}{1} + \binom{n}{0} = \frac{1}{n+1}\sum_{r=0}^{n}\binom{n+1}{r+1}$$

By simplifying the sum, we have:

$$\frac{1}{n+1}\left(2^{n+1}-1\right) = \frac{1023}{10}$$

Multiplying both sides by 10:

$$\frac{10}{n+1}\left(2^{n+1}-1\right) = 1023$$

Now solving for n:

$$n + 1 = 10 \quad \Rightarrow \quad n = 9$$

Quick Tip

For problems involving sums of binomial coefficients, you can use the identity $\sum_{r=0}^{n} \binom{n}{r} = 2^n$ to simplify the expressions and find the value of n.

---

17. Let C be the circle in the complex plane with centre $z_0 = \frac{1}{2}(1+3i)$ and radius $r = 1$. Let $z_1 = 1 + i$ and the complex number z_2 be outside the circle C such that $|z_1 - z_0| = |z_2 - z_0| = 1$. If z_0, z_1 and z_2 are collinear, then the smaller value of $|z_2|^2$ is equal to:

(1) $\frac{13}{2}$

(2) $\frac{5}{2}$

(3) $\frac{3}{2}$

(4) $\frac{7}{2}$

Correct Answer: (2) $\frac{5}{2}$

Solution: Step 1: Use the condition $|z_1 - z_0| = 1$. We are given that $z_1 = 1 + i$ and $z_0 = \frac{1}{2}(1 + 3i)$. Now, calculate $|z_1 - z_0|$:

$$|z_1 - z_0| = \left|(1+i) - \frac{1}{2}(1+3i)\right|$$

Simplifying:

$$|z_1 - z_0| = \left|\frac{1}{2} + \frac{1}{2}i\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Thus, $|z_1 - z_0| = \frac{1}{\sqrt{2}}$, satisfying the condition.

Step 2: Use the condition $|z_2 - z_0| = 1$. Now, we know that z_2 is outside the circle C, and we are given that $|z_2 - z_0| = 1$.

Step 3: Use the collinearity condition. The points z_0, z_1, and z_2 are collinear, so the angle between the points must be $135°$ (from the geometry).

$$\tan\theta = -1 \quad \Rightarrow \quad \theta = 135°$$

Now, calculate z_2 in polar form:

$$z_2 = \left(\frac{1}{2} + \sqrt{2}\cos 135^\circ, \frac{3}{2} - \sqrt{2}\sin 135^\circ\right)$$

$$z_2 = \left(-\frac{1}{2}, \frac{5}{2}\right)$$

Step 4: Find $|z_2|^2$. Thus, the value of $|z_2|^2$ is:

$$|z_2|^2 = \left(-\frac{1}{2}\right)^2 + \left(\frac{5}{2}\right)^2 = \frac{1}{4} + \frac{25}{4} = \frac{26}{4} = \frac{5}{2}$$

Quick Tip

For complex numbers, when dealing with geometric properties such as distances and collinearity, use the polar form of complex numbers. Additionally, use the distance formula to find the modulus of the complex number and apply geometric properties for collinearity.

---

18. If the point $\left(\alpha, \frac{7\sqrt{3}}{3}\right)$ lies on the curve traced by the mid-points of the line segments of the lines $x\cos\theta + y\sin\theta = 7$, $\theta \in \left(0, \frac{\pi}{2}\right)$ between the coordinates axes, then α is equal to

(1) 7

(2) -7

(3) $-7\sqrt{3}$

(4) $7\sqrt{3}$

Correct Answer: (1) 7

Solution: Let the equation of the line be:

$$x\cos\theta + y\sin\theta = 7$$

The line intersects the axes at:

$$x\text{- intercept} = \frac{7}{\cos\theta}, \quad y\text{- intercept} = \frac{7}{\sin\theta}$$

Thus, the points of intersection are:

$$A\left(\frac{7}{\cos\theta}, 0\right), \quad B\left(0, \frac{7}{\sin\theta}\right)$$

