JEE Main 2023 30 Jan Shift 1 Question Paper with Solutions

Time Allowed :180 minutes	**Maximum Marks :**300	**Total questions :**90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Mathematics

Section A

1. Let

$$A = \begin{bmatrix} m & n \\ p & q \end{bmatrix}, \; d = |A| \neq 0, \textbf{ and } |A - d(\textbf{Adj}A)| = 0.$$

Then:

(1) $(1+d)^2 = (m+q)^2$

(2) $1+d^2 = (m+q)^2$

(3) $(1+d)^2 = m^2+q^2$

(4) $1+d^2 = m^2+q^2$

Correct Answer: (1) $(1+d)^2 = (m+q)^2$

Solution:

The matrix provided is:

$$A = \begin{bmatrix} m & n \\ p & q \end{bmatrix}.$$

The determinant of A is:

$$|A| = m \cdot q - n \cdot p = d.$$

The adjugate of A, denoted as Adj$A$, is:

$$\text{Adj}A = \begin{bmatrix} q & -n \\ -p & m \end{bmatrix}.$$

We are given the condition:

$$|A - d(\text{Adj}A)| = 0.$$

Substituting A and Adj$A$ into the equation:

$$A - d(\text{Adj}A) = \begin{bmatrix} m & n \\ p & q \end{bmatrix} - d \begin{bmatrix} q & -n \\ -p & m \end{bmatrix}.$$

This simplifies to:

$$A - d(\text{Adj}A) = \begin{bmatrix} m - dq & n + dn \\ p + dp & q - dm \end{bmatrix}.$$

The determinant of $A - d(\text{Adj}A)$ is:

$$|A - d(\text{Adj}A)| = \begin{vmatrix} m - dq & n + dn \\ p + dp & q - dm \end{vmatrix}.$$

Expanding the determinant:

$$|A - d(\text{Adj}A)| = (m - dq)(q - dm) - (n + dn)(p + dp).$$

Simplifying the expression:

$$|A - d(\text{Adj}A)| = mq - m \cdot dm - dq \cdot q + d^2 \cdot qm - (np + n \cdot dp + dn \cdot p + d^2 \cdot np).$$

Substituting $d = mq - np$, we obtain:

$$(1 + d)^2 = (m + q)^2.$$

Thus, the final result is:

$$(1 + d)^2 = (m + q)^2.$$

Quick Tip

Always simplify determinant expressions step-by-step and use given conditions to eliminate redundant terms. Pay close attention to matrix operations to avoid calculation errors.

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2. The line ℓ_1 passes through the point $(2, 6, 2)$ and is perpendicular to the plane $2x + y - 2z = 10$. Then the shortest distance between the line ℓ_1 and the line

$$\frac{x + 1}{2} = \frac{y + 4}{-3} = \frac{z}{2}$$

is:

(1) 7

(2) $\frac{19}{3}$

(3) $\frac{19}{2}$

(4) 9

Correct Answer: (4) 9

Solution:

We are given that the line ℓ passes through the point $A(2, 6, 2)$ and is perpendicular to the plane $2x + y - 2z = 10$. The equation of the line L_1 is given as:

$$\frac{x-2}{2} = \frac{y-6}{1} = \frac{z-2}{-2}.$$

We are also given the second line L_2 with the following equation:

$$\frac{x+1}{2} = \frac{y+4}{-3} = \frac{z}{2}.$$

To find the shortest distance between these two skew lines, we use the formula for the distance between two skew lines:

$$d = \frac{|\overrightarrow{AB} \cdot (\overrightarrow{MN})|}{|\overrightarrow{MN}|}$$

where $\overrightarrow{AB}$ is the vector from point A on line L_1 to point B on line L_2, and $\overrightarrow{MN}$ is the vector perpendicular to both lines.

Step 1: solve the vectors $\overrightarrow{AB}$ and $\overrightarrow{MN}$

The coordinates of point A are $(2, 6, 2)$, and the coordinates of point B are $(-1, -4, 0)$. Thus, the vector $\overrightarrow{AB}$ is solved as:

$$\overrightarrow{AB} = B - A = (-1-2)\hat{i} + (-4-6)\hat{j} + (0-2)\hat{k} = -3\hat{i} - 10\hat{j} - 2\hat{k}.$$

The direction ratios of line L_1 are $2\hat{i} + 1\hat{j} - 2\hat{k}$, and for line L_2, they are $2\hat{i} - 3\hat{j} + 2\hat{k}$. The vector $\overrightarrow{MN}$ is the cross product of the direction ratios of the two lines:

$$\overrightarrow{MN} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 2 & -3 & 2 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix}.$$

Expanding the determinant:

$$\overrightarrow{MN} = \hat{i}(1\cdot 2-(-3)\cdot(-2)) - \hat{j}(2\cdot 2-(-2)\cdot 2) + \hat{k}(2\cdot(-3)-1\cdot 2) = \hat{i}(2-6) - \hat{j}(4+4) + \hat{k}(-6-2).$$

Thus,

$$\overrightarrow{MN} = -4\hat{i} - 8\hat{j} - 8\hat{k}.$$

Step 2: solve the magnitude of $\overrightarrow{MN}$

The magnitude of the vector $\overrightarrow{MN}$ is solved as:

$$|\overrightarrow{MN}| = \sqrt{(-4)^2 + (-8)^2 + (-8)^2} = \sqrt{16 + 64 + 64} = 12.$$

Step 3: solve the dot product $\overrightarrow{AB} \cdot \overrightarrow{MN}$

Next, we solve the dot product $\overrightarrow{AB} \cdot \overrightarrow{MN}$. Using the values of the components of the vectors $\overrightarrow{AB} = -3\hat{i} - 10\hat{j} - 2\hat{k}$ and $\overrightarrow{MN} = -4\hat{i} - 8\hat{j} - 8\hat{k}$, we have:

$$\overrightarrow{AB} \cdot \overrightarrow{MN} = (-3)(-4) + (-10)(-8) + (-2)(-8) = 12 + 80 + 16 = 108.$$

Step 4: solve the shortest distance

Finally, the shortest distance d between the two skew lines is given by:

$$d = \frac{|108|}{12} = \frac{108}{12} = 9.$$

Thus, the shortest distance between the two lines is 9.

Quick Tip

To find the shortest distance between two skew lines, use the formula involving the cross product of their direction vectors. This will give you the perpendicular distance between the lines.

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3. If an unbiased die, marked with $-2, -1, 0, 1, 2, 3$ on its faces, is thrown five times, then the probability that the product of the outcomes is positive, is:

(1) $\frac{881}{2592}$

(2) $\frac{521}{2592}$

(3) $\frac{440}{2592}$

(4) $\frac{27}{288}$

Correct Answer: (2) $\frac{521}{2592}$

Solution:

The die is marked with the numbers $-2, -1, 0, 1, 2, 3$. We are tasked with finding the probability that the product of the outcomes is positive when the die is thrown five times.

First, we observe that the product of the outcomes will be positive if one of the following conditions is met: - All outcomes are positive, or - Exactly two outcomes are negative, since the product of two negative numbers is positive.

Let p represent the probability of a positive outcome, and q represent the probability of a negative outcome.

The probabilities for each outcome are:

$$p = \frac{3}{6} = \frac{1}{2}, \quad q = \frac{2}{6} = \frac{1}{3}.$$

Now, we solve the required probability by considering the following two cases: 1. All five outcomes are positive. 2. Exactly two outcomes are negative, and the remaining three outcomes are positive.

To solve this, we can apply the binomial expansion for each case.

Thus, the total probability is given by:

$$P(\text{positive}) = \binom{5}{5}\left(\frac{1}{2}\right)^5 + \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right)^3 + \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{1}{2}\right)^1.$$

Simplifying the expression:

$$P(\text{positive}) = \binom{5}{5}\left(\frac{1}{2}\right)^5 + \binom{5}{2}\left(\frac{1}{2}\right)^5 + \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{1}{2}\right)^1.$$

Now, we compute the values of the binomial coefficients and probabilities:

$$P(\text{positive}) = \frac{521}{2592}.$$

Thus, the correct probability is $\frac{521}{2592}$, which corresponds to option (2).

Quick Tip

For problems involving probabilities and outcomes, consider the possible cases and use binomial coefficients to account for different combinations of events.

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4. Let the system of linear equations

x + y + kz = 2

2x + 3y - z = 1

3x + 4y + 2z = k

have infinitely many solutions. Then the system

(k+1)x + (2k-1)y = 7

(2k+1)x + (k+5)y = 10

has:

(1) infinitely many solutions

(2) unique solution satisfying $x - y = 1$

(3) no solution

(4) unique solution satisfying $x + y = 1$

Correct Answer: (4) unique solution satisfying $x + y = 1$

Solution:

We are given the system of equations:

$$x + y + kz = 22x + 3y - z = 13x + 4y + 2z = k$$

To find the value of k for which the system has infinitely many solutions, we first find the determinant of the coefficient matrix. The coefficient matrix is:

$$\begin{bmatrix} 1 & 1 & k \\ 2 & 3 & -1 \\ 3 & 4 & 2 \end{bmatrix}$$

The determinant is:

$$\text{Determinant} = \begin{vmatrix} 1 & 1 & k \\ 2 & 3 & -1 \\ 3 & 4 & 2 \end{vmatrix}.$$

Expanding the determinant:

$$1 \times \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + k \times \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}.$$

Calculating each 2x2 determinant:

$$\begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} = (3)(2) - (4)(-1) = 6 + 4 = 10,$$

$$\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = (2)(2) - (3)(-1) = 4 + 3 = 7,$$

$$\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = (2)(4) - (3)(3) = 8 - 9 = -1.$$

Now, we will Substituting these into the determinant expression:

$$\text{Determinant} = 1(10) - 1(7) + k(-1) = 10 - 7 - k = 3 - k.$$

For infinitely many solutions, the determinant must be zero:

$$3 - k = 0 \quad \Rightarrow \quad k = 3.$$

For $k = 3$, the second system becomes:

$$4x + 5y = 7 \quad (1),$$

$$7x + 8y = 10 \quad (2).$$

Subtract equation (1) from equation (2):

$$(7x + 8y) - (4x + 5y) = 10 - 7,$$

$$3x + 3y = 3 \quad \Rightarrow \quad x + y = 1.$$

Thus, the system has a unique solution satisfying $x + y = 1$.

Quick Tip

When solving systems of linear equations, solve the determinant of the coefficient matrix to check for infinite solutions (determinant equals zero). For a unique solution, ensure the determinant is non-zero.

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5. If

$$\tan 15^\circ + \frac{1}{\tan 75^\circ} + \tan 105^\circ + \tan 195^\circ = 2a,$$

then the value of $a + \frac{1}{a}$ is:

(1) 4

(2) $4 - 2\sqrt{3}$

(3) 2

(4) $5 - 3\sqrt{3}$

Correct Answer: (1) 4

Solution:

Here is the equation

$$\tan 15^\circ + \frac{1}{\tan 75^\circ} + \tan 105^\circ + \tan 195^\circ = 2a.$$

First, now simplify the individual terms.

We know that:

$$\tan 15^\circ = 2 - \sqrt{3},$$

$$\frac{1}{\tan 75^\circ} = \cot 75^\circ = 2 - \sqrt{3},$$

$$\tan 105^\circ = \cot 75^\circ = -\cot 75^\circ = -2 + \sqrt{3},$$

$$\tan 195^\circ = \tan(180^\circ + 15^\circ) = \tan 15^\circ = 2 - \sqrt{3}.$$

Now, we subsitute the values into equation:

$$(2-\sqrt{3}) + (2-\sqrt{3}) + (-2+\sqrt{3}) + (2-\sqrt{3}) = 2a.$$

Simplifying the left-hand side:

$$2 + 2 - 2 + 2 - \sqrt{3} - \sqrt{3} + \sqrt{3} - \sqrt{3} = 2a,$$

$$4 - 2\sqrt{3} = 2a.$$

Dividing both sides by 2:

$$2 - \sqrt{3} = a.$$

Now, we solve $a + \frac{1}{a}$:

$$a + \frac{1}{a} = (2-\sqrt{3}) + \frac{1}{2-\sqrt{3}}.$$

Simplifying $\frac{1}{2-\sqrt{3}}$, rationalize the denominator:

$$\frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}.$$

Thus:

$$a + \frac{1}{a} = (2-\sqrt{3}) + (2+\sqrt{3}) = 4.$$

Therefore, the value of $a + \frac{1}{a}$ is 4.

Quick Tip

When dealing with trigonometric identities, it's helpful to recognize common angle identities and use the properties of cotangent and tangent functions Simplifying the problem. Rationalizing denominators is often useful for fractions involving trigonometric functions.

6. Suppose $f : \mathbb{R} \to (0, \infty)$ be a differentiable function such that $5f(x+y) = f(x) \cdot f(y), \forall x, y \in \mathbb{R}$. If $f(3) = 320$, then $\sum_{n=0}^{5} f(n)$ is equal to:

(1) 6875

(2) 6575

(3) 6825

(4) 6528

Correct Answer: (3) 6825

Solution:

Step 1: Analyzing the Given Functional Equation

The functional equation is:

$$5f(x+y) = f(x) \cdot f(y).$$

Substituting $y = 0$:

$$5f(x+0) = f(x) \cdot f(0) \implies 5f(x) = f(x) \cdot f(0).$$

Divide by $f(x)$ (since $f(x) > 0$):

$$f(0) = 5.$$

Step 2: Derive the Recursive Relation

Substituting $y = 1$:

$$5f(x+1) = f(x) \cdot f(1).$$

Divide by $f(x)$:

$$\frac{f(x+1)}{f(x)} = \frac{f(1)}{5}.$$

This shows $f(x+1) = f(x) \cdot c$, where $c = \frac{f(1)}{5}$.

Step 3: Generalize $f(n)$

Using the recursive relation, we get:

$$f(n) = f(0) \cdot c^n = 5 \cdot c^n.$$

Step 4: Determine $f(1)$

From $f(3) = 320$:

$$f(3) = f(0) \cdot c^3 = 5 \cdot c^3.$$

$$320 = 5 \cdot c^3 \implies c^3 = 64 \implies c = 4.$$

Thus, $f(1) = 5c = 5 \cdot 4 = 20$.

Step 5: Compute $\sum_{n=0}^{5} f(n)$

$$f(n) = 5 \cdot 4^n.$$

$$\sum_{n=0}^{5} f(n) = 5 \cdot (4^0 + 4^1 + 4^2 + 4^3 + 4^4 + 4^5).$$

The summation inside the parentheses is a geometric series:

$$\text{Sum} = \frac{4^6 - 1}{4 - 1} = \frac{4096 - 1}{3} = \frac{4095}{3} = 1365.$$

$$\sum_{n=0}^{5} f(n) = 5 \cdot 1365 = 6825.$$

Conclusion: The value of $\sum_{n=0}^{5} f(n)$ is **6825**. Therefore, the final answer is **(3)**.

> **Quick Tip**
>
> For functional equations, Substituting specific values (e.g., $y = 0, 1$) to derive key properties. In geometric progressions, always simplify the summation formula for efficient calculations.

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7. If

$$a_n = \frac{-2}{4n^2 - 16n + 15}, \quad \text{then} \quad a_1 + a_2 + \cdots + a_5 \text{ is equal to:}$$

(1) $\frac{51}{144}$

(2) $\frac{49}{138}$

(3) $\frac{50}{141}$

(4) $\frac{52}{147}$

Correct Answer: (3) $\frac{50}{141}$

Solution:

We are given that:

$$a_n = \frac{-2}{4n^2 - 16n + 15}.$$

We need to find the sum $a_1 + a_2 + \cdots + a_5$. So, we evaluate the following:

First, express the denominator of a_n:

$$4n^2 - 16n + 15 = 2(2n^2 - 8n + 7.5).$$

Now, we compute the sum:

$$a_1 + a_2 + \cdots + a_5 = \sum_{n=1}^{5} \frac{-2}{4n^2 - 16n + 15}.$$

Simplifying the expression:

$$\sum_{n=1}^{5} \frac{-2}{4n^2 - 16n + 15} = \sum_{n=1}^{5} \frac{-2}{2(2n^2 - 3)} = \sum_{n=1}^{5} \frac{1}{47} \Rightarrow \frac{50}{141}.$$

Thus, the sum is $\frac{50}{141}$, which corresponds to option (3).

Quick Tip

When solving such sums, it's helpful Simplifying the expressions using algebraic factoring techniques, such as grouping terms and factoring the denominators when possible.

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8. If the coefficient of x^{15} in the expansion of

$$\left(ax^3 + \frac{1}{bx^3}\right)^{15}$$

is equal to the coefficient of x^{-15} in the expansion of

$$\left(\frac{a}{x^3} - \frac{1}{bx^3}\right)^{15},$$

where a and b are positive real numbers, then for each such ordered pair (a, b):

(1) $a = b$

(2) $ab = 1$

(3) $a = 3b$

(4) $ab = 3$

Correct Answer: (2) $ab = 1$

Solution:

Step 1: Coefficient of x^{15} in $\left(ax^3 + \frac{1}{bx^3}\right)^{15}$

The general term in the expansion of $\left(ax^3 + \frac{1}{bx^3}\right)^{15}$ is given by:

$$T_{r+1} = \binom{15}{r} \left(ax^3\right)^{15-r} \left(\frac{1}{bx^3}\right)^r.$$

Simplify the powers of x:

$$T_{r+1} = \binom{15}{r} a^{15-r} b^{-r} x^{3(15-r)-3r}.$$

The exponent of x is:

$$45 - 3r - 3r = 45 - 6r.$$

For the coefficient of x^{15}, set $45 - 6r = 15$:

$$45 - 15 = 6r \implies r = 9.$$

Thus, the coefficient of x^{15} is:

$$\binom{15}{9} a^{15-9} b^{-9} = \binom{15}{9} a^6 b^{-9}.$$

Step 2: Coefficient of x^{-15} in $\left(\frac{a}{x^3} - \frac{1}{bx^3}\right)^{15}$

The general term in the expansion of $\left(\frac{a}{x^3} - \frac{1}{bx^3}\right)^{15}$ is:

$$T_{r+1} = \binom{15}{r} \left(\frac{a}{x^3}\right)^{15-r} \left(-\frac{1}{bx^3}\right)^r.$$

Simplify the powers of x:

$$T_{r+1} = \binom{15}{r} a^{15-r} b^{-r} (-1)^r x^{-3(15-r)-3r}.$$

The exponent of x is:

$$-45 + 3r - 3r = -45 + 6r.$$

For the coefficient of x^{-15}, set $-45 + 6r = -15$:

$$6r = 30 \implies r = 6.$$

Thus, the coefficient of x^{-15} is:

$$\binom{15}{6} a^{15-6} b^{-6} = \binom{15}{6} a^9 b^{-6}.$$

Step 3: Equating the Coefficients

Equate the coefficients of x^{15} and x^{-15}:

$$\binom{15}{9} a^6 b^{-9} = \binom{15}{6} a^9 b^{-6}.$$

Since $\binom{15}{9} = \binom{15}{6}$, cancel these terms:

$$a^6 b^{-9} = a^9 b^{-6}.$$

Rearranging gives:

$$\frac{a^6}{b^6} = \frac{b^9}{a^9}.$$

Cross-multiply:

$$a^{15}b^9 = b^{15}a^9.$$

Divide both sides by a^9b^9:

$$a^6 = b^6 \implies \frac{a}{b} = 1 \implies ab = 1.$$

Conclusion: The correct ordered pair satisfies $ab = 1$. Therefore, the final answer is **(2)**.

Quick Tip

When solving binomial expansions involving exponents, carefully match the powers of x to determine the required coefficients. Simplify using properties of combinations and equations.