The midpoint M of the line segment AB is:

$$M(h,k) = \left(\frac{7}{2\cos\theta}, \frac{7}{2\sin\theta}\right)$$

Given that the point $\left(\alpha, \frac{7\sqrt{3}}{3}\right)$ lies on the curve, we have:

$$k = \frac{7\sqrt{3}}{3}$$

Substituting into the equation for k:

$$\frac{7}{2\sin\theta} = \frac{7\sqrt{3}}{3}$$

$$\Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

Now, substituting $\theta = \frac{\pi}{3}$ into the equation for h:

$$h = \frac{7}{2\cos\theta} = \frac{7}{2 \cdot \frac{1}{2}} = 7$$

Thus, $\alpha = 7$.

Quick Tip

For problems involving midpoints and line intersections, use the standard formula for the midpoint of a line segment. Additionally, trigonometric relationships can help in determining the values of intercepts and coordinates.

---

19. Two dice A and B are rolled, Let the numbers obtained on A and B be α and β respectively. If the variance of $\alpha - \beta$ is $\frac{p}{q}$, where p and q are coprime, then the sum of the positive divisors of p is equal to:

(1) 36

(2) 48

(3) 31

(4) 72

Correct Answer: (2) 48

Solution: We start by calculating the possible values of $\alpha - \beta$ and the corresponding probabilities P.

$\alpha - \beta$	Case	P
5	$(6, 1)$	$\frac{1}{36}$
4	$(6, 2), (5, 1)$	$\frac{2}{36}$
3	$(6, 3), (5, 2), (4, 1)$	$\frac{3}{36}$
2	$(6, 4), (5, 3), (4, 2), (3, 1)$	$\frac{4}{36}$
1	$(6, 5), (5, 4), (4, 3), (3, 2), (2, 1)$	$\frac{5}{36}$
0	$(6, 6), (5, 5), (4, 4), (3, 3), (2, 2), (1, 1)$	$\frac{6}{36}$
-1	$(6, 5), (5, 4), (4, 3), (3, 2), (2, 1)$	$\frac{5}{36}$
-2	$(6, 4), (5, 3), (4, 2), (3, 1)$	$\frac{4}{36}$
-3	$(6, 3), (5, 2), (4, 1)$	$\frac{3}{36}$
-4	$(2, 6), (1, 5)$	$\frac{2}{36}$
-5	$(1, 6)$	$\frac{1}{36}$

The expected value of x^2 is calculated as follows:

$$\sum x^2 = \sum x^2 P(x) = \frac{25}{36} + \frac{32}{36} + \frac{27}{36} + \frac{16}{36} + \frac{5}{36}$$
$$= \frac{105}{18} = \frac{35}{6}$$

The mean μ is 0 as the data is symmetric.

Now, calculate the variance σ^2:

$$\sigma^2 = \sum x^2 P(x) = \frac{35}{6}$$

Thus, $p = 35$ and $q = 7$, so the sum of divisors of $p = 35$ is $(5^0 + 5^1)(7^0 + 7^1) = 6 \times 8 = 48$.

Quick Tip

When calculating the variance, use the formula $\sigma^2 = \sum x^2 P(x)$, where $P(x)$ is the probability for each value. To find the sum of divisors of a number, use the formula based on its prime factorization.

---

20. In a triangle ABC, if $\cos A + 2\cos B + \cos C = 2$ and the lengths of the sides opposite to the angles A and C are 3 and 7 respectively, then $\cos A - \cos C$ is equal to:

(1) $\frac{3}{7}$

(2) $\frac{9}{7}$

(3) $\frac{10}{7}$

(4) $\frac{5}{7}$

Correct Answer: (3) $\frac{10}{7}$

Solution: We are given the equation:

$$\cos A + 2\cos B + \cos C = 2$$

Also, the lengths of the sides opposite to angles A and C are $a = 3$ and $c = 7$, respectively.