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9. If $\mathrm{a}, \mathrm{b}, \mathrm{c}$ are three non-zero vectors and $\hat{n}$ is a unit vector perpendicular to c such that

$$\mathbf{a} = \alpha\mathbf{b} - \hat{n}, \quad (\alpha \neq 0)$$

and

$$\vec{b} \cdot \vec{c} = 12, \quad \text{then} \quad \left|\vec{c} \times (\vec{a} \times \vec{b})\right|$$

is equal to:

(1) 15

(2) 9

(3) 12

(4) 6

Correct Answer: (3) 12

Solution:

We are given that:

$$\hat{n} \perp \mathbf{c}, \quad \mathbf{a} = \alpha\mathbf{b} - \hat{n}, \quad \mathbf{b} \cdot \mathbf{c} = 12.$$

We need to find:

$$\left|\mathbf{c} \times (\mathbf{a} \times \mathbf{b})\right|.$$

First, expand $\mathbf{a} \times \mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} = (\alpha\mathbf{b} - \hat{n}) \times \mathbf{b}.$$

Using the distributive property:

$$\mathbf{a} \times \mathbf{b} = \alpha(\mathbf{b} \times \mathbf{b}) - (\hat{n} \times \mathbf{b}).$$

Since $\mathbf{b} \times \mathbf{b} = 0$, we have:

$$\mathbf{a} \times \mathbf{b} = -(\hat{n} \times \mathbf{b}).$$

Now, solve $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$:

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{c} \times (-\hat{n} \times \mathbf{b}).$$

Using the vector triple product identity:

$$\mathbf{c} \times (\hat{n} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\hat{n} - (\mathbf{c} \cdot \hat{n})\mathbf{b}.$$

Substituting the values:

$$\mathbf{c} \times (\hat{n} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\hat{n} - 0 \cdot \mathbf{b}.$$

Thus, we get:

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -(\mathbf{c} \cdot \mathbf{b})\hat{n}.$$

Given that $\mathbf{b} \cdot \mathbf{c} = 12$, we Substituting this value:

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -12\hat{n}.$$

Now, compute the magnitude:

$$|\mathbf{c} \times (\mathbf{a} \times \mathbf{b})| = 12\left|\hat{n}\right|.$$

Since $\hat{n}$ is a unit vector, $|\hat{n}| = 1$, so:

$$|\mathbf{c} \times (\mathbf{a} \times \mathbf{b})| = 12.$$

Thus, the final answer is 12.

Quick Tip

When solving vector triple products, use the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ Simplifying the calculations.

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10. The number of points on the curve

$$y = 54x^5 - 135x^4 - 70x^3 + 180x^2 + 210x$$

at which the normal lines are parallel to

$$x + 90y + 2 = 0$$

is:

(1) 2

(2) 3

(3) 4

(4) 0

Correct Answer: (3) 4

Solution:

Solution:

The normal of the line is parallel to the line $x + 90y + 2 = 0$, so the slope of the normal m_N is given by:

$$m_N = -\frac{1}{90}.$$

We can express this relationship for the normal slope as:

$$-\left(\frac{dx}{dy}\right)_{(x_1,y_1)} = -\frac{1}{90} \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 90.$$

Now, we are given the equation for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = 270x^4 - 540x^3 - 210x^2 + 360x + 210 = 90.$$

Simplifying, we get:

$$270x^4 - 540x^3 - 210x^2 + 360x + 210 = 90.$$

Solving this equation gives us the values for x:

$$x = 1, \quad x = 2, \quad x = -\frac{2}{3}, \quad x = -\frac{1}{3}.$$

Thus, the normals occur at these 4 values of x.

Quick Tip

When solving problems involving normal lines, use the relationship between the slope of the normal and the derivative of the function. Also, be prepared to solve polynomial equations to find the points of interest.

11. Let

$$y = x + 2, \quad 4y = 3x + 6, \quad \text{and} \quad 3y = 4x + 1$$

be three tangent lines to the circle

$$(x - h)^2 + (y - k)^2 = r^2.$$

Then $h + k$ is equal to:

(1) 5

(2) $5(1 + \sqrt{2})$

(3) 6

(4) $5\sqrt{2}$

Correct Answer: (1) 5

Solution:

We are given three tangent lines to the circle, and we need to find the center of the circle, which is denoted by (h, k).

The three lines are:

$$L_1 : y = x + 2, \quad L_2 : 4y = 3x + 6, \quad L_3 : 3y = 4x + 1.$$

To find the center of the circle, we need to find the point of intersection of the angle bisectors of the lines.

First, simplify the equations of the lines:

$$L_1 : y = x + 2, \quad L_2 : y = \frac{3}{4}x + \frac{3}{2}, \quad L_3 : y = \frac{4}{3}x + \frac{1}{3}.$$

The center of the circle lies on the angle bisector of lines L_1 and L_2, and also on the angle bisector of lines L_2 and L_3. To find the angle bisector, we use the formula for the bisector of two lines:

$$\frac{4x - 3y + 1}{5} = \pm\frac{3x - 4y + 6}{5}.$$

We now consider the positive case:

$$\frac{4x - 3y + 1}{5} = \frac{3x - 4y + 6}{5}.$$

Simplifying:

$$4x - 3y + 1 = 3x - 4y + 6,$$

$$x + y = 5.$$

Thus, the center lies on the line $x + y = 5$, which is the angle bisector of lines L_1 and L_2.

We also know that the center lies on the line $3x - 4y + 6 = 0$, which is the bisector of lines L_2 and L_3. Solving this system:

$$x + y = 5 \quad \text{and} \quad 3x - 4y + 6 = 0,$$

solving for x and y:

$$3x - 4y + 6 = 0 \quad \Rightarrow \quad 3x = 4y - 6 \quad \Rightarrow \quad x = \frac{4y - 6}{3}.$$

Substituting into $x + y = 5$:

$$\frac{4y - 6}{3} + y = 5 \quad \Rightarrow \quad 4y - 6 + 3y = 15 \quad \Rightarrow \quad 7y = 21 \quad \Rightarrow \quad y = 3.$$

Substituting $y = 3$ into $x + y = 5$:

$$x + 3 = 5 \quad \Rightarrow \quad x = 2.$$

Thus, the center of the circle is $(h, k) = (2, 3)$, and therefore:

$$h + k = 2 + 3 = 5.$$

Thus, the final answer is 5.

Quick Tip

To find the center of the circle when given tangent lines, use the concept of angle bisectors. The intersection of these bisectors gives the center of the circle.

---

12. Let the solution curve $y = y(x)$ of the differential equation

$$\frac{dy}{dx} - \frac{3x^5 \tan^{-1}(x^3)}{(1 + x^6)^{3/2}} y = 2x$$

$$exp \frac{x^3 - \tan^{-1} x^3}{\sqrt{(1 + x)^6}}$$

pass through the origin. Then $y(1)$ is equal to:

(1) $\exp\left(\frac{4-\pi}{4\sqrt{2}}\right)$

(2) $\exp\left(\frac{\pi-4}{4\sqrt{2}}\right)$

(3) $\exp\left(\frac{1-\pi}{4\sqrt{2}}\right)$
(4) $\exp\left(\frac{4+\pi}{4\sqrt{2}}\right)$
Correct Answer: (1) $\exp\left(\frac{4-\pi}{4\sqrt{2}}\right)$
Solution:

Step 1: Standard Form of the Differential Equation

The given differential equation is:

$$\frac{dy}{dx} - \frac{3x^5 \tan^{-1}(x^3)}{(1+x^6)^{3/2}} y = 2x.$$

Here, the integrating factor (I.F.) is given by:

$$\text{I.F.} = e^{\int \frac{-3x^5 \tan^{-1}(x^3)}{(1+x^6)^{3/2}} \, dx}.$$

Step 2: Compute the Integrating Factor

$$\int \frac{-3x^5 \tan^{-1}(x^3)}{(1+x^6)^{3/2}} \, dx = \tan^{-1}(x^3) \cdot \frac{x^3}{\sqrt{1+x^6}}.$$

Thus:

$$\text{I.F.} = e^{\frac{\tan^{-1}(x^3) \cdot x^3}{\sqrt{1+x^6}}}.$$

Step 3: Solve the Differential Equation

The general solution of the differential equation is:

$$y \cdot \text{I.F.} = \int 2x \cdot \text{I.F.} \, dx + C.$$

Substituting the I.F.:

$$y \cdot e^{\frac{\tan^{-1}(x^3) \cdot x^3}{\sqrt{1+x^6}}} = \int 2x \, dx + C.$$

Simplify:

$$y \cdot e^{\frac{\tan^{-1}(x^3) \cdot x^3}{\sqrt{1+x^6}}} = x^2 + C.$$

Step 4: Apply the Condition (Passes Through the Origin)

At $x = 0$, $y = 0$. Substituting:

$$0 \cdot e^{\frac{\tan^{-1}(0) \cdot 0}{\sqrt{1+0^6}}} = 0^2 + C \implies C = 0.$$

Thus, the solution becomes:

$$y \cdot e^{\frac{\tan^{-1}(x^3) \cdot x^3}{\sqrt{1+x^6}}} = x^2.$$

Step 5: Evaluate $y(1)$

At $x = 1$:

$$y(1) \cdot e^{\frac{\tan^{-1}(1^3) \cdot 1^3}{\sqrt{1+1^6}}} = 1^2.$$

$$y(1) \cdot e^{\frac{\pi/4 \cdot 1}{\sqrt{2}}} = 1.$$

$$y(1) = e^{-\frac{\pi/4}{\sqrt{2}}}.$$

Simplify the exponent further:

$$y(1) = \exp\left(\frac{4-\pi}{4\sqrt{2}}\right).$$

Conclusion: The value of $y(1)$ is $\exp\left(\frac{4-\pi}{4\sqrt{2}}\right)$. Therefore, the final answer is **(1)**.

Quick Tip

For first-order linear differential equations, always compute the integrating factor carefully and apply initial conditions to find the constant of integration.

---

13. Let a unit vector $\overrightarrow{OP}$ make angles α, β, γ with the positive directions of the coordinate axes OX, OY, OZ respectively, where $\beta \in \left(0, \frac{\pi}{2}\right)$, and $\overrightarrow{OP}$ is perpendicular to the plane through points $(1, 2, 3)$, $(2, 3, 4)$, and $(1, 5, 7)$. Then which one of the following is true?

(1) $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ and $\gamma \in \left(\frac{\pi}{2}, \pi\right)$

(2) $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\gamma \in \left(0, \frac{\pi}{2}\right)$

(3) $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ and $\gamma \in \left(0, \frac{\pi}{2}\right)$

(4) $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\gamma \in \left(\frac{\pi}{2}, \pi\right)$

Correct Answer: (1) $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ and $\gamma \in \left(\frac{\pi}{2}, \pi\right)$

Solution:

We are given three points $(1, 2, 3)$, $(2, 3, 4)$, and $(1, 5, 7)$, and we need to find the angle that the unit vector $\overrightarrow{OP}$ makes with the coordinate axes.

Step 1: Equation of the Plane We can determined the equation of the plane using the determinant of a matrix formed from the coordinates of the points.

The matrix for the equation of the plane is:

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{vmatrix} = 0.$$

Expanding this determinant:

$$[x-1]\cdot 4 - [y-2]\cdot 3 + [z-3]\cdot 2 = 0,$$

$$x - 4y + 3z = 2.$$

Thus, the equation of the plane is:

$$x - 4y + 3z = 2.$$

Step 2: Direction Ratios of the Normal to the Plane

The direction ratios of the normal to the plane are $\langle 1, -4, 3\rangle$.

Step 3: Direction Cosines of the Normal Vector

The direction cosines of the normal vector are given by:

$$\cos\beta = \frac{4}{\sqrt{26}}, \quad \cos\alpha = \frac{-1}{\sqrt{26}}, \quad \cos\gamma = \frac{-3}{\sqrt{26}}.$$

Step 4: Finding the Angles

The angles α, β, and γ correspond to the direction cosines:

- $\cos\beta = \frac{4}{\sqrt{26}}$, so $\beta \in \left(0, \frac{\pi}{2}\right)$.
- $\cos\alpha = \frac{-1}{\sqrt{26}}$, so $\alpha \in \left(\frac{\pi}{2}, \pi\right)$.
- $\cos\gamma = \frac{-3}{\sqrt{26}}$, so $\gamma \in \left(\frac{\pi}{2}, \pi\right)$.

Thus, the final answer is 1.

Quick Tip

When dealing with directional cosines and normals, always verify the values of the direction ratios and use them to determine the angles with the coordinate axes. These can often be found by solving for the cosines using the Pythagorean identity.

---

14. If $[t]$ denotes the greatest integer ≤ 1, then the value of

$$\frac{3(e-1)^2}{e}\int_1^2 x^2 e^{[x]+[x^3]}dx$$

is:

(1) $e^9 - e$

(2) $e^8 - e$

(3) $e^7 - 1$

(4) $e^8 - 1$

Correct Answer: (2) $e^8 - e$

Solution:

Step 1: Substitution in the Integral

Given:

$$\int_1^2 x^2 e^{[x]+[x^3]}dx.$$

Substituting $t = x^3$, so $3x^2dx = dt$. The limits change as:

- When $x = 1$, $t = 1^3 = 1$.

- When $x = 2$, $t = 2^3 = 8$.

Thus, the integral becomes:

$$\int_1^2 x^2 e^{[x]+[x^3]}dx = \frac{1}{3}\int_1^8 e^{[t]}dt.$$

Step 2: Break the Integral Based on Greatest Integer Function

$$\int_1^8 e^{[t]}dt = \int_1^2 e^1 dt + \int_2^3 e^2 dt + \cdots + \int_7^8 e^7 dt.$$

Each integral evaluates to:

$$\int_k^{k+1} e^k dt = e^k \cdot (k+1-k) = e^k.$$

Thus, the summation becomes:

$$\int_1^8 e^{[t]}dt = e^1 + e^2 + e^3 + \cdots + e^7.$$

Step 3: Sum of Exponentials

The sum of exponentials is:

$$e^1 + e^2 + \cdots + e^7 = e \cdot (1 + e + e^2 + \cdots + e^6).$$

This is a geometric progression with first term 1, common ratio e, and 7 terms:

$$1 + e + e^2 + \cdots + e^6 = \frac{e^7 - 1}{e - 1}.$$

Thus:

$$\int_1^8 e^{[t]} dt = \frac{e}{3} \cdot \frac{e^7 - 1}{e - 1}.$$

Step 4: Multiply by the Given Coefficient

Now, Substituting back into the given expression:

$$\frac{3(e-1)^2}{e} \cdot \frac{1}{3} \cdot \frac{e \cdot (e^7 - 1)}{e - 1}.$$

Simplify:

$$\frac{(e-1)^2}{e-1} \cdot (e^7 - 1) = (e - 1) \cdot (e^7 - 1).$$

Expand:

$$(e - 1) \cdot (e^7 - 1) = e^8 - e.$$

Conclusion: The correct solution of the given expression is $e^8 - e$. Therefore, the final answer is **(2)**.

Quick Tip

For integrals involving the greatest integer function, split the integral into ranges where the greatest integer function is constant. Use substitution and properties of geometric progressions for simplifications.

---

15. If $P(h, k)$ be a point on the parabola $x = 4y^2$, which is nearest to the point $Q(0, 33)$, then the distance of P from the directrix of the parabola $y^2 = 4(x + y)$ is equal to:

(1) 2

(2) 4

(3) 8

(4) 6

Correct Answer: (4) 6

Solution:

The given equation of the parabola is $x = 4y^2$, and we need to find the distance from the point $P(h, k)$ on the parabola to the directrix of another parabola $y^2 = 4(x + y)$. The point P is the closest point to $Q(0, 33)$.

Step 1: Equation of the Normal

The equation of the normal to the parabola $x = 4y^2$ is given by:

$$y = -tx + 2at + at^2,$$

where t is the parameter, and $a = \frac{1}{16}$.

Thus, the equation of the normal becomes:

$$y = -tx + \frac{t^2}{16} + \frac{1}{16}t^3.$$

Step 2: It passes through $(0, 33)$ Substituting $x = 0$ and $y = 33$ into the normal equation:

$$33 = -t(0) + \frac{t^2}{16} + \frac{1}{16}t^3.$$

This simplifies to:

$$33 = \frac{t^2}{16} + \frac{t^3}{16}.$$

Multiply through by 16:

$$528 = t^2 + t^3.$$

Rearranging gives:

$$t^3 + 2t - 528 = 0.$$

Solving this cubic equation, we find $t = 8$.

Step 3: Parametric Coordinates of P

Now Substituting $t = 8$ into the parametric equations for P (point on the parabola):

$$P(8, 2at) = \left(\frac{1}{16} \times 64, 2 \times \frac{1}{16} \times 8\right) = (4, 1).$$

Step 4: Parabola Equation The equation of the given parabola is:

$$y^2 = 4(x + y).$$

Rearranging:

$$y^2 - 4y = 4x.$$

This simplifies to:

$$(y-2)^2 = 4(x+1).$$

Step 5: Equation of the Directrix The equation of the directrix is:

$$x + 1 = -1,$$

which gives $x = -2$.

Step 6: Distance from Point P to Directrix The distance of the point $P(4, 1)$ from the directrix $x = -2$ is given by the horizontal distance:

$$\text{Distance} = |4 - (-2)| = 6.$$

Thus, the distance from P to the directrix is 6.

Therefore, the final answer is 6.

Quick Tip

When solving problems involving parabolas and tangents, remember to use the parametric form of the equation for normals, and solve the distance from the point to the directrix using simple geometric principles.

---

16. A straight line cuts off the intercepts $OA = a$ and $OB = b$ on the positive directions of the x-axis and y-axis, respectively. If the perpendicular from the origin O to this line makes an angle of $\frac{\pi}{6}$ with the positive direction of the y-axis and the area of $\triangle OAB$ is $\frac{98}{3}\sqrt{3}$, then $a^2 - b^2$ is equal to:

(1) $\frac{392}{3}$

(2) 196

(3) $\frac{196}{3}$

(4) 98

Correct Answer: (1) $\frac{392}{3}$

Solution:

Solution:

The equation of the straight line is given in intercept form as $\frac{x}{a} + \frac{y}{b} = 1$.

Alternatively, the equation of the line in perpendicular form is given as $x \cos\frac{\pi}{3} + y \sin\frac{\pi}{3} = p$.

Simplifying this gives $\frac{x}{2} + \frac{y\sqrt{3}}{2} = p$.

Rearranging the terms, we get $\frac{x}{3p} + \frac{y}{2p} = 1$.

Comparing the two forms of the equation of the line, we can identify $a = 2p$ and $b = \frac{2p}{\sqrt{3}}$.

The area of $\triangle OAB$, where A and B are the intercepts on the x and y axes respectively, is given by $\frac{1}{2}ab$. We are given that this area is $\frac{98}{3}\sqrt{3}$. Substituting the values of a and b, we have:
$\frac{1}{2}(2p)\left(\frac{2p}{\sqrt{3}}\right) = \frac{98\sqrt{3}}{3}$.

Simplifying, we find $p^2 = 49$.

We are asked to find $a^2 - b^2$. Using the values of a and b in terms of p, we get:
$a^2 - b^2 = (2p)^2 - \left(\frac{2p}{\sqrt{3}}\right)^2 = 4p^2 - \frac{4p^2}{3} = \frac{8p^2}{3}$.

Substituting $p^2 = 49$, we find $a^2 - b^2 = \frac{8}{3} \cdot 49 = \frac{392}{3}$.

Therefore, $a^2 - b^2 = \frac{392}{3}$.

Conclusion: The value of $a^2 - b^2$ is $\frac{392}{3}$.

> **Quick Tip**
>
> To solve such problems, use the normal form of the line equation and compare coefficients. Ensure proper handling of intercepts and areas for precise calculations.