We can use the identity:

$$\cos\left(\frac{A+C}{2}\right) = \sin\left(\frac{B}{2}\right)$$

which simplifies to:

$$\cos A - \cos C = 2\sin\frac{B}{2}\cos\frac{B}{2}$$

Next, we use the given identity $2\cos B/2\cos A$ and simplify the calculation to the sum of $A + C$ and thus $\cos A - \cos C$.

So, after calculating the entire expression:

$$\boxed{\cos A - \cos C = \frac{10}{7}}$$

Quick Tip

For triangle trigonometry problems, use the relationship between the sides and angles, as well as the trigonometric identities for the sum and difference of angles. In this case, use the cosine and sine half-angle identities to simplify.

SECTION-B

21. A fair $n > 1$ faces die is rolled repeatedly until a number less than n appears. If the mean of the number of tosses required is $\frac{n}{9}$, then n is equal to:

Correct Answer: (10)

Solution: Let the mean number of tosses be M. We have:

$$M = 1 \times \frac{n-1}{n} + 2 \times \frac{n-1}{n} + 3 \times \left(\frac{1}{n}\right)^2 + \dots$$

$$\Rightarrow \frac{n}{9} = \left(\frac{n-1}{n}\right)\left(1 + 2\left(\frac{1}{n}\right) + 3\left(\frac{1}{n}\right)^2 + \dots\right)$$

$$\Rightarrow n = 10$$

Quick Tip

For mean problems with repeated trials, it's helpful to express the problem as a summation of probabilities for each possible outcome and then solve for the variable.

---

22. Let the digits a, b, c be in A.P. Nine-digit numbers are to be formed using each of these three digits thrice such that three consecutive digits are in A.P. at least once. How many such numbers can be formed?

Correct Answer: (1260)

Solution: We are asked to form nine-digit numbers using the digits a, b, c in an arithmetic progression (A.P.). The possible numbers are of the form abc or cba.

We have to choose 7 positions from the 9-digit number where the three digits will appear thrice, such that at least one group of three consecutive digits is in A.P.

So, the number of ways to select the digits is:

$$7C1 \times 2 \times 6! = 1260$$

Quick Tip

When forming numbers with repetitions, use combinations and factorials to account for all possible arrangements. In cases where the digits are restricted to specific positions, ensure you correctly calculate the number of valid groupings.

---

23. Let $[x]$ be the greatest integer $\leq x$. Then the number of points in the interval $(-2, 1)$, where the function $f(x) = |[x]| + \sqrt{x - [x]}$ is discontinuous is:

Correct Answer: 3

Solution:

We need to check for the points where the function is discontinuous. The function is defined as:

$$f(x) = |[x]| + \sqrt{x - [x]}$$

We check for discontinuity at the following points in the interval $(-2, 1)$:

1. $x = -1$:

At $x = -1$, $f(x)$ is continuous because both terms $|[x]|$ and $\sqrt{x - [x]}$ are continuous.

2. $x = 0$:

At $x = 0$, $f(x)$ is continuous.

3. $x = 1$:

At $x = 1$, the term $\sqrt{x - [x]}$ causes a discontinuity because as $x \to 1^-$, $\sqrt{x - [x]} \to 0$, while at $x = 1$, $\sqrt{x - [x]} = 1$. Therefore, the function is discontinuous at $x = 1$.

Thus, there are 3 points where the function is discontinuous.

Quick Tip

For functions involving the greatest integer and square roots, check the points where the greatest integer function causes jumps, and where the square root function might create discontinuities.

---

24. Let the plane $x + 3y - 2z + 6 = 0$ meet the co-ordinate axes at the points A, B, C. If the orthocentre of the triangle ABC is $(\alpha, \beta, \frac{6}{7})$, then $98(\alpha + \beta)^2$ is equal to:

Correct Answer: 288

Solution: Given the equation of the plane $x + 3y - 2z + 6 = 0$, we find the points where the plane intersects the axes.

The points of intersection with the axes are:

$A(-6, 0, 0)$

$B(0, -2, 0)$

$C(0, 0, 3)$

Now, calculate the vectors:

$$\overrightarrow{AB} = 6\hat{i} - 2\hat{j}, \quad \overrightarrow{BC} = 2\hat{j} + 3\hat{k}, \quad \overrightarrow{AC} = 6\hat{i} + 3\hat{k}$$

The orthocenter $H(\alpha, \beta, \frac{6}{7})$ satisfies the following conditions:

$$\overrightarrow{AH} \cdot \overrightarrow{BC} = 0, \quad \overrightarrow{CH} \cdot \overrightarrow{AB} = 0$$

Solving for α and β, we get:

$$\alpha = -\frac{3}{7}, \quad \beta = -\frac{9}{7}$$

Thus:

$$98(\alpha+\beta)^2 = 98\left(-\frac{3}{7} - \frac{9}{7}\right)^2 = 98\left(-\frac{12}{7}\right)^2 = 98 \times \frac{144}{49} = 288$$

Quick Tip

When solving for the orthocenter, use the vector approach and ensure you apply the condition of perpendicularity between the vectors. Also, pay attention to the geometry of the situation for accurate computations.

---

25. Let I(x) = $\int \sqrt{\frac{x+7}{x}}$ dx and I(9) = 12 + 7 $\log_e$ 7. If I(1) = $\alpha + 7\log_e(1+2\sqrt{2})$, then α^4 is equal to:

(1) 64

(2) 128

(3) 32

(4) 16

Correct Answer: (1) 64

Solution: We are given the integral:

$$I(x) = \int \sqrt{\frac{x+7}{x}}\,dx$$

Rewriting the integrand:

$$\sqrt{\frac{x+7}{x}} = \sqrt{1+\frac{7}{x}}$$

Let's make the substitution $u = x+7$, so that $du = dx$ and the integral becomes:

$$I(x) = \int \sqrt{\frac{u}{x}}\,du$$

However, a more straightforward way to solve the problem would be to simplify the integral directly. We can proceed by noticing that:

$$I(x) = \sqrt{x+7} + 7\ln\left|\sqrt{x+7}+\sqrt{x}\right| + C$$

Now, use the given condition $I(9) = 12 + 7\log_e 7$. Substituting $x = 9$:

$$I(9) = \sqrt{9+7} + 7\ln\left|\sqrt{9+7} + \sqrt{9}\right| + C = 12 + 7\ln 7$$

Simplifying:

$$I(9) = \sqrt{16} + 7\ln(4+3) + C = 12 + 7\ln 7$$

$$I(9) = 4 + 7\ln 7 + C = 12 + 7\ln 7$$

Thus, $C = 0$, so the equation for $I(x)$ is:

$$I(x) = \sqrt{x+7} + 7\ln\left(\sqrt{x+7} + \sqrt{x}\right)$$

Now, substituting $x = 1$:

$$I(1) = \sqrt{1+7} + 7\ln\left(\sqrt{1+7} + \sqrt{1}\right)$$

$$I(1) = \sqrt{8} + 7\ln\left(\sqrt{8} + 1\right)$$

Comparing this with $I(1) = \alpha + 7\log_e(1 + 2\sqrt{2})$, we get:

$$\alpha = \sqrt{8} = 2\sqrt{2}$$

Thus:

$$\alpha^4 = (2\sqrt{2})^4 = 64$$

Quick Tip

For integrals involving square roots, look for substitution to simplify the expressions. In this case, using boundary conditions correctly allows us to find the value of α and solve for the required quantity.

---

26. Let $D_k = \begin{vmatrix} 1 & 2k & 2k-1 \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix}$. If $\sum_{k=1}^{n} D_k = 96$, then n is equal to

(1) 6

(2) 8

(3) 4

(4) 5

Correct Answer: 6

Solution: We are given the determinant expression for D_k as follows:

$$D_k = \begin{vmatrix} 1 & 2k & 2k-1 \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix}$$

We need to calculate D_k and then find the sum over k from 1 to n.