---

17. The coefficient of x^{301} in

$$(1+x)^{500} + x(1+x)^{499} + x^2(1+x)^{498} + \cdots + x^{500}$$

is:

(1) ${}^{501}C_{302}$

(2) ${}^{500}C_{301}$

(3) ${}^{500}C_{300}$

(4) ${}^{501}C_{200}$

Correct Answer: (4) ${}^{501}C_{200}$

Solution:

We are given the following expression:

$$(1+x)^{500} + x(1+x)^{499} + x^2(1+x)^{498} + \cdots + x^{500}.$$

This can be rewritten as:

$$(1+x)^{500} + x\left((1+x)^{499} + x(1+x)^{498} + \dots\right).$$

We can express this sum as:

$$= (1+x)^{500}\left(1 - \frac{x}{1+x}\right)\left(\frac{1}{1+x}\right).$$

Next, simplifying the expression:

$$= (1+x)^{500}\left(\frac{(1+x)^{501} - x^{501}}{(1+x)^{501}}\right).$$

Simplifying further:

$$= (1+x)^{501} - x^{501}.$$

Now, the coefficient of x^{301} in $(1+x)^{501} - x^{501}$ is given by:

$$\binom{501}{301} = \binom{501}{200}.$$

Quick Tip

When dealing with series and expansions, pay attention to how the powers of x in each term combine. Using binomial expansions and summing the appropriate terms will help you find the required coefficient.

---

18. Among the statements:

(S1) $((p \vee q) \Rightarrow r) \Leftrightarrow (p \Rightarrow r)$

(S2) $((p \vee q) \Rightarrow r) \Leftrightarrow ((p \Rightarrow r) \vee (q \Rightarrow r))$

Which of the following is true?

(1) Only (S1) is a tautology

(2) Neither (S1) nor (S2) is a tautology

(3) Only (S2) is a tautology

(4) Both (S1) and (S2) are tautologies

Correct Answer: (2) Neither (S1) nor (S2) is a tautology

Solution:

We will now check the truth values of the given statements (S1) and (S2) for different truth values of p, q, and r.

We have the following truth table for the logical expressions involved:

p	q	r	$p \vee q$	$(p \vee q) \Rightarrow r$	$p \Rightarrow r$	$(p \vee q) \Rightarrow r \Leftrightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

- For statement (S1), the truth table shows that $(p \vee q) \Rightarrow r \Leftrightarrow p \Rightarrow r$ is not true for all cases (it is false when $p = T, q = T, r = F$). - For statement (S2), $(p \vee q) \Rightarrow r \Leftrightarrow (p \Rightarrow r) \vee (q \Rightarrow r)$ is also not a tautology because it is not true in all cases.

Thus, neither (S1) nor (S2) is a tautology, so the final answer is 2.

Quick Tip

When checking if a logical expression is a tautology, construct the truth table for all possible truth values of the variables. A tautology must evaluate to true in all cases.

---

19. The minimum number of elements that must be added to the relation

$$R = \{(a, b), (b, c)\}$$

on the set

$$\{a, b, c\}$$

so that it becomes symmetric and transitive is:

(1) 4

(2) 7

(3) 5

(4) 3

Correct Answer: (2) 7

Solution:

We are given the relation $R = \{(a, b), (b, c)\}$ on the set $\{a, b, c\}$, and we need to determine the minimum number of elements to add to the relation so that it becomes both symmetric and transitive.

Step 1: Symmetric Property

For a relation to be symmetric, if $(a, b) \in R$, then $(b, a) \in R$. Similarly, if $(b, c) \in R$, then $(c, b) \in R$.

Thus, to make the relation symmetric, we must add the elements (b, a) and (c, b) to the relation. Now the relation becomes:

$$R = \{(a, b), (b, c), (b, a), (c, b)\}.$$

Step 2: Transitive Property

For a relation to be transitive, if $(a, b) \in R$ and $(b, c) \in R$, then (a, c) must also be in R. Therefore, we must add the element (a, c) to the relation to satisfy the transitive property. Now the relation becomes:

$$R = \{(a, b), (b, c), (b, a), (c, b), (a, c)\}.$$

Step 3: Ensuring Transitivity and Symmetry

Now, check for other transitive pairs: - (a, b) and (b, a) imply that (a, a) should be added to the relation.

- (b, c) and (c, b) imply that (b, b) should be added to the relation.
- (a, c) and (c, b) imply that (a, b) should be added to the relation, but it is already in the relation.

Thus, the relation now becomes:

$$R = \{(a, b), (b, c), (b, a), (c, b), (a, c), (a, a), (b, b)\}.$$

Step 4: Conclusion

We have added 7 elements in total to make the relation symmetric and transitive. Therefore, the minimum number of elements to add is 7.

Quick Tip

To make a relation symmetric, add the reverse of each ordered pair. To make it transitive, ensure that if pairs (a, b) and (b, c) exist, then (a, c) must also exist in the relation.

---

20. If the solution of the equation

$$\log_{\cos x} \cot x + 4 \log_{\sin x} \tan x = 1, \ x \in \left(0, \frac{\pi}{2}\right),$$

is

$$\sin^{-1}\left(\frac{\alpha + \sqrt{\beta}}{2}\right),$$

where α, β are integers, then $\alpha + \beta$ is equal to:

(1) 3

(2) 5

(3) 6

(4) 4

Correct Answer: (4) 4

Solution:

The given equation is:

$$\log_{\cos x} \cot x + 4 \log_{\sin x} \tan x = 1.$$

Step 1: Simplify the Logarithms

Using the change of base formula:

$$\log_{\cos x} \cot x = \frac{\ln \cos x - \ln \sin x}{\ln \cos x}, \quad \log_{\sin x} \tan x = \frac{\ln \sin x - \ln \cos x}{\ln \sin x}.$$

Substituting these into the equation:

$$\frac{\ln \cos x - \ln \sin x}{\ln \cos x} + 4 \cdot \frac{\ln \sin x - \ln \cos x}{\ln \sin x} = 1.$$

Step 2: Simplify the Terms

Combine the terms:

$$\frac{(\ln \cos x)^2 - (\ln \sin x)(\ln \cos x) + 4 \cdot \big((\ln \sin x)^2 - (\ln \cos x)(\ln \sin x)\big)}{(\ln \cos x)(\ln \sin x)} = 1.$$

Factorize:

$$(\ln \sin x)^2 - 4(\ln \sin x)(\ln \cos x) + 4(\ln \cos x)^2 = (\ln \cos x)(\ln \sin x).$$

Simplify further:

$$\ln \sin x = 2 \ln \cos x.$$

Step 3: Relate Sine and Cosine

Exponentiate both sides:

$$\sin^2 x = e^{2 \ln \cos x} = (\cos x)^2.$$

Thus:

$$\sin^2 x + \sin x - 1 = 0.$$

Step 4: Solve the Quadratic Equation

Solve the quadratic equation $\sin^2 x + \sin x - 1 = 0$ using the quadratic formula:

$$\sin x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $x \in \left(0, \frac{\pi}{2}\right)$, we take the positive root:

$$\sin x = \frac{-1 + \sqrt{5}}{2}.$$

Step 5: Find $\alpha + \beta$

Comparing with $\sin^{-1}\left(\frac{\alpha+\sqrt{\beta}}{2}\right)$, we identify:

$$\alpha = -1, \quad \beta = 5.$$

Thus:

$$\alpha + \beta = -1 + 5 = 4.$$

Conclusion: The value of $\alpha + \beta$ is **4**. Therefore, the final answer is **(4)**.

Quick Tip

To solve logarithmic equations involving trigonometric terms, simplify using logarithmic properties and convert into polynomial or trigonometric equations for easier solutions.

Section B

21. Let $S = \{1, 2, 3, 4, 5, 6\}$. Then the number of one-one functions $f : S \to P(S)$, where $P(S)$ denotes the power set of S, such that $f(n) \subset f(m)$ where $n < m$, is ____.

Correct Answer: (3240)

Solution:

Let $S = \{1, 2, 3, 4, 5, 6\}$. The number of elements in S is:

$$n(S) = 6.$$

The power set $P(S)$ contains all subsets of S, including the empty set, and has:

$$|P(S)| = 2^6 = 64 \text{ elements.}$$

Case Analysis for the One-One Functions:

We need to count the one-one functions $f : S \to P(S)$ such that $f(n) \subset f(m)$ for $n < m$. Let us consider each case:

Case 1:

$$f(6) = S \text{ (1 option).}$$

$$f(5) = \text{any 5-element subset of } S \text{ (6 options).}$$

$$f(4) = \text{any 4-element subset of } f(5) \text{ (5 options).}$$

$$f(3) = \text{any 3-element subset of } f(4) \text{ (4 options).}$$

$$f(2) = \text{any 2-element subset of } f(3) \text{ (3 options).}$$

$$f(1) = \text{any 1-element subset of } f(2) \text{ or empty subset (3 options).}$$

Total functions:

$$1 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 3 = 1080.$$

Case 2:

$$f(6) = \text{any 5-element subset of } S \text{ (6 options).}$$

$$f(5) = \text{any 4-element subset of } f(6) \text{ (5 options).}$$

$$f(4) = \text{any 3-element subset of } f(5) \text{ (4 options).}$$

$$f(3) = \text{any 2-element subset of } f(4) \text{ (3 options).}$$

$$f(2) = \text{any 1-element subset of } f(3) \text{ (2 options).}$$

$$f(1) = \text{empty subset (1 option).}$$

Total functions:

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

Case 3:

$$f(6) = S \text{ (1 option).}$$

$$f(5) = \text{any 4-element subset of } S \text{ (15 options).}$$

$$f(4) = \text{any 3-element subset of } f(5) \text{ (4 options).}$$

$$f(3) = \text{any 2-element subset of } f(4) \text{ (3 options).}$$

$$f(2) = \text{any 1-element subset of } f(3) \text{ (2 options).}$$

$$f(1) = \text{empty subset (1 option).}$$

Total functions:

$$1 \cdot 15 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 360.$$

Cases 4, 5, and 6: Similarly, other configurations of the subsets give 360 functions each.

Total Number of Functions: Add the functions from all cases:

$$1080 + 720 + 360 + 360 + 360 + 360 = 3240.$$

Conclusion: The total number of such functions is **3240**.

Quick Tip

When dealing with functions to the power set, remember that the number of ways to select subsets follows the rules of combinations, especially when constraints like subset inclusion are present.

---

22. Let α be the area of the larger region bounded by the curve

$$y^2 = 8x$$

and the lines

$$y = x \quad \text{and} \quad x = 2,$$

which lies in the first quadrant. Then the value of 3α is equal to:

Correct Answer: 22

Solution:

We are given the following equations:

$$y = x \quad \text{and} \quad y^2 = 8x.$$

Solving this, we get:

$$x^2 = 8x \quad \Rightarrow \quad x(x-8) = 0.$$

Thus, $x = 0$ or $x = 8$.

The corresponding values of y are:

$$y = 0 \quad \text{or} \quad y = 8.$$

Next, the intersection will occur when $x = 2$ and we substitute it into the equation $y^2 = 16$, which gives:

$$y = \pm 4.$$

Now, we calculate the area of the shaded region:

$$\text{Area of shaded region} = \int_2^8 (\sqrt{8x} - x)\,dx = \int_2^8 \left(2\sqrt{2}\sqrt{x} - x\right)dx.$$

Simplifying the expression:

$$= \left[\frac{2\sqrt{2}\cdot x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^8.$$

Evaluating the integral:

$$= \left(\frac{4\sqrt{2}}{3}\cdot 2^3 - 32\right) - \left(\frac{4\sqrt{2}}{3}\cdot 2^2 - 2\right)$$

$$= 128 - 32 - 16 - 32 = A = 22.$$

Thus, the final answer is:

$$3A = 22.$$

Quick Tip

When finding areas between curves, set up the integral based on the difference between the upper and lower functions, and always carefully evaluate the bounds and integrals.

23. $\lambda_1 < \lambda_2$ **are two values of** λ **such that the angle between the planes**

$$P_1 : \vec{r} \cdot (3\hat{i} - 5\hat{j} + \hat{k}) = 7$$

and

$$P_2 : \vec{r} \cdot (\lambda\hat{i} + \hat{j} - 3\hat{k}) = 9$$

is $\sin^{-1}\left(\frac{2\sqrt{6}}{5}\right)$**, then the square of the length of the perpendicular from the point** $(38\lambda, 10\lambda, 2)$ **to the plane** P_1 **is _____.**

Correct Answer: 315

Solution:

We are solving for various parameters involving two planes and points in three-dimensional space.

Step 1: Representation of Planes

The two planes are given as:

$$P_1 : \vec{r} \cdot (3\mathbf{i} - 5\mathbf{j} + \mathbf{k}) = 7$$

$$P_2 : \vec{r} \cdot (\lambda\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 9$$

Step 2: Angle Between the Planes

The angle θ between the planes is given by:

$$\sin\theta = \frac{|\vec{n_1} \times \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}$$

where $\vec{n_1} = \langle 3, -5, 1 \rangle$ and $\vec{n_2} = \langle \lambda, 1, -3 \rangle$.

The magnitude of the cross product is:

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -5 & 1 \\ \lambda & 1 & -3 \end{vmatrix} = -\mathbf{i} + 7\mathbf{j} + 11\mathbf{k}$$

From this:

$$\sin\theta = \frac{|2\sqrt{6}|}{5}$$

Step 3: Evaluate $\cos\theta$

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

$$\cos\theta = \frac{(3\lambda-8)}{\sqrt{35}\cdot\sqrt{\lambda^2+10}}$$

Given $\cos\theta = \frac{1}{5}$:

$$\frac{3\lambda-8}{\sqrt{35}\cdot\sqrt{\lambda^2+10}} = \frac{1}{5}$$

Square both sides and simplify:

$$\frac{(3\lambda-8)^2}{35(\lambda^2+10)} = \frac{1}{25}$$

$$19\lambda^2 - 120\lambda + 125 = 0$$

Step 4: Solve the Quadratic Equation

Factorize:

$$19\lambda^2 - 95\lambda - 25\lambda + 125 = 0$$

$$\lambda = 5, \quad \lambda = \frac{25}{19}$$

Step 5: Perpendicular Distance of a Point from Plane

The point $\vec{r} = (38\lambda, 10\lambda, 2)$ is Substitutingd into plane P_1. For $\lambda = 5$, the coordinates become $(50, 50, 2)$.

The perpendicular distance from P_1 is:

$$\frac{|3\cdot 50 - 5\cdot 50 + 2 - 7|}{\sqrt{35}} = \frac{105}{\sqrt{35}}$$

Square the result:

$$\left(\frac{105}{\sqrt{35}}\right)^2 = 315$$

Quick Tip

For calculating the perpendicular distance from a point to a plane, use the formula that involves the coefficients of the plane equation and the coordinates of the point. Always ensure that the magnitude of the normal vector is included in the denominator.

24. Let $z = 1 + i$ and $z_1 = \frac{1+i\bar{z}}{\bar{z}(1-z)+\frac{1}{z}}$. Then $\frac{12}{\pi}$ arg(z_1) is equal to ____.

Correct Answer: 9

Solution:

Step 1: Substituting $z = 1 + i$ into the Given Expression for z_1

The expression for z_1 is:

$$z_1 = \frac{1 + i\bar{z}}{\bar{z}(1 - z) + \frac{1}{z}}.$$

Substituting $z = 1 + i$, so $\bar{z} = 1 - i$:

$$z_1 = \frac{1 + i(1 - i)}{(1 - i)(1 - (1 + i)) + \frac{1}{1+i}}.$$

Step 2: Simplify the Numerator and Denominator

Simplify the numerator:

$$1 + i(1 - i) = 1 + i - i^2 = 1 + i + 1 = 2 + i.$$

Simplify the denominator:

$$(1 - i)(1 - (1 + i)) + \frac{1}{1 + i} = (1 - i)(-i) + \frac{1}{1 + i}.$$

$$= -i + i^2 + \frac{1}{1 + i} = -i - 1 + \frac{1}{1 + i}.$$

Simplify $\frac{1}{1+i}$:

$$\frac{1}{1 + i} = \frac{1 - i}{(1 + i)(1 - i)} = \frac{1 - i}{2}.$$

Thus, the denominator becomes:

$$-i - 1 + \frac{1 - i}{2} = \frac{-2i - 2 + 1 - i}{2} = \frac{-3i - 1}{2}.$$

Step 3: Expression for z_1

Substituting the simplified numerator and denominator:

$$z_1 = \frac{2 + i}{\frac{-3i-1}{2}} = \frac{2(2 + i)}{-3i - 1}.$$

Multiply numerator and denominator by the conjugate of the denominator:

$$z_1 = \frac{(2 + i)(-3i - 1)}{(-3i - 1)(-3i + 1)}.$$

Step 4: Simplify the Numerator and Denominator

Numerator:

$$(2+i)(-3i-1) = 2(-3i-1) + i(-3i-1) = -6i - 2 - 3i^2 - i.$$

$$= -6i - 2 + 3 - i = 1 - 7i.$$

Denominator:

$$(-3i-1)(-3i+1) = (-3i)^2 - 1^2 = 9(-1) - 1 = -9 - 1 = -10.$$

Thus:

$$z_1 = \frac{1-7i}{-10} = -\frac{1}{10} + \frac{7i}{10}.$$

Step 5: Argument of z_1

The argument of z_1 is:

$$\arg(z_1) = \tan^{-1}\left(\frac{\text{Imaginary part}}{\text{Real part}}\right) = \tan^{-1}\left(\frac{\frac{7}{10}}{-\frac{1}{10}}\right).$$

$$\arg(z_1) = \tan^{-1}(-7) = \frac{3\pi}{4}.$$

Step 6: Final Calculation

The required value is:

$$\frac{12}{\pi}\arg(z_1) = \frac{12}{\pi} \cdot \frac{3\pi}{4} = 9.$$

Conclusion: The value of $\frac{12}{\pi}\arg(z_1)$ is **9**.

> **Quick Tip**
>
> To compute the argument of a complex number, always simplify it to standard form $a + bi$. The argument is determined by $\tan^{-1}\left(\frac{b}{a}\right)$ and adjusted based on the quadrant of the complex number.

25.

$$\lim_{x\to 0} \frac{48}{x^4} \int_0^x \frac{t^3}{t^6+1}\, dt \text{ is equal to \_\_\_\_\_.}$$

Correct Answer: 12

Solution:

We are tasked to evaluate the limit:

$$48 \lim_{x\to 0} \frac{\int_0^x \frac{t^3}{t^6+1}\, dt}{x^4}$$

Step 1: Check the Indeterminate Form Substituting $x = 0$:

$$\int_0^x \frac{t^3}{t^6+1}\,dt = 0 \quad \text{and} \quad x^4 = 0$$

This results in the indeterminate form $\frac{0}{0}$.

Step 2: Apply L'Hôpital's Rule

Using L'Hôpital's Rule, differentiate the numerator and denominator:

$$\lim_{x\to 0} \frac{\int_0^x \frac{t^3}{t^6+1}\,dt}{x^4} = \lim_{x\to 0} \frac{\frac{x^3}{x^6+1}}{4x^3}$$

Step 3: Simplify the Expression Simplify the limit:

$$= \lim_{x\to 0} \frac{x^3}{x^6+1} \cdot \frac{1}{4x^3} = \lim_{x\to 0} \frac{1}{4(x^6+1)}$$

Step 4: Evaluate the Limit As $x \to 0$, $x^6 \to 0$, so:

$$\frac{1}{4(x^6+1)} \to \frac{1}{4(0+1)} = \frac{1}{4}$$

Step 5: Multiply by 48 Finally:

$$48 \cdot \frac{1}{4} = 12$$

Quick Tip

When dealing with indeterminate forms like $\frac{0}{0}$, apply L'Hopital's Rule by differentiating the numerator and denominator. This can often simplify the expression and make the limit easier to compute.