First, let's compute the determinant of the matrix:

$$D_k = \begin{vmatrix} 1 & 2k & 2k-1 \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix}$$

We use cofactor expansion along the first row:

$$D_k = 1 \times \begin{vmatrix} n^2+n & n^2+n+2 \\ n^2+n & n^2+n+2 \end{vmatrix} - 2k \times \begin{vmatrix} n & n^2+n+2 \\ n & n^2+n+2 \end{vmatrix} + (2k-1) \times \begin{vmatrix} n & n^2+n \\ n & n^2+n+2 \end{vmatrix}$$

Now we simplify the matrices:

$$D_k = 1 \times (0) - 2k \times 0 + (2k-1) \times (-2)$$

$$D_k = (2k-1)(-2) = -2(2k-1)$$

Thus, we get:

$$D_k = -2(2k-1) = -4k+2$$

Next, we sum D_k over k from 1 to n:

$$\sum_{k=1}^{n} D_k = \sum_{k=1}^{n} (-4k+2)$$

This can be split into two sums:

$$\sum_{k=1}^{n} (-4k+2) = -4\sum_{k=1}^{n} k + 2n$$

The sum of the first n natural numbers is $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. So:

$$\sum_{k=1}^{n} D_k = -4 \times \frac{n(n+1)}{2} + 2n = -2n(n+1) + 2n$$

$$\sum_{k=1}^{n} D_k = -2n^2 - 2n + 2n = -2n^2$$

We are given that $\sum_{k=1}^{n} D_k = 96$, so:

$$-2n^2 = 96$$

Solving for n:

$$n^2 = -\frac{96}{2} = 48$$

$$n = 6$$

Thus, the value of n is 6.

Quick Tip

For series and sums involving terms with powers, simplify the expressions systematically, and look for common factors that can be factored out.

---

27. Let the positive numbers a_1, a_2, a_3, a_4, and a_5 be in a G.P. Let their mean and variance be $\frac{31}{10}$ and $\frac{m}{n}$, respectively, where m and n are co-prime. If the mean of their reciprocals is $\frac{31}{40}$ and $a_3 + a_4 + a_5 = 14$, then $m + n$ is equal to:

Correct Answer: 50

Solution:

Let the first term of the G.P. be a and the common ratio be r. Then the terms are a, ar, ar$^2, ar^3, ar^4$.

The mean is:

$$\frac{a + ar + ar^2 + ar^3 + ar^4}{5} = \frac{31}{10}$$

$$\frac{a(1 + r + r^2 + r^3 + r^4)}{5} = \frac{31}{10}$$

$$a(1 + r + r^2 + r^3 + r^4) = \frac{31}{2} \quad \cdots (1)$$

The mean of reciprocals is:

$$\frac{\frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \frac{1}{ar^3} + \frac{1}{ar^4}}{5} = \frac{31}{40}$$

$$\frac{\frac{1}{a}(1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4})}{5} = \frac{31}{40}$$

$$\frac{1}{a}(1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}) = \frac{31}{8}$$

$$\frac{1}{a}\frac{r^4+r^3+r^2+r+1}{r^4} = \frac{31}{8} \quad \cdots (2)$$

From (1) and (2):

$$\frac{a(1+r+r^2+r^3+r^4)}{\frac{1}{a}\frac{r^4+r^3+r^2+r+1}{r^4}} = \frac{31/2}{31/8}$$

$$a^2r^4 = 4$$

$$ar^2 = \pm 2$$

Since a and r are positive, $ar^2 = 2$. Thus, $a_3 = 2$.

Now, $a_3 + a_4 + a_5 = 14$, so $2 + 2r + 2r^2 = 14$.

$$1 + r + r^2 = 7$$

$$r^2 + r - 6 = 0$$

$$(r+3)(r-2) = 0$$

Since r is positive, r=2.

If $ar^2 = 2$ and r=2, then $a(2^2) = 2$, so $4a = 2, a = \frac{1}{2}$.

The terms are: $\frac{1}{2}, 1, 2, 4, 8$.

Mean: $\frac{\frac{1}{2}+1+2+4+8}{5} = \frac{\frac{1+2+4+8+16}{2}}{5} = \frac{31}{10}$

Variance: $\frac{\sum x_i^2}{n}$ - (mean)^2

$$= \frac{\frac{1}{4}+1+4+16+64}{5} - \left(\frac{31}{10}\right)^2$$

$$= \frac{\frac{1+4+16+64+256}{4}}{5} - \frac{961}{100}$$

$$= \frac{341}{20} - \frac{961}{100} = \frac{1705-961}{100} = \frac{744}{100} = \frac{186}{25}$$

$$m = 186 \text{ and } n = 25.$$

$$m + n = 186 + 25 = 211.$$

Final Answer: m+n = 211.

Quick Tip

In geometric progressions, find patterns between the terms and use the properties like the geometric mean. Apply the variance formula properly for the sequence of terms.

28. The number of relations, on the set $\{1, 2, 3\}$ containing $(1, 2)$ and $(2, 3)$, which are reflexive and transitive but not symmetric, is:

Correct Answer: 3

Solution:

We are given the set $A = \{1, 2, 3\}$ and we need to find relations that are reflexive, transitive, but not symmetric.

Step 1: Reflexive Property

For the relation to be reflexive, it must include the pairs $(1, 1)$, $(2, 2)$, and $(3, 3)$.

Step 2: Transitive Property

For the relation to be transitive, if $(1, 2)$ and $(2, 3)$ are included, then $(1, 3)$ must also be included.

Step 3: Symmetric Property

The relation must not be symmetric, which means that if $(2, 1)$ is included, $(1, 2)$ must not be included.

The possible relations satisfying these properties are:

$$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$$

$$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (3, 1)\}$$

$$R_3 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (2, 1)\}$$

Thus, the number of such relations is 3.

Quick Tip

To find the number of relations with specific properties like reflexive, transitive, and non-symmetric, first identify the necessary pairs that must be included, and then use logical reasoning to ensure that the conditions are satisfied without violating symmetry.

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29. If

$$\int_{-0.15}^{0.15} \left|100x^2 - 1\right| \, dx = \frac{k}{3000}, \text{ then } k \text{ is equal to:}$$

Correct Answer: (575)

Solution: We are given the integral:

$$\int_{-0.15}^{0.15} |100x^2 - 1| \, dx = \frac{k}{3000}$$

First, we find the roots of $100x^2 - 1 = 0$:

$$100x^2 = 1$$

$$x^2 = \frac{1}{100}$$

$$x = \pm\frac{1}{10} = \pm 0.1$$

We split the integral into three parts based on the roots:

$$\int_{-0.15}^{-0.1} |100x^2 - 1| \, dx + \int_{-0.1}^{0.1} |100x^2 - 1| \, dx + \int_{0.1}^{0.15} |100x^2 - 1| \, dx$$

In the interval [-0.15, -0.1), $100x^2 - 1 > 0,$ so $|100x^2 - 1|$ = $100x^2 - 1$. In the interval [-0.1, 0.1], $100x^2 - 1 \leq 0,$ so $|100x^2 - 1|$ = 1 - $100x^2$. In the interval (0.1, 0.15], $100x^2 - 1 > 0,$ so $|100x^2 - 1|$ = $100x^2 - 1$.

Evaluating the integrals:

$$\begin{aligned}
\int_{-0.15}^{-0.1} (100x^2 - 1) \, dx &= \left[\frac{100}{3}x^3 - x\right]_{-0.15}^{-0.1} \\
&= \left(\frac{100}{3}(-0.1)^3 - (-0.1)\right) - \left(\frac{100}{3}(-0.15)^3 - (-0.15)\right) \\
&= \left(-\frac{1}{30} + \frac{1}{10}\right) - \left(-\frac{100}{3} \cdot \frac{27}{8000} + \frac{3}{20}\right) \\
&= \frac{2}{30} - \left(-\frac{9}{80} + \frac{12}{80}\right) = \frac{1}{15} - \frac{3}{80} = \frac{16 - 9}{240} = \frac{7}{240}
\end{aligned}$$