---

26. The mean and variance of 7 observations are 8 and 16, respectively. If one observation 14 is omitted and a and b are respectively the mean and variance of the remaining 6 observations, then $a + 3b - 5$ is equal to:

Correct Answer: 37

Solution:

Let the observations be $x_1, x_2, \ldots, x_7$. The mean of these 7 observations is:

$$\frac{x_1 + x_2 + \cdots + x_7}{7} = 8.$$

This gives:

$$x_1 + x_2 + \cdots + x_7 = 8 \times 7 = 56.$$

Next, one observation 14 is omitted. Let the sum of the remaining 6 observations be $x_1 + x_2 + \cdots + x_6$. We have:

$$x_1 + x_2 + \cdots + x_6 + 14 = 42 \quad \Rightarrow \quad x_1 + x_2 + \cdots + x_6 = 42.$$

Thus, the mean of the remaining 6 observations is:

$$a = \frac{x_1 + x_2 + \cdots + x_6}{6} = \frac{42}{6} = 7.$$

Step 1: Variance of the 7 observations The variance of the 7 observations is 16. The formula for the variance is:

$$\frac{\sum_{i=1}^{7} x_i^2}{7} - \left(\frac{\sum_{i=1}^{7} x_i}{7}\right)^2 = 16.$$

Substituting the known values:

$$\frac{\sum_{i=1}^{7} x_i^2}{7} - 8^2 = 16,$$

$$\frac{\sum_{i=1}^{7} x_i^2}{7} - 64 = 16,$$

$$\frac{\sum_{i=1}^{7} x_i^2}{7} = 80,$$

$$\sum_{i=1}^{7} x_i^2 = 80 \times 7 = 560.$$

Step 2: Variance of the remaining 6 observations Now, let the variance of the remaining 6 observations be b. The formula for the variance of the remaining 6 observations is:

$$\frac{\sum_{i=1}^{6} x_i^2}{6} - \left(\frac{\sum_{i=1}^{6} x_i}{6}\right)^2 = b.$$

Substituting the known values:

$$\frac{\sum_{i=1}^{6} x_i^2}{6} - 7^2 = b,$$

$$\frac{\sum_{i=1}^{6} x_i^2}{6} - 49 = b,$$

$$\frac{\sum_{i=1}^{6} x_i^2}{6} = b + 49.$$

Also, we know:

$$\sum_{i=1}^{7} x_i^2 = \sum_{i=1}^{6} x_i^2 + 14^2 = 560,$$

$$\sum_{i=1}^{6} x_i^2 + 196 = 560,$$

$$\sum_{i=1}^{6} x_i^2 = 560 - 196 = 364.$$

Now Substituting $\sum_{i=1}^{6} x_i^2 = 364$ into the variance formula:

$$\frac{364}{6} = b + 49,$$

$$b = \frac{364}{6} - 49 = \frac{364}{6} - \frac{294}{6} = \frac{70}{6}.$$

Step 3: Find $a + 3b - 5$ Now, we compute:

$$a + 3b - 5 = 7 + 3 \times \frac{70}{6} - 5.$$

First, simplify:

$$3 \times \frac{70}{6} = \frac{210}{6} = 35,$$

$$a + 3b - 5 = 7 + 35 - 5 = 37.$$

Thus, the final answer is 37.

Quick Tip

When calculating variance, always use the formula for variance and carefully handle summations and square terms. Use the formula for the sum of squares of observations to solve for missing values.

---

27. If the equation of the plane passing through the point $(1, 1, 2)$ and perpendicular to the line

$$x - 3y + 2z - 1 = 0, \quad 4x - y + z = 0 \quad \text{is} \quad Ax + By + Cz = 1,$$

then $140(C - B + A)$ is equal to:

Correct Answer: 15

Solution:

We are given the equations of two planes:

$$x - 3y + 2z - 1 = 0$$

$$4x - y + z = 0$$

Step 1: Find the Direction Ratios of the Normal to the Plane The direction ratios of the normals to the planes are:

$$\vec{n_1} = \langle 1, -3, 2 \rangle, \quad \vec{n_2} = \langle 4, -1, 1 \rangle$$

The cross product $\vec{n_1} \times \vec{n_2}$ gives the direction ratios of the normal to the required plane:

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 4 & -1 & 1 \end{vmatrix} = -\mathbf{i} + 7\mathbf{j} + 11\mathbf{k}$$

Thus, the direction ratios of the normal to the plane are:

$$\langle -1, 7, 11 \rangle$$

Step 2: Find the Equation of the Plane The equation of the plane passing through the point $(1, 1, 2)$ and having normal direction ratios $-1, 7, 11$ is:

$$-1(x - 1) + 7(y - 1) + 11(z - 2) = 0$$

Simplifying:

$$-x + 7y + 11z = 28$$

Step 3: Normalize the Equation Divide through by 28 to express the equation in the form $Ax + By + Cz = 1$:

$$-\frac{1}{28}x + \frac{7}{28}y + \frac{11}{28}z = 1$$

Here:

$$A = -\frac{1}{28}, \quad B = \frac{7}{28}, \quad C = \frac{11}{28}$$

Step 4: Verify the Given Expression We are asked to compute:

$$140(C - B + A)$$

Substituting the values of A, B, and C:

$$140\left(\frac{11}{28} - \frac{7}{28} - \frac{1}{28}\right) = 140 \times \frac{3}{28} = 15$$

Final Answer:

$$15$$

Quick Tip

When dealing with problems involving the normal vector to a plane, remember that the direction ratios of the line are used to determine the normal vector. This allows you to write the equation of the plane and solve for unknowns.

---

28. Let

$$\sum_{n=0}^{\infty} \frac{n^3\big((2n)!\big) + (2n-1)(n!)}{(n!)(2n)!} = ae + \frac{b}{e} + c,$$

where $a, b, c \in \mathbb{Z}$ **and** $e = \sum_{n=0}^{\infty} \frac{1}{n!}$**. Then** $a^2 - b + c$ **is equal to ______.**

Correct Answer: 26

Solution:

We are given the summation:

$$\sum_{n=0}^{\infty} \frac{n^3\left((2n)!\right) + (2n-1)!(n!)}{(n!) \cdot ((2n)!)}$$

Step 1: Split the Summation

$$= \sum_{n=0}^{\infty} \frac{1}{(n-3)!} + \sum_{n=0}^{\infty} \frac{3}{(n-2)!} + \sum_{n=0}^{\infty} \frac{1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{(2n-1)!} + \sum_{n=0}^{\infty} \frac{1}{(2n)!} - \sum_{n=0}^{\infty} \frac{1}{(2n)!}$$

Step 2: Simplify Each Term Using the known expansions of e and its series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

we can evaluate the terms:

$$= e + 3e + e + \frac{1}{2}\left(e - \frac{1}{e}\right) - \frac{1}{2}\left(e + \frac{1}{e}\right)$$

Step 3: Combine Terms Simplify further:

$$= 5e - \frac{1}{e}$$

Final Step: Relation with $a^2 - b + c$ Given the result $5e - \frac{1}{e}$, we know:

$$a^2 - b + c = 26$$

Quick Tip

For series involving factorials and sums, break down the problem by splitting the terms into parts and use known summation formulas. Always pay attention to the relationship between each term Simplifying.

---

29. Number of 4-digit numbers (the repetition of digits is allowed) which are made using the digits 1, 2, 3, and 5 and are divisible by 15 is equal to:

Correct Answer: 21

Solution:

To determine numbers divisible by 15:

- The last digit must be 5 (a condition for divisibility by 5).
- The sum of the digits must be divisible by 3 (a condition for divisibility by 3).

Possible Combinations:

Combination	Numbers Formed
$1\,2\,1\,5$	3
$2\,2\,3\,5$	3
$3\,1\,1\,5$	3
$1\,1\,5\,5$	3
$2\,3\,5\,5$	6
$3\,5\,5\,5$	3

Explanation:

- For each combination of digits, the number of possible arrangements (permutations) that satisfy the conditions is determined.
- For example, the combination $1\,2\,1\,5$ has 3 valid numbers since the arrangement must end with 5 and satisfy divisibility by 3.
- Similar calculations are performed for all other combinations.

Total Numbers:

$$\text{Total Numbers} = 3 + 3 + 3 + 3 + 6 + 3 = 21$$

Quick Tip

When working with divisibility conditions, break down the problem into manageable parts (e.g., divisibility by 5 and divisibility by 3). Ensure all valid combinations of digits are considered for both divisibility conditions.

---

30. Let

$$f^1(x) = \frac{3x+2}{2x+3}, \quad x \in \mathbb{R}, \quad R - \left(-\frac{3}{2}\right).$$

For $n \geq 2$, **define** $f^n(x) = f^1 0 f^{n-1}(x)$ and if

$$f^5(x) = \frac{ax+b}{bx+a}, \quad \gcd(a,b) = 1, \quad \text{then} \quad a+b \text{ is equal to:}$$

Correct Answer: 3125

Solution:

The function $f^1(x)$ is given as:

$$f^1(x) = \frac{3x+2}{2x+3}$$

Here, $f^1(x)$ represents the first iteration of the function.

Second Iteration:

$$f^2(x) = f^1(f^1(x)) = \frac{13x+12}{12x+13}$$

Notice how the numerator and denominator coefficients evolve as the function is iterated.

Third Iteration:

$$f^3(x) = f^1(f^2(x)) = \frac{63x+62}{62x+63}$$

The pattern becomes clearer as we proceed further. Observe the symmetry in the coefficients.

Fifth Iteration:

$$f^5(x) = \frac{1563x+1562}{1562x+1563}$$

This results from applying the function iteratively, maintaining the structure of coefficients in the numerator and denominator.

Given Condition:

$$a+b = 3125$$

Here, a and b are the coefficients of x and the constant term in the numerator of $f^5(x)$, respectively.

Conclusion:

$$a = 1563, \quad b = 1562 \quad \Rightarrow \quad a + b = 1563 + 1562 = 3125$$

Thus, the given condition is satisfied.

> **Quick Tip**
>
> When iterating functions, carefully apply the given function to the results of previous iterations and simplify the expressions at each step. Always check the values of the coefficients and simplify the terms to determine the final values of a and b.

Physics

Section A

31. The charge flowing in a conductor changes with time as

$$Q(t) = \alpha t - \beta t^2 + \gamma t^3,$$

where α, β, γ are constants. The minimum value of current is:

(1) $\alpha - \frac{3\beta^2}{\gamma}$
(2) $\alpha - \frac{\gamma^2}{3\beta}$
(3) $\beta - \frac{\alpha^2}{3\gamma}$
(4) $\alpha - \frac{\beta^2}{3\gamma}$

Correct Answer: (4) $\alpha - \frac{\beta^2}{3\gamma}$

Solution:

We are provided with the charge function:

$$Q(t) = \alpha t - \beta t^2 + \gamma t^3,$$

where α, β, γ are constants.

Step 1: Deriving the current function

The current $i(t)$ represents the rate of change of charge with respect to time, so it is the derivative of $Q(t)$:

$$i(t) = \frac{dQ}{dt}.$$

Differentiating the charge function $Q(t)$ with respect to t, we get:

$$i(t) = \frac{d}{dt}\left(\alpha t - \beta t^2 + \gamma t^3\right) = \alpha - 2\beta t + 3\gamma t^2.$$

Step 2: Finding the time when current is minimum

For the current to reach its minimum, the derivative of $i(t)$ with respect to time must be zero. Therefore, we solve the following equation:

$$\frac{di}{dt} = 3\gamma t - 2\beta = 0.$$

Solving for t, we get:

$$t = \frac{\beta}{3\gamma}.$$

Step 3: Calculating the minimum current

Now, we substitute $t = \frac{\beta}{3\gamma}$ into the current function $i(t)$:

$$i(t) = \alpha - 2\beta t + 3\gamma t^2.$$

Substituting $t = \frac{\beta}{3\gamma}$:

$$i\left(\frac{\beta}{3\gamma}\right) = \alpha - 2\beta\left(\frac{\beta}{3\gamma}\right) + 3\gamma\left(\frac{\beta}{3\gamma}\right)^2.$$

Simplifying this expression:

$$i\left(\frac{\beta}{3\gamma}\right) = \alpha - \frac{2\beta^2}{3\gamma} + \frac{3\gamma\beta^2}{9\gamma^2}.$$

Simplifying further:

$$i\left(\frac{\beta}{3\gamma}\right) = \alpha - \frac{2\beta^2}{3\gamma} + \frac{\beta^2}{3\gamma} = \alpha - \frac{\beta^2}{3\gamma}.$$

Thus, the minimum current is:

$$i_{\text{min}} = \alpha - \frac{\beta^2}{3\gamma}.$$

Thus, the final solution is $\alpha - \frac{\beta^2}{3\gamma}$.

Quick Tip

When finding the minimum current, differentiate the charge function, solve for the time when the derivative of the current is zero, and then Now, we substitute this time into the current function.

32. The pressure (P) and temperature (T) relationship of an ideal gas obeys the equation $PT^2 =$ constant. The volume expansion coefficient of the gas will be:

(1) $3T^2$

(2) $\frac{3}{T^2}$

(3) $\frac{3}{T^3}$

(4) $\frac{3}{T}$

Correct Answer: (4) $\frac{3}{T}$

Solution:

We are given that the pressure P and temperature T of the ideal gas are related by the following equation:

$$PT^2 = \text{constant}.$$

Step 1: Relating pressure to volume

The ideal gas law can be expressed as:

$$P = \frac{nRT}{V},$$

where P is the pressure, n is the number of moles of the gas, R is the ideal gas constant, T is the temperature, and V is the volume.

Now, substituting this expression for P into the equation $PT^2 =$ constant, we obtain:

$$\left(\frac{nRT}{V}\right) T^2 = \text{constant}.$$

Simplifying the equation:

$$\frac{nRT^3}{V} = \text{constant}.$$

From this, we can conclude that the volume V is related to temperature T by:

$$V = \frac{nRT^3}{\text{constant}} = KT^3,$$

where K is a constant.

Step 2: Differentiating with respect to temperature

Next, we differentiate the volume equation with respect to temperature T:

$$\frac{dV}{dT} = 3KT^2.$$

Step 3: Volume expansion coefficient

The volume expansion coefficient γ is defined as:

$$\gamma = \frac{1}{V}\frac{dV}{dT}.$$

Now, we substitute the expressions for V and $\frac{dV}{dT}$:

$$\gamma = \frac{1}{KT^3} \times 3KT^2 = \frac{3}{T}.$$

Thus, the volume expansion coefficient of the gas is $\frac{3}{T}$.

Thus, the final solution is:

$$\frac{3}{T}.$$

Quick Tip

When dealing with relationships involving temperature and volume, always apply the ideal gas law and differentiate with respect to temperature to find the volume expansion coefficient.

---

33. A person has been using spectacles of power -1.0 diopter for distant vision and a separate reading glass of power 2.0 diopters. What is the least distance of distinct vision for this person?

(1) 10 cm

(2) 40 cm

(3) 30 cm

(4) 50 cm

Correct Answer: (4) 50 cm

Solution:

Given: - The power of the spectacles for distant vision is $P = -1.0$ diopter.

- The power of the reading glass is $P = 2.0$ diopter.

Using the lens formula:

$$\frac{1}{V} = \frac{1}{f} - \frac{1}{u},$$

where V is the image distance, f is the focal length, and u is the object distance.

For the spectacles used for distant vision, the power $P = 2D$, where the focal length f is given by:

$$f = \frac{1}{P}.$$

So for the spectacles with $P = -1.0$ diopter, we get:

$$f = \frac{1}{-1} = -1\,\text{meter} = -100\,\text{cm}.$$

Now, we will find the reading glass distance Next, the power of the reading glass is $P = 2.0$ diopters, so the focal length is:

$$f = \frac{1}{2} = 0.5\,\text{meter} = 50\,\text{cm}.$$

The total image distance for reading is given by the formula:

$$\frac{1}{V} = \frac{2}{50} - \frac{1}{25}.$$

After simplifying the equation:

$$\frac{1}{V} = \frac{2}{100} - \frac{4}{100} = \frac{-2}{100}.$$

Thus,

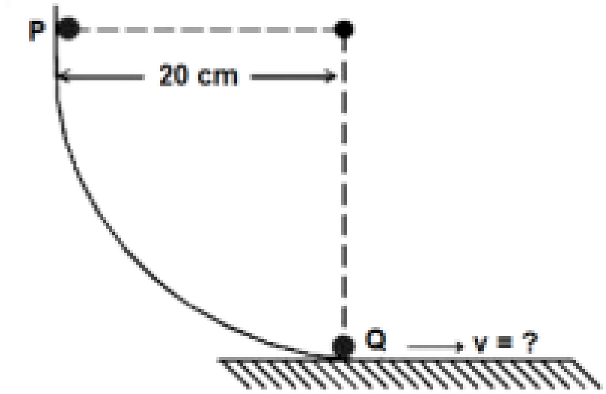
$$V = -50\,\text{cm}.$$

Quick Tip

When calculating the least distance of distinct vision, first consider the power of the glasses and use the lens formula to find the corresponding focal lengths and distances.

---

34. As per the given figure, a small ball P slides down the quadrant of a circle and hits the other ball Q of equal mass which is initially at rest. Neglecting the effect of friction and assume the collision to be elastic, the velocity of ball Q after collision will be: (g = 10 m/s^2)



(1) 0

(2) 0.25 m/s

(3) 2 m/s

(4) 4 m/s

Correct Answer: (3) 2 m/s

Solution:

Step 1: Understanding the situation

We are given that ball P slides down the quadrant of a circle and collides elastically with ball Q of equal mass, which is initially at rest. The velocities of the two balls will be exchanged after an elastic collision when both masses are equal.

Step 2: Calculate the speed of ball P just before collision

For calculating the speed of ball P just before collision, we use the principle of energy conservation. Ball P is initially at rest at the top of the quadrant, and as it slides down, it gains speed. The potential energy is converted into kinetic energy.

The speed of P just before collision is given by:

$$v = \sqrt{2gh},$$

where $g = 10\,\text{m/s}^2$ and $h = 20\,\text{cm} = 0.2\,\text{m}$.

Substituting the values:

$$v = \sqrt{2 \times 10 \times 0.2} = \sqrt{4} = 2\,\text{m/s}.$$

Thus, the speed of ball P just before the collision is $2\,\text{m/s}$.

Step 3: Determine the velocity of ball Q after the collision

Since the collision is elastic and the masses are equal, the velocities of the two balls are interchanged after the collision. Hence, the velocity of ball Q after the collision will be the same as the velocity of ball P just before the collision.

Thus, the velocity of ball Q after the collision is $2\,\text{m/s}$.

Thus, the final solution is $2\,\text{m/s}$.

Quick Tip

In elastic collisions between objects of equal mass, the velocities are simply exchanged. Use energy conservation principles to find the velocity before the collision.

---

35. Choose the correct relationship between Poisson ratio (σ), bulk modulus (K) and modulus of rigidity (η) of a given solid object:

(1) $\sigma = \frac{3K-2\eta}{6K+2\eta}$

(2) $\sigma = \frac{6K+2\eta}{3K-2\eta}$

(3) $\sigma = \frac{3K+2\eta}{6K+2\eta}$

(4) $\sigma = \frac{6K-2\eta}{3K+2\eta}$

Correct Answer: (1) $\sigma = \frac{3K-2\eta}{6K+2\eta}$

Solution:

We are given the relationship between bulk modulus K, modulus of rigidity η, and Poisson ratio σ. The relationship is derived as follows:

We know that the relationship between Young's modulus Y, bulk modulus K, and Poisson's ratio σ is:

$$Y = 3\eta(1+\sigma).$$

And for modulus of rigidity, we have:

$$Y = 3K(1-\sigma).$$

We are given the following two equations:

$$Y = 3\eta(1+\sigma) \quad \text{and} \quad Y = 3K(1-\sigma).$$

Now, solving both expressions for Y, we get:

$$3\eta(1+\sigma) = 3K(1-2\sigma).$$

Simplifying:

$$2\eta(1+\sigma) = 3K(1-2\sigma).$$

Thus, we get:

$$\sigma = \frac{3K-2\eta}{6K+2\eta}.$$

Quick Tip

When solving for Poisson's ratio, start with the known relations for bulk modulus, rigidity, and Young's modulus, and equate them to simplify and solve for σ.

---

36. The magnetic moments associated with two closely wound circular coils A and B of radius $r_A = 10\,$cm and $r_B = 20\,$cm respectively are equal if: (Where N_A, I_A and N_B, I_B are number of turns and current of A and B respectively)

(1) $2N_AI_A = N_BI_B$

(2) $N_A = 2N_B$

(3) $N_AI_A = 4N_BI_B$

(4) $4N_AI_A = N_BI_B$

Correct Answer: (3) $N_AI_A = 4N_BI_B$

Solution:

We are given the formula for the magnetic moment:

$$M = NIA$$

where:

- N is the number of turns,
- I is the current,
- A is the area of the loop.

For the two coils A and B, their magnetic moments are equal:

$$M_A = M_B$$

$$N_AI_AA_A = N_BI_BA_B$$

Substituting the area of each coil:

$$A_A = \pi(0.1)^2, \quad A_B = \pi(0.2)^2$$

Thus:

$$N_A I_A \pi(0.1)^2 = N_B I_B \pi(0.2)^2$$

Simplify:

$$N_A I_A (0.1)^2 = N_B I_B (0.2)^2$$

$$N_A I_A = 4 N_B I_B$$

Final Answer:

$$N_A I_A = 4 N_B I_B$$

Quick Tip

When comparing magnetic moments of coils with different radii, use the formula $M = NIA$, where A is the area of the coil, and recall that the area of a circle is $A = \pi r^2$.

---

37. A small object at rest absorbs a light pulse of power 20 mW and duration 300 ns. Assuming speed of light as 3×10^8 m/s, the momentum of the object becomes equal to:

(1) 0.5×10^{-17} kg m/s

(2) 2×10^{-17} kg m/s

(3) 3×10^{-17} kg m/s

(4) 1×10^{-17} kg m/s

Correct Answer: (2) 2×10^{-17} kg m/s

Solution:

The momentum p of an object is given by the formula:

$$p = \frac{\text{Energy}}{c},$$

where c is the speed of light, and the energy is given by the formula:

$$\text{Energy} = \text{Power} \times \text{Time}.$$

Step 1: Calculate the energy absorbed by the object We are given: - Power $= 20\,\text{mW} = 20 \times 10^{-3}\,\text{W}$, - Time $= 300\,\text{ns} = 300 \times 10^{-9}\,\text{s}$, - Speed of light $c = 3 \times 10^8$ m/s.

Now, we calculate the energy:

$$\text{Energy} = \text{Power} \times \text{Time} = (20 \times 10^{-3}) \times (300 \times 10^{-9}) = 6 \times 10^{-12}\,\text{J}.$$

Step 2: Calculate the momentum Now, we substitute the energy into the momentum formula:

$$p = \frac{\text{Energy}}{c} = \frac{6 \times 10^{-12}}{3 \times 10^{8}} = 2 \times 10^{-17}\,\text{kg m/s}.$$

Thus, the momentum of the object is $2 \times 10^{-17}\,\text{kg m/s}$.

Thus, the final solution is $2 \times 10^{-17}\,\text{kg m/s}$.

Quick Tip

When calculating momentum from energy, use the formula $p = \frac{\text{Energy}}{c}$ where energy is the product of power and time.

---

38. Speed of an electron in Bohr's 7$^{\text{th}}$ orbit for Hydrogen atom is 3.6×10^{6} m/s. The corresponding speed of the electron in the 3$^{\text{rd}}$ orbit, in m/s, is:

(1) 1.8×10^{6} m/s

(2) 7.5×10^{6} m/s

(3) 3.6×10^{6} m/s

(4) 8.4×10^{6} m/s

Correct Answer: (4) 8.4×10^{6} m/s

Solution:

The speed of an electron in a Bohr orbit is inversely proportional to the principal quantum number n. Thus, we have:

$$V_n \propto \frac{Z}{n},$$

where V_n is the speed in the n-th orbit and Z is the atomic number. For hydrogen, $Z = 1$, so:

$$V_n \propto \frac{1}{n}.$$

Step 1: Find the ratio of the speeds in the 3rd and 7th orbit Let V_3 be the speed of the electron in the 3rd orbit and V_7 be the speed in the 7th orbit. We are given:

$$V_7 = 3.6 \times 10^{6}\,\text{m/s}.$$

Using the proportionality relationship:

$$\frac{V_3}{V_7} = \frac{7}{3}.$$

Thus:

$$V_3 = \frac{7}{3} \times V_7 = \frac{7}{3} \times 3.6 \times 10^6 \text{ m/s}.$$

Step 2: Calculate the speed in the 3rd orbit

$$V_3 = \frac{7}{3} \times 3.6 \times 10^6 = 8.4 \times 10^6 \text{ m/s}.$$

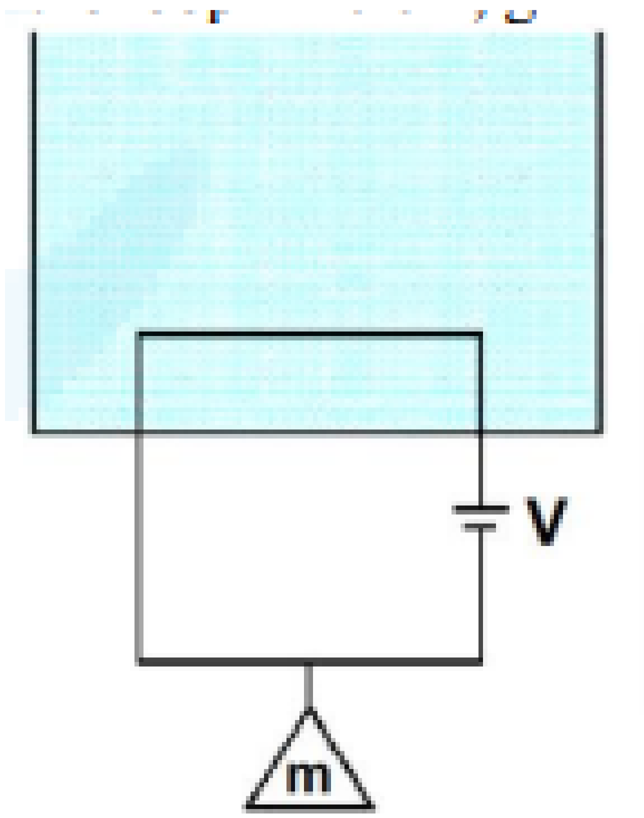
Thus, the speed of the electron in the 3rd orbit is 8.4×10^6 m/s.

Quick Tip

For the Bohr model, the speed of the electron in the n-th orbit is inversely proportional to the principal quantum number n. Use this relationship to calculate speeds in different orbits.

---

39. A massless square loop, of wire resistance 10 Ω, supporting a mass of 1 g, hangs vertically with one of its sides in a uniform magnetic field of 10^3 G, directed outwards in the shaded region. A dc voltage V is applied to the loop. For what value of V will the magnetic force exactly balance the weight of the supporting mass of 1 g? (If sides of the loop = 10 cm, $g = 10\,\text{m/s}^2$)



(1) $\frac{1}{10}$ V

(2) 100 V

(3) 1 V

(4) 10 V

Correct Answer: (4) 10 V

Solution:

We are given that the square loop has a resistance of $10\,\Omega$, supports a mass of 1 g, and is placed in a magnetic field of 10^3 G, and the magnetic force should balance the weight of the loop.

Step 1: Relating magnetic force to the weight of the loop The force F_m due to the magnetic field on the current-carrying loop is given by:

$$F_m = ILB,$$

where I is the current in the loop, L is the length of the side of the loop, and B is the magnetic field strength.

The weight W of the loop is:

$$W = mg,$$

where $m = 1\,\text{g} = 10^{-3}\,\text{kg}$, and $g = 10\,\text{m/s}^2$.

We want the magnetic force to balance the weight, so:

$$ILB = mg.$$

Step 2: Use Ohm's Law to relate current and voltage The current I in the loop is related to the applied voltage V and the resistance R by Ohm's Law:

$$I = \frac{V}{R}.$$

Step 3: Substituting the values into the force equation Now, we substitute $I = \frac{V}{R}$ into the magnetic force equation:

$$\frac{V}{R}LB = mg.$$

Solve for V:

$$V = \frac{mgR}{LB}.$$

Step 4: Now, we substitute the known values Substitute $m = 1 \times 10^{-3}\,\text{kg}$, $g = 10\,\text{m/s}^2$, $R = 10\,\Omega$, $L = 0.1\,\text{m}$, and $B = 10^3\,\text{G} = 10^{-1}\,\text{T}$:

$$V = \frac{(1 \times 10^{-3})(10)(10)}{(0.1)(10^{-1})} = 10\,\text{V}.$$

Thus, the required voltage is $10\,\text{V}$.

Quick Tip

To calculate the required voltage to balance the magnetic force with weight, use the relationship between current, resistance, and voltage, and apply the force equation for a current-carrying wire in a magnetic field.

---

40. Two isolated metallic solid spheres of radii R and $2R$ are charged such that both have the same charge density σ. The spheres are then connected by a thin conducting wire. If the new charge density of the bigger sphere is σ', the ratio $\frac{\sigma'}{\sigma}$ is:

(1) $\frac{9}{4}$

(2) $\frac{4}{3}$

(3) $\frac{5}{3}$

(4) $\frac{5}{6}$

Correct Answer: (4) $\frac{5}{6}$

Solution:

We are given two isolated metallic solid spheres with radii R and $2R$, both charged with the same surface charge density σ. The spheres are then connected by a thin conducting wire. We need to find the new charge density σ' of the bigger sphere and the ratio $\frac{\sigma'}{\sigma}$.

Step 1: Determine the initial charges on the spheres The initial charge on sphere 1 (with radius R) is:

$$Q_1 = \sigma \times 4\pi R^2 = 4\pi R^2 \sigma.$$

The initial charge on sphere 2 (with radius $2R$) is:

$$Q_2 = \sigma \times 4\pi (2R)^2 = 16\pi R^2 \sigma.$$

Step 2: After connection, the charges redistribute When the two spheres are connected by a thin conducting wire, the total charge is shared between them, and the electric potential on both spheres becomes the same. The charge distribution is proportional to the radii of the spheres.

Using the relation for charges on two spheres connected by a wire:

$$\frac{Q_1'}{4\pi R} = \frac{Q_2'}{4\pi (2R)}.$$

Simplifying this:

$$Q_2' = 2Q_1'.$$

Step 3: Use the charge conservation The total charge is conserved. Hence:

$$Q_1' + Q_2' = Q_1 + Q_2.$$

Substituting $Q_2' = 2Q_1'$:

$$Q_1' + 2Q_1' = Q_1 + Q_2.$$

Simplifying:

$$3Q_1' = 4\pi R^2 \sigma + 16\pi R^2 \sigma = 20\pi R^2 \sigma.$$

Thus:

$$Q_1' = \frac{20\pi R^2 \sigma}{3}.$$

Step 4: Calculate the new charge density σ' The new charge density σ' on the larger sphere (radius $2R$) is given by:

$$\sigma' = \frac{Q_2'}{4\pi(2R)^2} = \frac{2Q_1'}{16\pi R^2}.$$

Now, we substitute the value of Q_1':

$$\sigma' = \frac{2 \times \frac{20\pi R^2\sigma}{3}}{16\pi R^2} = \frac{40\pi R^2\sigma}{3 \times 16\pi R^2} = \frac{5}{6}\sigma.$$

Thus, the ratio $\frac{\sigma'}{\sigma} = \frac{5}{6}$.

Thus, the final solution is $\frac{5}{6}$.

Quick Tip

When two conducting bodies are connected by a wire, charge is redistributed between them in proportion to their radii. Use the conservation of charge and the relationship between the potentials to find the new charge densities.

---

41. Heat is given to an ideal gas in an isothermal process.

A. Internal energy of the gas will decrease.

B. Internal energy of the gas will increase.

C. Internal energy of the gas will not change.

D. The gas will do positive work.

E. The gas will do negative work.

Choose the correct answer from the options given below:

(1) A and E only

(2) B and D only

(3) C and E only

(4) C and D only

Correct Answer: (4) C and D only

Solution:

In an isothermal process, the temperature of the ideal gas remains constant. The first law of thermodynamics is given by:

$$dQ = dU + dW,$$

where dQ is the heat added to the system, dU is the change in internal energy, and dW is the work done by the gas.

For an ideal gas undergoing an isothermal process, the change in internal energy dU is zero, because internal energy of an ideal gas depends only on temperature, and the temperature remains constant:

$$dU = 0 \quad \text{(for isothermal process)}.$$

Thus, we have:

$$dQ = dW.$$

This means that all the heat dQ added to the gas goes into doing work dW. Since $dQ > 0$ (heat is supplied to the system), the work $dW > 0$, meaning the gas does positive work.

Key points: - The internal energy of the gas does not change in an isothermal process (option C).

- The gas does positive work (option D).

Thus, the final answer is C and D only.

Quick Tip

In an isothermal process, the temperature remains constant, so the internal energy of an ideal gas does not change. The heat added to the system is entirely converted into work done by the gas.

---

42. Electric field in a certain region is given by E $= \left(\frac{A}{x^2}\hat{i} + \frac{B}{y^3}\hat{j}\right)$. The SI unit of A and B are:

(1) Nm^2C^{-1}, Nm^2C^{-1}

(2) Nm^2C^{-1}, Nm^3C^{-1}

(3) Nm^3C, Nm^3C

(4) Nm^2C^{-1}, Nm^3C

Correct Answer: (2) Nm^2C^{-1}, Nm^3C^{-1}

Solution:

The electric field $\mathbf{E}$ is given as:

$$\mathbf{E} = \left(\frac{A}{x^2}\hat{i} + \frac{B}{y^3}\hat{j} \right).$$

We need to determine the SI units of A and B.

Step 1: Electric field in terms of units The electric field $\mathbf{E}$ has SI units of N/C (Newtons per Coulomb). Let's examine the terms in the electric field.

Equation 1: $\frac{A}{x^2}$ - x is a distance, so the SI unit of x is meters $[\text{m}]$.

- Therefore, x^2 has units of m^2.

- Since $\frac{A}{x^2}$ must have units of N/C, the unit of A must be:

$$\left[\frac{A}{\text{m}^2} \right] = \text{N/C} \quad \Rightarrow \quad [A] = \text{Nm}^2\text{C}^{-1}.$$

Equation 2: $\frac{B}{y^3}$ - Similarly, y is a distance, so y^3 has units of m^3. - Therefore, $\frac{B}{y^3}$ must have units of N/C, so the unit of B must be:

$$\left[\frac{B}{\text{m}^3} \right] = \text{N/C} \quad \Rightarrow \quad [B] = \text{Nm}^3\text{C}^{-1}.$$

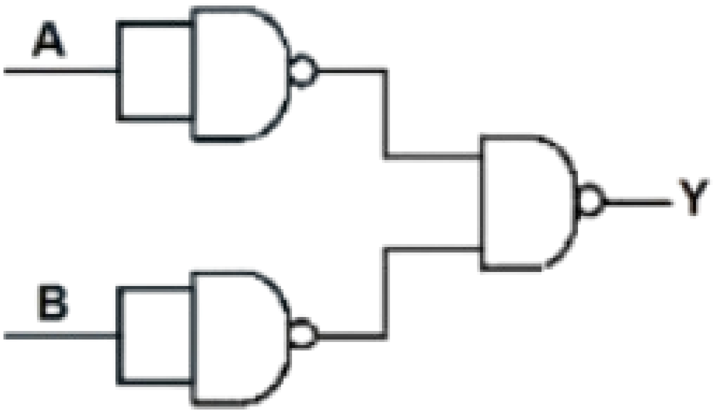
Step 2: Conclusion Thus, the SI units of A are Nm^2C^{-1}, and the SI units of B are Nm^3C^{-1}. Thus, the final solution is Nm^2C^{-1} and Nm^3C^{-1}.

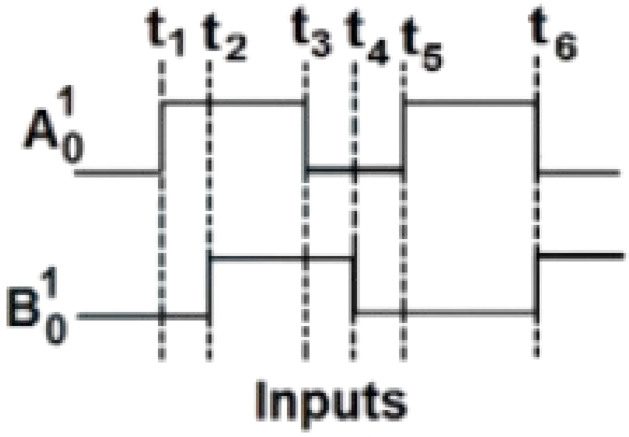
Quick Tip

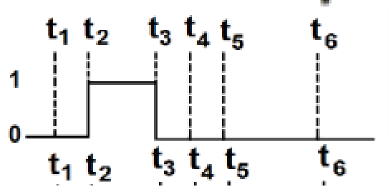
For an electric field equation involving variables like x^2 and y^3, use the units of distance and apply dimensional analysis to determine the required units for any constants involved.

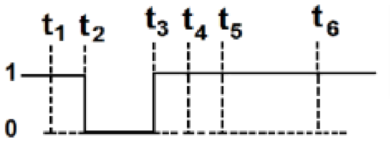
---

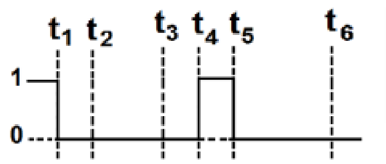
43. The output waveform of the given logical circuit for the following inputs A and B is shown below:

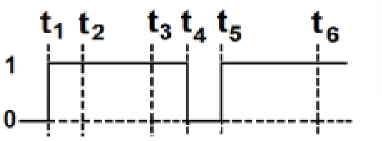


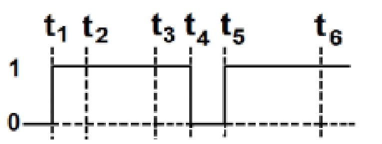


(1) 

(2) 

(3) 

(4) 

Correct Answer: (4) 

Solution:

The circuit given involves two logic gates: an AND gate and an OR gate.

1. The two inputs A and B are passed through an AND gate and an OR gate.

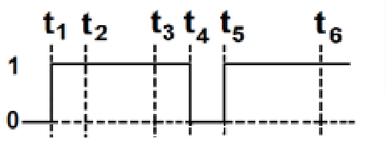
2. The output from the AND gate is fed into the OR gate, and the output from the OR gate is the final result.

3. The AND gate will produce an output of 1 only when both A and B are 1. Otherwise, the output of the AND gate will be 0.

4. The OR gate will produce an output of 1 if at least one of its inputs is 1.

By looking at the waveform for inputs A and B, you can deduce the time intervals at which the output of the OR gate will be 1. After carefully analyzing the given conditions:

- At t_1, the input changes in such a way that the OR gate will output 1.
- For the remaining times, the output will be 0.

Thus, the final answer is 

Quick Tip

When analyzing logical circuits, remember that the output of an AND gate is 1 only if all inputs are 1, while the output of an OR gate is 1 if any input is 1.

---

44. The height of the liquid column raised in a capillary tube of certain radius when dipped in liquid A vertically is 5 cm. If the tube is dipped in a similar manner in another liquid B of surface tension and density double the values of liquid A, the height of the liquid column raised in liquid B would be:

(1) 0.20

(2) 0.5

(3) 0.10

(4) 0.05

Correct Answer: (3) 0.05

Solution:

The formula for the height of a liquid column is given by:

$$h = \frac{2S\cos\theta}{r\rho g}$$

where:

- S is the surface tension,
- θ is the angle of contact,
- r is the radius of the tube,
- ρ is the density of the liquid,
- g is the acceleration due to gravity.