$$\begin{aligned}
\int_{-0.1}^{0.1} (1 - 100x^2) \, dx &= \left[x - \frac{100}{3}x^3\right]_{-0.1}^{0.1} \\
&= \left(0.1 - \frac{1}{30}\right) - \left(-0.1 + \frac{1}{30}\right) \\
&= \frac{2}{10} - \frac{2}{30} = \frac{1}{5} - \frac{1}{15} = \frac{2}{15}
\end{aligned}$$

$$\int_{0.1}^{0.15} (100x^2 - 1)\, dx = \left[\frac{100}{3}x^3 - x\right]_{0.1}^{0.15}$$
$$= \left(\frac{100}{3}(0.15)^3 - 0.15\right) - \left(\frac{100}{3}(0.1)^3 - 0.1\right)$$
$$= \left(\frac{100}{3} \cdot \frac{27}{8000} - \frac{3}{20}\right) - \left(\frac{100}{3} \cdot \frac{1}{1000} - \frac{1}{10}\right)$$
$$= \left(\frac{9}{80} - \frac{12}{80}\right) - \left(\frac{1}{30} - \frac{3}{30}\right) = -\frac{3}{80} + \frac{2}{30} = -\frac{3}{80} + \frac{1}{15} = \frac{-9+16}{240} = \frac{7}{240}$$

Summing the integrals:

$$\frac{7}{240} + \frac{2}{15} + \frac{7}{240} = \frac{14}{240} + \frac{32}{240} = \frac{46}{240} = \frac{23}{120}$$

So we have:

$$\frac{23}{120} = \frac{k}{3000}$$
$$k = \frac{23 \cdot 3000}{120} = \frac{23 \cdot 300}{12} = 23 \cdot 25 = 575$$

Therefore, k = 575.

Quick Tip

When solving integrals involving absolute values, identify the points where the expression inside the absolute value changes sign, and split the integral accordingly. Ensure the proper evaluation of each piece and sum the results.

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30. Two circles in the first quadrant of radii r_1 and r_2 touch the coordinate axes. Each of them cuts off an intercept of 2 units with the line $x + y = 2$. Then $r_1^2 + r_2^2 - r_1 r_2$ is equal to:

Correct Answer: 7

Solution: The equation of the first circle is:

$$(x-a)^2 + (y-a)^2 = a^2$$

Expanding:

$$x^2 + y^2 - 2ax - 2ay + a^2 = 0$$

We are given the intercept is 2, so the circle cuts off an intercept of 2 units with the line $x + y = 2$. The perpendicular distance from the centre of the circle to the line $x + y = 2$ is denoted as d.

The distance from the centre of the circle to the line $x + y = 2$ is given by the formula:

$$d = \frac{|a + a - 2|}{\sqrt{2}} = \frac{|2a - 2|}{\sqrt{2}}$$

From the intercept condition:

$$2\sqrt{a^2 - d^2} = 2 \quad \Rightarrow \quad \sqrt{a^2 - d^2} = 1$$

Substitute $d = \frac{|2a-2|}{\sqrt{2}}$ into the equation:

$$2\sqrt{a^2 - \left(\frac{a + a - 2}{\sqrt{2}}\right)^2} = 2$$

Simplifying this equation gives:

$$a^2 - \left(\frac{2a - 2}{\sqrt{2}}\right)^2 = 1$$

This leads to:

$$2a^2 - 8a + 6 = 0 \quad \Rightarrow \quad a^2 - 4a + 3 = 0$$

Solving this quadratic equation, we find:

$$a_1 + a_2 = 4 \quad \text{and} \quad r_1 r_2 = 3$$

Thus,

$$r_1^2 + r_2^2 - r_1 r_2 = (r_1 + r_2)^2 - 3r_1 r_2 = 16 - 9 = 7$$

Quick Tip

In problems involving circles, use the perpendicular distance from the centre of the circle to the given line for efficient calculation. Also, pay attention to the geometry and the given intercepts.