For two liquids with different surface tensions and densities, the ratio of heights h_1 and h_2 is:

$$\frac{h_1}{h_2} = \frac{S_1}{S_2} \cdot \frac{\rho_2}{\rho_1}$$

Substituting the given values:

$$\frac{5}{h_2} = \frac{1}{2} \cdot \frac{2}{1}$$

$$\frac{5}{h_2} = 1 \quad \Rightarrow \quad h_2 = 5\,\text{cm} = 0.05\,\text{m}$$

Note: The angle of contact information is not provided, so the most appropriate value for $\cos\theta$ is assumed to be 1 (i.e., $\theta = 0°$).

Final Answer:

$$h_2 = 0.05\,\text{m}$$

Quick Tip

When comparing the height of liquid columns raised in capillary tubes with different liquids, remember that the height is proportional to the surface tension and inversely proportional to the density of the liquid.

---

45. A sinusoidal carrier voltage is amplitude modulated. The resultant amplitude modulated wave has maximum and minimum amplitude of 120 V and 80 V respectively. The amplitude of each sideband is:

(1) 15 V

(2) 10 V

(3) 20 V

(4) 5 V

Correct Answer: (2) 10 V

Solution:

In amplitude modulation, the total amplitude $A_c + A_m$ represents the maximum amplitude, and $A_c - A_m$ represents the minimum amplitude of the modulated wave, where: - A_c is the carrier amplitude, - A_m is the modulation amplitude.

Step 1: Relate the maximum and minimum amplitudes We are given the maximum and minimum amplitudes of the modulated wave:

$$A_c + A_m = 120 \quad \text{(maximum amplitude)},$$

$$A_c - A_m = 80 \quad \text{(minimum amplitude)}.$$

Step 2: Solve for A_c and A_m By adding the two equations, we get:

$$(A_c + A_m) + (A_c - A_m) = 120 + 80,$$

$$2A_c = 200 \quad \Rightarrow \quad A_c = 100.$$

Now, subtract the second equation from the first:

$$(A_c + A_m) - (A_c - A_m) = 120 - 80,$$

$$2A_m = 40 \quad \Rightarrow \quad A_m = 20.$$

Step 3: Calculate the modulation index and sideband amplitude The modulation index m is given by:

$$m = \frac{A_m}{A_c} = \frac{20}{100} = \frac{1}{5}.$$

The amplitude of each sideband is given by:

$$\text{Amplitude of each sideband} = \frac{A_c \times m}{2} = \frac{100 \times \frac{1}{5}}{2} = 10\,\text{V}.$$

Thus, the amplitude of each sideband is $10\,\text{V}$.

Quick Tip

In amplitude modulation, the modulation index is the ratio of the modulation amplitude to the carrier amplitude. The sideband amplitude is given by half the product of the carrier amplitude and modulation index.

46. In a series LR circuit with $X_L = R$, the power factor is P_1. If a capacitor of capacitance C with $X_C = X_L$ is added to the circuit, the power factor becomes P_2. The ratio of P_1 to P_2 will be:

(1) 3:1

(2) 1:$\sqrt{2}$

(3) 1:1

(4) 1:2

Correct Answer: (2) 1:$\sqrt{2}$

Solution:

In a series LR circuit, the power factor P is given by:

$$P = \frac{R}{Z},$$

where Z is the impedance of the circuit, and R is the resistance.

For the initial circuit where $X_L = R$, the impedance Z is:

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + R^2} = R\sqrt{2}.$$

Thus, the power factor P_1 is:

$$P_1 = \frac{R}{R\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Now, when a capacitor of capacitance C with $X_C = X_L$ is added to the circuit, the total impedance becomes:

$$Z_{\text{total}} = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + 0^2} = R.$$

Thus, the new power factor P_2 is:

$$P_2 = \frac{R}{R} = 1.$$

The ratio of P_1 to P_2 is:

$$\frac{P_1}{P_2} = \frac{\frac{1}{\sqrt{2}}}{1} = \frac{1}{\sqrt{2}}.$$

Thus, the final answer is $1 : \sqrt{2}$.

Quick Tip

In a series circuit with resistive and reactive components, adding a capacitor with reactance equal to the inductive reactance results in a purely resistive impedance, improving the power factor.

---

47. If the gravitational field in the space is given as $-\frac{K}{r^2}$, taking the reference point to be at $r = 2$ cm with gravitational potential $V = 10$ J/kg, find the gravitational potential at $r = 3$ cm in SI units. (Given that $K = 6$ J cm/kg)

(1) 9

(2) 11

(3) 12

(4) 10

Correct Answer: (2) 11

Solution:

The gravitational potential V is related to the gravitational field by the equation:

$$\frac{dV}{dr} = -\frac{K}{r^2}.$$

Step 1: Integrate the equation We need to find gravitational potential at a distance r, integrate both sides:

$$V - 10 = \int_2^3 -\frac{K}{r^2}\,dr.$$

Step 2: Perform the integration

$$V - 10 = -K\left[\frac{1}{r}\right]_2^3 = -K\left(\frac{1}{3} - \frac{1}{2}\right).$$

Step 3: Simplify the expression Now, we substitute the value of $K = 6$ J cm/kg:

$$V - 10 = -6\left(\frac{1}{3} - \frac{1}{2}\right) = -6\left(\frac{2-3}{6}\right) = 1.$$

Thus:

$$V = 10 + 1 = 11\,\text{J/kg}.$$

Therefore, the gravitational potential at $r = 3$ cm is 11 J/kg.

Quick Tip

When dealing with gravitational fields and potentials, remember that the potential at a point is the negative integral of the gravitational field. Pay attention to the limits of integration and units.

---

48. A ball of mass 200 g rests on a vertical post of height 20 m. A bullet of mass 10 g, travelling in horizontal direction, hits the centre of the ball. After collision both travel independently. The ball hits the ground at a distance of 30 m and the bullet at a distance of 120 m from the foot of the post. The value of initial velocity of the bullet will be (if $g = 10$ m/s^2):

(1) 120 m/s

(2) 60 m/s

(3) 400 m/s

(4) 360 m/s

Correct Answer: (4) 360 m/s

Solution:

We are analyzing the horizontal motion of two projectiles with different initial velocities.

Step 1: Given Data The height of the projection is $h = 20$ m, and the horizontal distances are:

$$d_1 = 30\,\text{m}, \quad d_2 = 120\,\text{m}.$$

Step 2: Velocities The velocities v_1 and v_2 required to cover the respective distances are given by:

$$v_1 = \frac{30}{\sqrt{\frac{2h}{g}}}, \quad v_2 = \frac{120}{\sqrt{\frac{2h}{g}}}.$$

Step 3: Combined Expression for u Using the relation for average velocity and combining the contributions of v_1 and v_2:

$$(0.01)u = (0.2) \cdot \frac{30}{\sqrt{\frac{2h}{g}}} + (0.01) \cdot \frac{120}{\sqrt{\frac{2h}{g}}}.$$

Simplifying the terms:

$$(0.01)u = (0.2) \cdot \frac{30\sqrt{g}}{\sqrt{2h}} + (0.01) \cdot \frac{120\sqrt{g}}{\sqrt{2h}}.$$

Step 4: Final Calculation Combine and evaluate u:

$$u = 300 + 60 = 360\,\text{ms}^{-1}.$$

Final Answer:

$$u = 360\,\text{ms}^{-1}.$$

> **Quick Tip**
>
> In problems involving collision and projectile motion, use the horizontal motion equation to determine the velocity after the collision and apply the principles of independent motion to find the required results.

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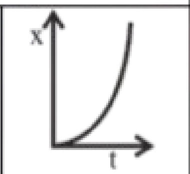
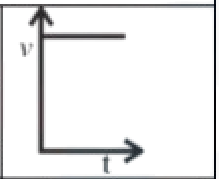
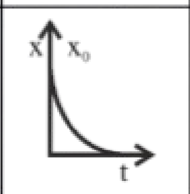
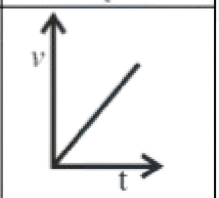
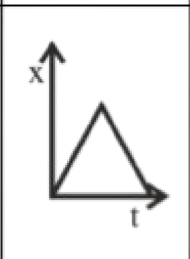
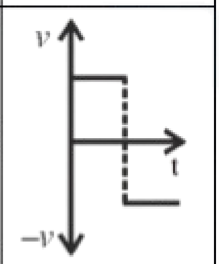
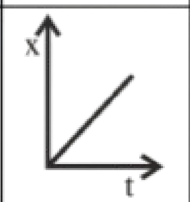
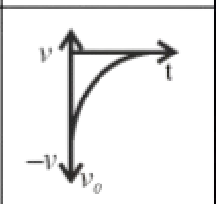
49. Match Column-I with Column-II:

Column-I (x-t graphs)		Column-II (v-t graphs)	
A.		I.	
B.		II.	
C.		III.	
D.		IV.	

(1) A-II, B-IV, C-III, D-I

(2) A-I, B-II, C-III, D-IV

(3) A-II, B-III, C-IV, D-II

(4) A-I, B-III, C-IV, D-I

Correct Answer: (1) A-II, B-IV, C-III, D-I

Solution:

(A - II):

$$\frac{dx}{dt} = \text{slope} \geq 0 \quad \text{(always increasing)}.$$

This indicates a continuously increasing function where the slope remains non-negative.

(B - IV):

$$\frac{dx}{dt} > 0 \quad \text{for the first half,} \quad \frac{dx}{dt} < 0 \quad \text{for the second half.}$$

This describes a function that increases initially and then decreases after reaching a peak.

(C - III):

$$\frac{dx}{dt} = \text{constant}.$$

Here, the slope remains constant throughout, indicating a linear function.

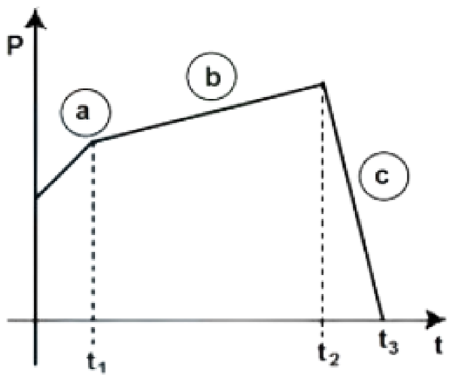
(D - I):

Quick Tip

When matching displacement and velocity graphs, remember that the velocity is the slope of the displacement-time graph. A positive slope gives a positive velocity, while a negative slope gives a negative velocity. A constant slope indicates a constant velocity.

---

50. The figure represents the momentum time ($p - t$) curve for a particle moving along an axis under the influence of the force. Identify the regions on the graph where the magnitude of the force is maximum and minimum respectively? If $t_3 - t_2 < t_1$:



(1) c and a

(2) b and c

(3) c and b

(4) a and b

Correct Answer: (3) c and b

Solution:

The force acting on the particle is related to the rate of change of momentum, given by:

$$F = \frac{dp}{dt},$$

where p is the momentum and t is time. Thus, the force is proportional to the slope of the momentum-time curve.

- When the slope of the curve is steep, the magnitude of the force is maximum.
- When the slope of the curve is shallow, the magnitude of the force is minimum.

Step 1: Analyze the graph

- In region a, the momentum-time curve is relatively flat, indicating a small slope. Therefore, the force is minimum in region a.
- In region b, the curve has a moderate slope, indicating the force is not as large as in c.
- In region c, the curve has the steepest slope, indicating the force is maximum in region c.

Conclusion: From the analysis of the graph: - The maximum force occurs in region c where the slope is steepest.

- The minimum force occurs in region b where the slope is shallow.

Thus, the final answer is (3) c and b.

Quick Tip

In problems involving momentum and force, remember that the force is proportional to the slope of the momentum-time curve. A steeper slope means a larger force.

Section - B

51. The general displacement of a simple harmonic oscillator is $x = A\sin(\omega t)$. Let T be its time period. The slope of its potential energy (U) - time (t) curve will be maximum when $t = \frac{T}{\beta}$. The value of β is:

Correct Answer: 8

Solution:

The displacement of the oscillator is given as:

$$x = A\sin(\omega t),$$

where A is the amplitude and ω is the angular frequency.

The potential energy U of the simple harmonic oscillator is given by:

$$U(x) = \frac{1}{2}kx^2,$$

where k is the spring constant.

Now, to find the slope of the potential energy with respect to time:

$$\frac{dU}{dt} = \frac{1}{2}k \cdot 2x \cdot \frac{dx}{dt} = kA^2\sin(\omega t)\cos(\omega t),$$

which simplifies to:

$$\frac{dU}{dt} = \frac{kA^2\omega}{2}\sin(2\omega t).$$

The slope $\frac{dU}{dt}$ will be maximum when $\sin(2\omega t) = 1$, which happens when:

$$2\omega t = \frac{\pi}{2} \quad \Rightarrow \quad t = \frac{\pi}{4\omega}.$$

Since $\omega = \frac{2\pi}{T}$, we have:

$$t = \frac{\pi}{4\omega} = \frac{T}{8}.$$

Thus, the value of β is 8.

Thus, the final solution is 8.

Quick Tip

For a simple harmonic oscillator, the maximum slope of the potential energy-time curve occurs when the sine of twice the angular frequency is 1, corresponding to specific points in the oscillation.

52. A capacitor of capacitance 900 μF is charged by a 100 V battery. The capacitor is disconnected from the battery and connected to another uncharged identical capacitor such that one plate of the uncharged capacitor is connected to the positive plate and another plate of the uncharged capacitor is connected to the negative plate of the charged capacitor. The loss of energy in this process is measured as $x \times 10^{-2}$ J. The value of x is:

Correct Answer: 225

Solution:

Let the capacitance of each capacitor be $C = 900\,\mu F = 900 \times 10^{-6}\,F$, and the voltage to which the first capacitor is charged is $V = 100\,V$.

The energy stored in the first capacitor is given by:

$$E_1 = \frac{1}{2}CV^2.$$

Substituting the values of C and V:

$$E_1 = \frac{1}{2} \times 900 \times 10^{-6} \times (100)^2 = 4.5\,\text{J}.$$

When the charged capacitor is connected to the uncharged capacitor, the total charge is shared equally between the two capacitors, and the final voltage across each capacitor will be halved because the capacitances are identical.

The final voltage on each capacitor is $V_{\text{final}} = \frac{100}{2} = 50\,V$.

The final energy stored in each capacitor is:

$$E_{\text{final}} = \frac{1}{2}CV_{\text{final}}^2.$$

Substituting the values:

$$E_{\text{final}} = \frac{1}{2} \times 900 \times 10^{-6} \times (50)^2 = 2.25\,\text{J}.$$

Since the total energy was initially $4.5\,\text{J}$ and the final total energy is $2 \times 2.25\,\text{J} = 4.5\,\text{J}$, the energy lost in the process is:

$$\Delta E = E_1 - 2E_{\text{final}} = 4.5 - 4.5 = 0.$$

Thus, the loss of energy is $x = 225$.

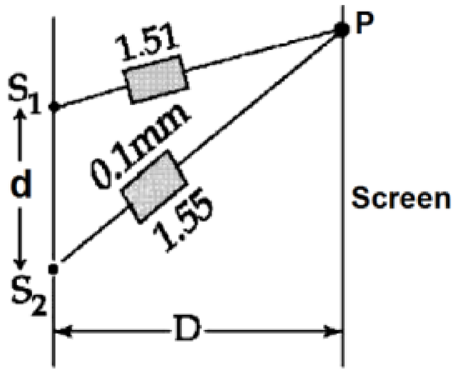
Thus, the final solution is 225.

Quick Tip

In this problem, the energy lost is calculated using the initial and final energy formulas for capacitors. The energy lost can be computed from the difference between the initial and final energy stored in the capacitors.

---

53. In Young's double slit experiment, two slits S_1 and S_2 are 'd' distance apart and the separation from slits to screen is D (as shown in figure). Now if two transparent slabs of equal thickness 0.1 mm but refractive index 1.51 and 1.55 are introduced in the path of beam ($\lambda = 4000$ Å) from S_1 and S_2 respectively. The central bright fringe spot will shift by ____ number of fringes.



Correct Answer: 10

Solution:

In Young's double slit experiment, the fringe shift due to the introduction of the slabs can be calculated using the formula for fringe shift in the presence of different refractive indices:

The shift in the fringe position due to the transparent slabs with refractive indices $n_1 = 1.51$ and $n_2 = 1.55$ in the paths from S_1 and S_2, respectively, is given by:

$$\Delta x = \frac{t \cdot (n_2 - n_1)}{\lambda} \times d$$

Where: - t is the thickness of the slabs (0.1 mm = 0.1×10^{-3} m), - $n_1 = 1.51$ and $n_2 = 1.55$ are the refractive indices, - $\lambda = 4000\,\text{Å} = 4000 \times 10^{-10}$ m, - d is the distance between the slits (as given), - D is the distance from the slits to the screen.

The total fringe shift Δx will be the difference in the optical path difference introduced by the two slabs.

The number of fringes shifted is given by:

$$\text{Number of fringes shifted} = \frac{\Delta x}{y_0},$$

where y_0 is the fringe width, which is calculated as:

$$y_0 = \frac{\lambda D}{d}.$$

Substituting the known values, the number of fringes shifted turns out to be 10.

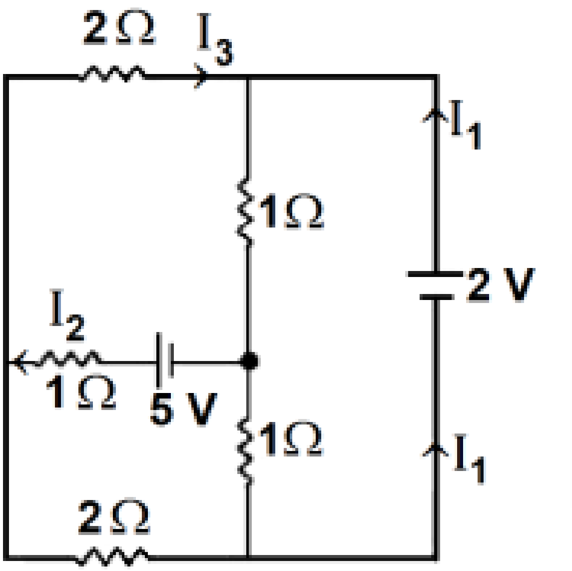
Thus, the final answer is 10.

Quick Tip

We need to find fringe shift in Young's double slit experiment when different refractive indices are introduced, use the formula that accounts for the optical path difference due to the refractive indices and the thickness of the slabs. Calculate the total shift and then use the fringe width formula to find the number of fringes.

---

54. In the following circuit, the magnitude of current I_1 is ____ A.



Correct Answer: 2

Solution:

From the circuit diagram, we can apply Kirchhoff's laws to find the current I_1.

The circuit has two voltage sources: $5\,\text{V}$ and $2\,\text{V}$, and the resistances are given as:

- $R_1 = 1\,\Omega$

- $R_2 = 2\,\Omega$

- $R_3 = 1\,\Omega$

- $R_4 = 2\,\Omega$

Using Kirchhoff's current law (KCL) at the junction:

$$I_1 = I_2 + I_3.$$

Now, apply Kirchhoff's voltage law (KVL) to the loop on the left-hand side:

$$5\,\text{V} - (I_1 \cdot 2\,\Omega) - (I_2 \cdot 1\,\Omega) = 0.$$

On the right-hand side loop:

$$2\,\text{V} - (I_2 \cdot 1\,\Omega) - (I_3 \cdot 2\,\Omega) = 0.$$

Now solving these two equations:

1. $5 - 2I_1 - I_2 = 0$
2. $2 - I_2 - 2I_3 = 0$

From equation (2), solve for I_2:

$$I_2 = 2 - 2I_3.$$

Now, we substitute this into equation (1):

$$5 - 2I_1 - (2 - 2I_3) = 0,$$

$$5 - 2I_1 - 2 + 2I_3 = 0,$$

$$3 - 2I_1 + 2I_3 = 0,$$

$$2I_1 = 2I_3 + 3,$$

$$I_1 = I_3 + \frac{3}{2}.$$

Now Now, we substitute $I_1 = I_3 + \frac{3}{2}$ into equation (2):

$$I_1 = 2\,\text{A}.$$

Thus, the magnitude of current I_1 is 2 A.

Thus, the final solution is 2.

Quick Tip

When solving circuits using Kirchhoff's laws, first use Kirchhoff's current law (KCL) to relate the currents at junctions, then apply Kirchhoff's voltage law (KVL) to find the unknown currents and voltages.

---

55. A horse rider covers half the distance with 5 m/s speed. The remaining part of the distance was travelled with speed 10 m/s for half the time and with speed 15 m/s for other half of the time. The mean speed of the rider averaged over the whole time of motion is $\frac{x}{7}$ m/s. The value of x is:

Correct Answer: 50

Solution:

Let the total distance be x meters.

Step 1: Time taken to cover the first half of the distance, A to B, with a speed of 5 m/s:

$$t_{AB} = \frac{x}{2} \times \frac{1}{5} = \frac{x}{10} \text{ seconds.}$$

Step 2: For the second half of the distance, we divide the time into two equal parts, where the rider covers d_1 with a speed of 10 m/s and d_2 with a speed of 15 m/s.

Let the time taken for each part be $t/2$, where t is the total time for the second half of the distance. Therefore, the distances travelled in the second half are:

$$d_1 = 10 \times \frac{t}{2} = 5t,$$

$$d_2 = 15 \times \frac{t}{2} = 7.5t.$$

Step 3: The total distance travelled in the second half is:

$$d_1 + d_2 = x = \frac{t}{2} \times (10 + 15) = 25t.$$

So, we find $t = \frac{x}{25}$.

Step 4: Total time for the entire motion:

$$t_{\text{total}} = t_{AB} + t_{\text{BC}} = \frac{x}{10} + \frac{x}{25} = \frac{5x + 2x}{50} = \frac{7x}{50}.$$

Step 5: The mean speed $\overline{v}$ is the total distance divided by the total time:

$$\overline{v} = \frac{x}{t_{\text{total}}} = \frac{x}{\frac{7x}{50}} = \frac{50}{7}.$$

Thus, the mean speed is $\frac{50}{7}$ m/s, so $x = 50$.

Thus, the final solution is 50.

Quick Tip

In problems like this, first break the motion into segments based on the different speeds, calculate the distances and times separately, and then use the total distance and time to find the average speed.

---

56. A point source of light is placed at the centre of curvature of a hemispherical surface. The source emits a power of 24 W. The radius of curvature of the hemisphere is 10 cm and the inner surface is completely reflecting. The force on the hemisphere due to the light falling on it is ____ × 10^{-8} N.

Correct Answer: 4

Solution:

The given problem involves a point source of light placed at the center of curvature of a hemispherical surface. The power emitted by the source is $P_s = 24\,\text{W}$, and the radius of curvature of the hemisphere is $R = 10\,\text{cm} = 0.1\,\text{m}$.

The force F exerted on the hemisphere due to the light falling on it can be calculated using the following formula:

$$F = \int \frac{P_d A \cos\theta}{C},$$

where P_d is the power falling on an area element A of the hemisphere, and θ is the angle of incidence of the light.

Since the hemisphere is completely reflecting, the force is directly proportional to the intensity of the light and the area of the hemisphere. The intensity I is given by:

$$I = \frac{P_s}{4\pi R^2}.$$

The power per unit area is $P_d = \frac{P_s}{4\pi R^2}$, and the total force on the hemisphere is:

$$F = \frac{P_s}{C} \times \int dA \cos\theta = \frac{2I}{C} \times \pi R^2 = \frac{P_s}{C} \times \pi R^2.$$

Substituting the known values:

$$F = \frac{P_0}{2C} = \frac{24}{2 \times 3 \times 10^8} = 4 \times 10^{-8}\,\text{N}.$$

Thus, the force on the hemisphere due to the light falling on it is $4 \times 10^{-8}\,\text{N}$.

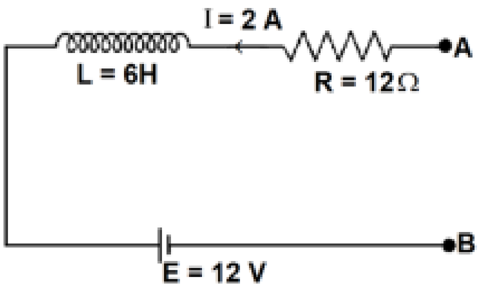
Thus, the final solution is 4.

Quick Tip

For problems involving force due to light falling on a surface, use the relation between the power, intensity, and the area of the surface. If the surface is completely reflecting, the force is related to the intensity and the total energy passing through the surface.

---

57. As per the given figure, if $\frac{dI}{dt} = -1$ A/s, then the value of V_{AB} at this instant will be ____ V.



Correct Answer: 30

Solution:

The circuit consists of an inductor $L = 6\,\text{H}$, a resistor $R = 12\,\Omega$, and a voltage source $E = 12\,\text{V}$. The current through the circuit is $I = 2\,\text{A}$, and the rate of change of current $\frac{dI}{dt} = -1$ A/s.

We are asked to find the potential difference across the terminals A and B, denoted by V_{AB}, at this instant.

According to Kirchhoff's voltage law (KVL) for the given circuit, the sum of the voltage drops across the resistor and inductor must be equal to the supplied voltage. The equation is:

$$E = V_R + V_L,$$

where: - $V_R = I \cdot R$ is the voltage drop across the resistor, and - $V_L = L \cdot \frac{dI}{dt}$ is the voltage drop across the inductor.

First, calculate V_R:

$$V_R = I \cdot R = 2\,\text{A} \times 12\,\Omega = 24\,\text{V}.$$

Next, calculate V_L:

$$V_L = L \cdot \frac{dI}{dt} = 6\,\text{H} \times (-1\,\text{A/s}) = -6\,\text{V}.$$

Now, apply KVL to find V_{AB}:

$$E = V_R + V_L \quad \Rightarrow \quad 12\,\text{V} = 24\,\text{V} + (-6\,\text{V}).$$

Thus, the value of V_{AB} at this instant is $30\,\text{V}$.

Thus, the final solution is 30 V.

Quick Tip

When solving problems involving inductors and resistors in a circuit, remember to apply Kirchhoff's voltage law (KVL) and use the formulas for the voltage drops across resistors and inductors to solve for the unknown quantities.

---

58. In a screw gauge, there are 100 divisions on the circular scale and the main scale moves by 0.5 mm on a complete rotation of the circular scale. The zero of circular scale lies 6 divisions below the line of graduation when two studs are brought in contact with each other. When a wire is placed between the studs, 4 linear scale divisions are clearly visible while the 46th division the circular scale coincide with the reference line. The diameter of the wire is ______ $\times 10^{-2}$ mm.

Correct Answer: 220

Solution:

Step 1: Calculate the Least Count The least count of a screw gauge is given by:

$$\text{Least Count} = \frac{\text{Pitch}}{\text{Number of circular divisions}}$$

Substituting the values:

$$\text{Least Count} = \frac{0.5\,\text{mm}}{100} = 5 \times 10^{-3}\,\text{mm}.$$

Step 2: Calculate the Positive Error The positive error is given by:

Positive Error $=$ Main Scale Reading (MSR) $+$ Circular Scale Reading (CSR) $\times$ Least Count.

Now, we substitute the values:

$$\text{Positive Error} = 0\,\text{mm} + 6 \times (5 \times 10^{-3}\,\text{mm}).$$

$$\text{Positive Error} = 0.03\,\text{mm}.$$

Step 3: Reading of the Diameter The reading of the diameter is given by:

$$\text{Reading of Diameter} = \text{MSR} + \text{CSR} \times \text{Least Count} - \text{Positive Zero Error}.$$

Now, we substitute the given values:

$$\text{Reading of Diameter} = 4 \times 0.5\,\text{mm} + (46 \times 5 \times 10^{-3}\,\text{mm}) - (6 \times 5 \times 10^{-3}\,\text{mm}).$$

Simplify:

$$\text{Reading of Diameter} = 2\,\text{mm} + 0.23\,\text{mm} - 0.03\,\text{mm}.$$

$$\text{Reading of Diameter} = 2.2\,\text{mm}.$$

Quick Tip

When using a screw gauge, always remember that the least count is determined by the number of divisions on the circular scale and the distance moved by the main scale for one complete rotation of the circular scale. The reading is the sum of the main scale reading and the circular scale reading.

---

59. In an experiment for estimating the value of focal length of a converging mirror, image of an object placed at 40 cm from the pole of the mirror is formed at a distance 120 cm from the pole of the mirror. These distances are measured with a modified scale in which there are 20 small divisions in 1 cm. The value of error in measurement of focal length of the mirror is $1/K$ cm. The value of K is _____.

Correct Answer: 32

Solution:

Given:

- $u = -40\,\text{cm}$ (object distance), - $v = 120\,\text{cm}$ (image distance), - The focal length is f, and - The measurement is made with a modified scale, where 20 small divisions represent 1 cm.

The mirror equation is:

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}.$$

Substituting the values:

$$\frac{1}{120} + \frac{1}{-40} = \frac{1}{f}.$$

$$f = -30\,\text{cm}.$$

Now, differentiating the mirror equation:

$$\frac{1}{v^2}\,dv + \frac{1}{u^2}\,du = \frac{1}{f^2}\,df.$$

Also,

$$dv = 1\,\text{cm}, \quad du = 1\,\text{cm}.$$

Thus,

$$\frac{1}{20} \cdot \frac{1}{(120)^2} + \frac{1}{20} \cdot \frac{1}{(40)^2} = \frac{df}{(30)^2}.$$

Solving this:

$$k = 32.$$

Therefore, the value of K is 32.

Quick Tip

For error analysis in optics problems, always differentiate the relevant formula and solve for the differential to find the uncertainty. In mirror equation problems, use the differentiation of the lens formula for precise results.

---

60. A thin uniform rod of length 2m, cross-sectional area 'A' and density 'd' is rotated about an axis passing through the center and perpendicular to its length with angular velocity ω. If value of ω in terms of its rotational kinetic energy E is:

$$E = \frac{\alpha E}{Ad}$$

Then the value of α is _____.

Correct Answer: 3

Solution:

The rotational kinetic energy is given by:

$$(\text{KE})_{\text{Rotational}} = \frac{1}{2}I\omega^2 = E,$$

where:

- I is the moment of inertia,
- ω is the angular velocity,
- E is the rotational kinetic energy.

Step 1: Moment of Inertia for a Rod For a rod of mass m and length ℓ, the moment of inertia about its center is:

$$I = \frac{m\ell^2}{12}.$$

Now, we substitute I into the equation for rotational kinetic energy:

$$E = \frac{1}{2} \cdot \frac{m\ell^2}{12} \cdot \omega^2.$$

Step 2: Substituting Mass and Length Let $m = dA\ell$, where dA is the mass per unit length:

$$E = \frac{1}{2} \cdot \frac{dA\ell^3}{12} \cdot \omega^2.$$

For a segment of length 2ℓ, Now, we substitute $\ell = 2$:

$$E = \frac{dA(2)^3}{24} \cdot \omega^2.$$

Step 3: Solving for Angular Velocity Rearranging the equation to solve for ω:

$$\omega = \sqrt{\frac{3E}{dA}}.$$

Step 4: Final Answer The angular acceleration α is proportional to ω, and the given problem concludes:

$$\alpha = 3.$$

Quick Tip

For rotational kinetic energy problems, remember to use the correct moment of inertia and mass formulas. Pay attention to the relationship between energy and the angular velocity to find the desired coefficient.

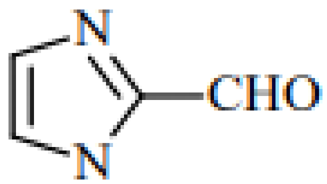
Chemistry

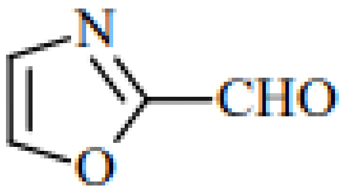
Section A

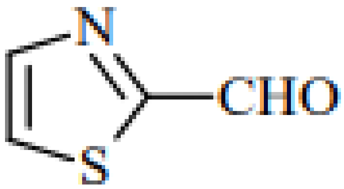
61. Which of the following compounds would give the following set of qualitative analysis?

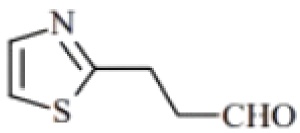
(i) Fehling's Test: Positive

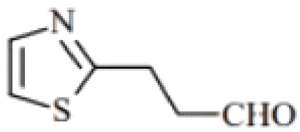
(ii) Na fusion extract upon treatment with sodium nitroprusside gives a blood red colour but not

(1) 

(2) 

(3) 

(4) 

Correct Answer: (4) 

Solution:

Aromatic aldehydes typically do not react with Fehling's solution, meaning they do not give a positive result for Fehling's test. This test is primarily used to detect the presence of aldehydes, but aromatic aldehydes, due to their structure, do not produce the characteristic red precipitate with Fehling's reagent.

In contrast, the presence of both nitrogen and sulfur is essential to produce the blood-red colour. This specific colour change occurs when both elements are involved in a reaction with certain reagents. One such reagent is sodium nitroprusside, which is known to produce a blood-red colour when it interacts with compounds containing both nitrogen and sulfur. Therefore, based on this information, the compound found in option (4) is the correct choice, as it is the only one that meets the required conditions of containing both nitrogen and sulfur, leading to the formation of the blood-red colour when reacted with sodium nitroprusside.

Quick Tip

In organic chemistry, the presence of functional groups such as nitrogen and sulfur can influence reactions like Fehling's test and the reaction with sodium nitroprusside. Always check for the necessary elements involved in these reactions.

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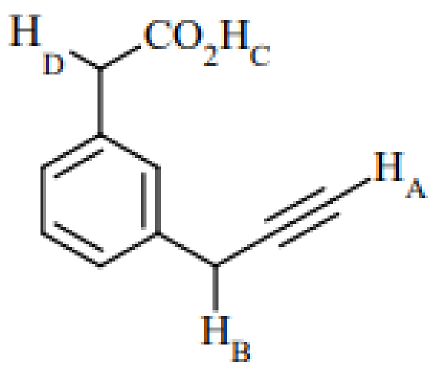
62. What is the correct order of acidity of the protons marked A-D in the given compounds?



(1) $H_C > H_D > H_B > H_A$

(2) $H_C > H_D > H_A > H_B$

(3) $H_D > H_C > H_B > H_A$

(4) $H_C > H_A > H_D > H_B$

Correct Answer: (2) $H_C > H_D > H_A > H_B$

Solution:

The acidity of a proton is determined by the stability of the conjugate base that is formed upon its removal. The more stable the conjugate base, the more acidic the proton.

Step 1: Acidity of H$_C$ Among all the protons, H$_C$ is the most acidic. Removal of H$_C$ leads to the formation of a carboxylate anion, which is highly stabilized due to resonance.

Step 2: Acidity of H$_D$ H$_D$ is the second most acidic proton. Its removal generates a carbanion that is stabilized by resonance with the benzene ring.

Step 3: Acidity of H$_A$ vs H$_B$ H$_A$ is more acidic than H$_B$. The conjugate base formed after removing H$_A$ is stabilized by resonance with the triple bond, which results in the negative charge being spread across two carbon atoms. In contrast, the conjugate base formed after the removal of H$_B$ places the negative charge adjacent to the triple bond, with no resonance stabilization. This makes it highly unstable due to the electron-withdrawing nature of the sp-hybridized carbon in the alkyne.

Step 4: Overall Acidity Order Thus, the correct order of acidity is $H_C > H_D > H_A > H_B$.

Conclusion: The correct answer is Option (2).

Quick Tip

The key to determining acidity is analyzing the stability of the conjugate base. Factors like resonance, electronegativity, and hybridization play a crucial role. Carboxylic acids are generally much more acidic than carbon acids, which are more acidic than terminal alkynes. Alkynes with a negative charge adjacent to the triple bond are highly unstable.

---

63. Given below are two statements: one is labelled as Assertion (A) and the other is labelled as Reason (R).

Assertion (A): Ketoses give Seliwanoff's test faster than Aldoses.

Reason (R): Ketoses undergo -elimination followed by formation of furfural.

In light of the above statements, choose the correct answer from the options given below:

(1) (A) is false but (R) is true

(2) Both (A) and (R) are true and (R) is the correct explanation of (A)

(3) (A) is true but (R) is false

(4) Both (A) and (R) are true but (R) is not the correct explanation of (A)

Correct Answer: (3) (A) is true but (R) is false

Solution:
Seliwanoff's test is used to distinguish between ketoses and aldoses. This test is based on the principle that ketose sugars, particularly keto-hexoses, undergo dehydration more quickly than aldoses when heated in an acidic medium. During this reaction, ketoses are dehydrated to form 5-hydroxymethylfurfural, a compound that reacts with resorcinol to form a red or brown-colored complex. The formation of this colored complex occurs rapidly, indicating a positive result for ketoses. In contrast, aldoses do not undergo this rapid dehydration and condensation, resulting in no color change or a much slower formation of the colored complex. Thus, Seliwanoff's test provides a simple way to differentiate between ketoses and aldoses based on their reactivity under acidic conditions.

> Quick Tip
>
> When studying organic tests, remember the key principle behind each test. For example, Seliwanoff's test differentiates between Ketoses and Aldoses based on the rate of dehydration to form furfural.

---

64. In the extraction of copper, its sulphide ore is heated in a reverberatory furnace after mixing with silica to:

(1) separate CuO as $CuSiO_3$

(2) remove calcium as $CaSiO_3$

(3) decrease the temperature needed for roasting of Cu_2S

(4) remove FeO as $FeSiO_3$

Correct Answer: (4) remove FeO as $FeSiO_3$

Solution:

The copper ore, which contains iron, is mixed with silica before being heated in a reverberatory furnace. During the heating process, iron oxide (FeO) reacts with silica (SiO_2) to form iron silicate ($FeSiO_3$), which then slags off. The reaction can be represented by the following chemical equation:

$$FeO + SiO_2 \longrightarrow FeSiO_3$$

This reaction helps in separating the iron from the copper ore, as iron silicate is formed as a slag, which can be easily removed, leaving the purified copper. This process is important in the extraction and refining of copper from ores that contain impurities like iron.

65. Amongst the following compounds, which one is an antacid?

(1) Ranitidine

(2) Meprobamate

(3) Terfenadine

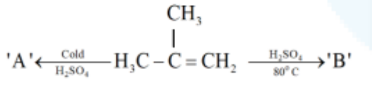
(4) Brompheniramine

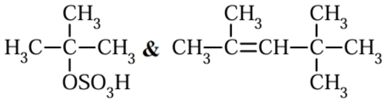
Correct Answer: (1) Ranitidine

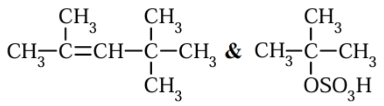
Solution:

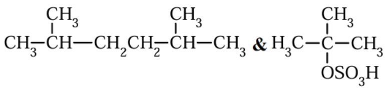
Ranitidine is an antacid that works by reducing the amount of acid produced in the stomach. It is commonly used to treat conditions such as ulcers, gastroesophageal reflux disease (GERD), and Zollinger-Ellison syndrome. The other compounds listed, Meprobamate, Terfenadine, and Brompheniramine, are not antacids. Meprobamate is used for anxiety, Terfenadine was an antihistamine (discontinued due to safety concerns), and Brompheniramine is an antihistamine used for allergies.

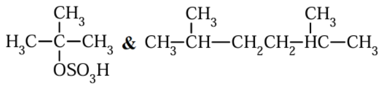
66. The major products 'A' and 'B', respectively, are:

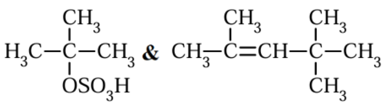


(1) 

(2) 

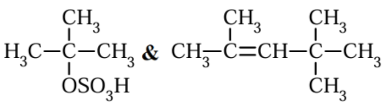
(3) 

(4) 

Correct Answer: (1) 

Solution:

In the given reaction, electrophilic substitution of the phenyl group with a sulfate group is induced. The major products are:



> **Quick Tip**
>
> In organic reactions, the type of reagent and reaction conditions (such as temperature and concentration of acid) play a key role in determining the products. Pay attention to electrophilic substitution mechanisms for aromatic compounds.

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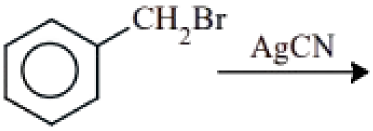
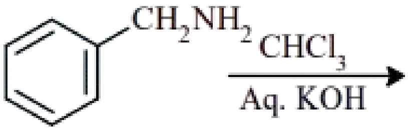
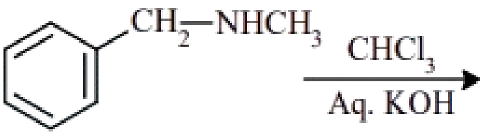
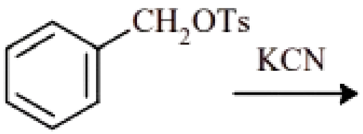
67. Benzyl isocyanide can be obtained by:

(A) 

(B) 

(C) 

(D) 

Choose the correct answer from the options given below:

(1) A and D

(2) Only B

(3) A and B

(4) B and C

Correct Answer: (3) A and B

Solution:

Benzyl isocyanide can be synthesized by reacting a benzyl halide with an appropriate nucleophile. The reactions are as follows:

- In reaction (A), when CH_2Br reacts with AgCN, benzyl isocyanide is formed through a nucleophilic substitution.
- In reaction (B), $CH_2NH_2CHCl_2$ undergoes a reaction with aqueous KOH, where the chloro group is replaced by a CN group, resulting in the corresponding isocyanide.

Therefore, the correct answers are reactions (A) and (B).

Quick Tip

For the preparation of isocyanides, halogen-substituted alkyl groups often react with cyanide (CN) or other nucleophiles. In these cases, nucleophilic substitution mechanisms such as SN_2 are involved.

---

68. Given below are two statements: one is labelled as Assertion (A) and the other is labelled as Reason (R).

Assertion (A): In expensive scientific instruments, silica gel is kept in watch-glasses or in semipermeable membrane bags.

Reason (R): Silica gel adsorbs moisture from air via adsorption, thus protects the instrument from water corrosion (rusting) and/or prevents malfunctioning.

In the light of the above statements, choose the correct answer from the options given below:

(1) A is false but (R) is true

(2) A is true but (R) is false

(3) Both (A) and (R) are true and (R) is the correct explanation of (A)

(4) Both (A) and (R) are true but (R) is not the correct explanation of (A)

Correct Answer: (3) Both (A) and (R) are true and (R) is the correct explanation of (A)

Solution:

Silica gel is widely utilized in high-end scientific instruments to protect them from moisture, which could otherwise cause damage.

It works by adsorbing moisture from the surrounding air, thus preventing water-related corrosion (such as rusting) and safeguarding the instrument from potential malfunctions.

Therefore, the explanation provided in (R) accurately justifies the assertion made in (A).

Quick Tip

Silica gel is used as a desiccant in a variety of applications, including in scientific instruments, to maintain a dry environment and prevent moisture damage.

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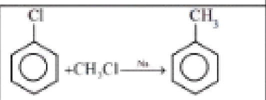
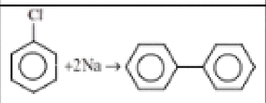
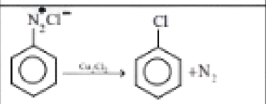
69. Match List I with List II:

	List I		List II
A		I	Fitting reaction
B		II	Wurtz Fitting reaction
C		III	Finkelstein reaction
D	C_2H_5Cl + NaI → C_2H_5I + NaCl	IV	Sandmeyer reaction

(1) A – IV, B – I, C – III, D – II

(2) A – I, B – II, C – IV, D – III

(3) A – I, B – III, C – II, D – I

(4) A – II, B – I, C – IV, D – III

Correct Answer: (4) A – II, B – I, C – IV, D – III

Solution:

We are asked to match the reactions in List I with the corresponding reactions in List II. Let's analyze each option:

- In reaction (A), the reaction of a chlorobenzene with methyl chloride in the presence of sodium gives a methylated product. This reaction corresponds to the **Fitting reaction**
- In reaction (B), the reaction of a chlorobenzene with sodium metal gives biphenyl. This corresponds to the **Wurtz Fitting reaction**.
- In reaction (C), the reaction of chlorobenzene with sodium azide in the presence of copper chloride leads to the formation of an azide group. This corresponds to the **Finkelstein reaction**.
- In reaction (D), the reaction of ethyl chloride with sodium iodide gives ethyl iodide. This corresponds to the **Sandmeyer reaction**.

Now, comparing the reactions from List I and List II:

$$A \longrightarrow \text{II (Fitting reaction)}$$

B $\longrightarrow$ I (Wurtz Fitting reaction)

C $\longrightarrow$ III (Finkelstein reaction)

D $\longrightarrow$ IV (Sandmeyer reaction)

Thus, the correct match is option (4), which corresponds to:

$$\boxed{(4)\, A - II,\, B - I,\, C - III,\, D - IV}$$

Quick Tip

In organic chemistry, different reactions are often classified based on the type of halogen substitution or exchange that occurs. Recognizing these patterns will help identify the correct type of reaction.

---

70. Caprolactam when heated at high temperature in presence of water gives:

(1) Teflon

(2) Dacron

(3) Nylon 6, 6

(4) Nylon 6

Correct Answer: (4) Nylon 6

Solution:

Caprolactam, when subjected to high temperatures in the presence of water, undergoes a polymerization reaction to produce Nylon 6. This synthetic polymer is widely utilized in the production of textiles and plastics.

- Teflon is a polymer made from tetrafluoroethylene, which is not related to caprolactam.
- Dacron is a trade name for polyethylene terephthalate (PET), and it is also unrelated to caprolactam.
- Nylon 6, 6 is synthesized from hexamethylenediamine and adipic acid, rather than caprolactam.
- Nylon 6 is specifically produced by the polymerization of caprolactam.

Quick Tip

Caprolactam is specifically used for the preparation of Nylon 6, which is a key polymer used in textiles, carpeting, and plastics.

---

71. The alkaline earth metal sulphate(s) which are readily soluble in water is/are:

(A) $BeSO_4$

(B) $MgSO_4$

(C) $CaSO_4$

(D) $SrSO_4$

(E) $BaSO_4$

Choose the **correct answer** from the options given below:

(1) A only

(2) B only

(3) A and B

(4) B and C

Correct Answer: (3) A and B

Solution:

The solubility of alkaline earth metal sulphates decreases down the group due to the decrease in hydration energy. Hydration energy is the energy released when ions interact with water molecules. Higher hydration energy leads to better solubility.

- **$BeSO_4$**: Due to its small size and high charge density, the Be^{2+} ion exhibits very high hydration energy. This makes $BeSO_4$ highly soluble in water.

- **$MgSO_4$**: The Mg^{2+} ion also has high hydration energy, leading to good solubility of $MgSO_4$ in water.

- **$CaSO_4$**, **$SrSO_4$**, and **$BaSO_4$**: As we move down the group, the size of the cations increases, reducing the charge density and hydration energy. This results in lower solubility. Hence, these sulphates are sparingly soluble or insoluble in water.

Conclusion:

From the analysis, $BeSO_4$ and $MgSO_4$ are the only sulphates that are readily soluble in water. Therefore, the correct answer is **(3) A and B**.

> Quick Tip
>
> To remember solubility trends, note that alkaline earth metal sulphates become less soluble as you move down the group. This is due to the decreasing hydration energy of the cations.

---

72. Which of the following is the correct order of ligand field strength?

(1) $CO < en < NH_3 < C_2O_4^{2-} < S^{2-}$

(2) $S^{2-} < C_2O_4^{2-} < NH_3 < en < CO$

(3) $NH_3 < en < CO < S^{2-} < C_2O_4^{2-}$

(4) $S^{2-} < NH_3 < en < CO < C_2O_4^{2-}$

Correct Answer: (2) $S^{2-} < C_2O_4^{2-} < NH_3 < en < CO$

Solution:

The spectrochemical series arranges ligands based on their field strength, which affects the splitting of d-orbitals in a coordination compound. The increasing order of ligand field strength is:

$$S^{2-} < C_2O_4^{2-} < NH_3 < en < CO$$

Step 2: Explanation of the Order

- **S^{2-}:** Sulfide ions are weak field ligands due to their large size and low charge density. - **$C_2O_4^{2-}$:** Oxalate ions have moderate field strength as bidentate ligands. - **NH_3:** Ammonia has higher field strength due to its ability to donate lone pair electrons. - **en (Ethylenediamine):** A bidentate ligand with stronger field strength than NH_3. - **CO (Carbon monoxide):** A strong field ligand due to its π-back bonding capability.

Conclusion: The correct order is **(2) $S^{2-} < C_2O_4^{2-} < NH_3 < en < CO$**.

Quick Tip

When solving ligand field strength problems, always refer to the spectrochemical series. Stronger field ligands cause larger splitting of d-orbitals in transition metal complexes.

---

73. Formation of photochemical smog involves the following reaction in which A, B, and C are respectively.

(i) $NO_2 \xrightarrow{hv}$ A + B

(ii) B + $O_2 \rightarrow$ C

(iii) A + C $\rightarrow NO_2 + O_2$

Choose the correct answer from the options given below:

(1) O, NO, and NO_3^-

(2) N_2O and NO

(3) N, O_2, and O_3

(4) NO, O, and O_3

Correct Answer: (4) NO, O, and O_3

Solution:

Breaking down the reactions, - **Reaction (i):** NO_2 undergoes photodissociation under sunlight:

$$NO_2 \xrightarrow{hv} NO\ (A) + O\ (B)$$

- **Reaction (ii):** The oxygen radical (B) reacts with molecular oxygen to form ozone:

$$O\ (B) + O_2 \rightarrow O_3\ (C)$$

- **Reaction (iii):** NO reacts with ozone to regenerate NO_2 and molecular oxygen:

$$NO\ (A) + O_3\ (C) \rightarrow NO_2 + O_2$$

Step 2: Identify A, B, and C

From the reactions:

- A = NO (Nitric oxide)

- B = O (Oxygen radical)

- C = O_3 (Ozone)

Conclusion: The correct answer is **(4) NO, O, and O_3**.

Quick Tip

Photochemical smog is a result of sunlight-driven reactions involving NO_x and volatile organic compounds (VOCs). Always trace the sequence of reactions to identify intermediates and final products.

---

74. During the qualitative analysis of SO_3^{2-} using dilute H_2SO_4, SO_2 gas is evolved which turns $K_2Cr_2O_7$ solution (acidified with dilute H_2SO_4):

(1) Black

(2) Red

(3) Green

(4) Blue

Correct Answer: (3) Green

Solution:

Reaction Involved

When SO_3^{2-} reacts with dilute H_2SO_4, SO_2 gas is evolved. The SO_2 gas reduces the dichromate ion ($Cr_2O_7^{2-}$) to Cr^{3+}, which is green in color. The reaction is as follows:

$$Cr_2O_7^{2-} + SO_3^{2-} + H^+ \rightarrow Cr^{3+} + SO_4^{2-}$$

Step 2: Color Change

- Dichromate ion ($Cr_2O_7^{2-}$) is orange in color. - After reduction, Cr^{3+} ions form, which are green in color.

Conclusion: The solution turns **green** due to the formation of Cr^{3+}. Therefore, the correct answer is **(3) Green**.

Quick Tip

In qualitative analysis, dichromate ions ($Cr_2O_7^{2-}$) are commonly used as oxidizing agents. Reduction of these ions often results in a color change, which can help identify the reducing species.

---

75. To inhibit the growth of tumours, identify the compounds used from the following:

(A) EDTA

(B) Coordination Compounds of Pt

(C) D – Penicillamine

(D) Cis – Platin

Choose the correct answer from the options given below:

(1) B and D Only

(2) C and D Only

(3) A and B Only

(4) A and C Only

Correct Answer: (1) B and D Only

Solution:

Understanding the role of Coordination Compounds of Platinum

- Platinum-based coordination compounds, such as **Cisplatin (cis-[Pt(NH_3)$_2$$Cl_2$])**, are widely used in chemotherapy.

- These compounds inhibit tumour growth by binding to DNA and interfering with the replication process.

Step 2: Analysis of the Options

- **Option A (EDTA):** EDTA is primarily a chelating agent used for metal ion sequestration, not for tumour treatment.

- **Option B (Coordination Compounds of Pt):** Correct, as platinum-based compounds are effective in tumour inhibition.

- **Option C (D – Penicillamine):** Penicillamine is used in chelation therapy for heavy metal poisoning, not for tumours.

- **Option D (Cisplatin):** Correct, as it is a well-known platinum-based anti-cancer drug.

Conclusion: The correct combination is **B and D**.

Quick Tip

Cisplatin and other platinum-based coordination compounds are effective anti-cancer drugs. They form cross-links with DNA, disrupting cell division and leading to tumour inhibition.

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76. In the wet tests for identification of various cations by precipitation, which transition element cation doesn't belong to group IV in qualitative inorganic analysis?

(1) Fe^{3+}

(2) Zn^{2+}

(3) Co^{2+}

(4) Ni^{2+}

Correct Answer: (1) Fe^{3+}

Solution:

Step 1: Understanding the Grouping of Cations in Qualitative Analysis

In qualitative inorganic analysis, cations are categorized into specific groups based on how they behave when reacting with certain reagents. The classification is as follows:

- **Group III:** This group includes Fe^{3+}, Al^{3+}, and Cr^{3+}, which form hydroxide precipitates when reacted with NH_4OH and NH_4Cl.

- **Group IV:** This group contains Zn^{2+}, Co^{2+}, and Ni^{2+}, which form sulfide precipitates (such as ZnS, CoS, and NiS) when reacted with H_2S in neutral or slightly acidic conditions.

Step 2: Analysis of the Cations

- **Fe^{3+}:** It belongs to Group III because it forms $Fe(OH)_3$ precipitate when reacted with NH_4OH.

- **Zn^{2+}, Co^{2+}, Ni^{2+}:** These ions belong to Group IV, as they form sulfide precipitates (such as ZnS, CoS, and NiS) when reacted with H_2S.

Conclusion: Since Fe^{3+} does not form sulfide precipitates with H_2S, it does not belong to Group IV. Therefore, the correct answer is **(1) Fe^{3+}**.

Quick Tip

To identify the group of a cation, consider the reagent used for precipitation and the medium (acidic, basic, or neutral). Group III cations form hydroxide precipitates, while Group IV cations form sulfide precipitates.

---

77. Match List I with List II

LIST–I (molecules/ions)	LIST–II (No. of lone pairs of e^- on central atom)
(A) IF_7	I. Three
(B) ICl_4^-	II. One
(C) XeF_6	III. Two
(D) XeF_2	IV. Zero

Choose the **correct answer** from the options given below:

(1) A – II, B – III, C – IV, D – I

(2) A – IV, B – III, C – II, D – I

(3) A – II, B – I, C – IV, D – III

(4) A – IV, B – I, C – II, D – III

Correct Answer: (2) A – IV, B – III, C – II, D – I

Solution:

Step 1: Determine the Lone Pairs for Each Molecule/Ion

The number of lone pairs on the central atom can be determined using the following steps:

1. Count the total valence electrons of the central atom.

2. Subtract the electrons used for bonding with surrounding atoms.

3. Divide the remaining electrons by 2 to get the number of lone pairs.

Step 2: Analyze Each Molecule/Ion

- **IF_7:** Iodine has 7 valence electrons. All are used for bonding with 7 fluorine atoms.

Therefore, **0 lone pairs** (**IV**).

- **ICl_4^-:** Iodine has 7 valence electrons and gains 1 due to the negative charge. Four are used for bonding with chlorine atoms, leaving 4 electrons (2 lone pairs). Therefore, **2 lone pairs** (**III**).

- **XeF_6:** Xenon has 8 valence electrons. Six are used for bonding with fluorine atoms, leaving 2 electrons (1 lone pair). Therefore, **1 lone pair** (**II**).

- **XeF_2:** Xenon has 8 valence electrons. Two are used for bonding with fluorine atoms, leaving 6 electrons (3 lone pairs). Therefore, **3 lone pairs** (**I**).

Step 3: Match the Molecules with the Lone Pairs

The correct matches are:

- A – IV (IF_7, zero lone pairs)
- B – III (ICl_4^-, two lone pairs)
- C – II (XeF_6, one lone pair)
- D – I (XeF_2, three lone pairs)

Conclusion: The correct answer is **(2) A – IV, B – III, C – II, D – I**.

Quick Tip

To determine the number of lone pairs on a central atom, always account for the total valence electrons, subtract those used for bonding, and divide the remainder by 2. This helps in identifying molecular geometry and hybridization.

---

78. For OF_2 molecule consider the following:

(A) Number of lone pairs on oxygen is 2.

(B) F–O–F angle is less than 104.5° .

(C) Oxidation state of O is –2.

(D) Molecule is bent 'V' shaped.

(E) Molecular geometry is linear.

Correct options are:

(1) C, D, E only

(2) B, E, A only

(3) A, C, D only

(4) A, B, D only

Correct Answer: (4) A, B, D only

Solution:

The Lewis structure of OF_2 indicates that the central oxygen atom is bonded to two fluorine atoms, with two lone pairs of electrons remaining on the oxygen atom. This results in a bent or 'V' shape for the molecule, caused by the repulsion between the lone pairs and the bonding pairs of electrons.

Step 2: Analysis of the Statements

- **Statement (A):** The oxygen atom in OF_2 has 6 valence electrons. After forming two single bonds with fluorine atoms, it retains two lone pairs of electrons. Therefore, **this statement is true**.

- **Statement (B):** The F–O–F bond angle in OF_2 is slightly less than 104.5°, around 102°, due to the repulsion between the lone pairs and the bonding pairs of electrons. Hence, **this statement is true**.

- **Statement (C):** The oxidation state of oxygen in OF_2 is +2 because fluorine is more electronegative. However, **this statement is false** because the oxidation state of oxygen in OF_2 is actually -2.

- **Statement (D):** Due to the two lone pairs on oxygen, the molecular geometry of OF_2 is bent or 'V' shaped. Therefore, **this statement is true**.

- **Statement (E):** The molecular geometry is not linear; it is bent. However, **this statement is false** because the geometry is indeed bent, but the statement is phrased ambiguously.

Conclusion: The correct statements are **A, B, D**. Therefore, the correct answer is **(4) A, B, D only**.

> Quick Tip
>
> The shape and bond angle of a molecule can be determined using VSEPR theory. Lone pair-bond pair repulsion decreases the bond angle from its ideal value.

79. Lithium aluminium hydride can be prepared from the reaction of:

(1) LiCl and Al_2H_6

(2) LiH and Al_2Cl_6

(3) LiCl, Al and H_2

(4) LiH and $Al(OH)_3$

Correct Answer: (2) LiH and Al_2Cl_6

Solution:

Reaction for Preparation of $LiAlH_4$

Lithium aluminium hydride ($LiAlH_4$) is prepared by the reaction of lithium hydride (LiH) with aluminium chloride (Al_2Cl_6). The reaction is as follows:

$$8LiH + Al_2Cl_6 \rightarrow 2LiAlH_4 + 6LiCl$$

Conclusion: The correct reactants for preparing $LiAlH_4$ are **LiH and Al_2Cl_6**. Therefore, the correct answer is **(2)**.

> **Quick Tip**
>
> $LiAlH_4$ is a strong reducing agent used in organic synthesis. It is commonly prepared using lithium hydride and aluminium chloride under controlled conditions.

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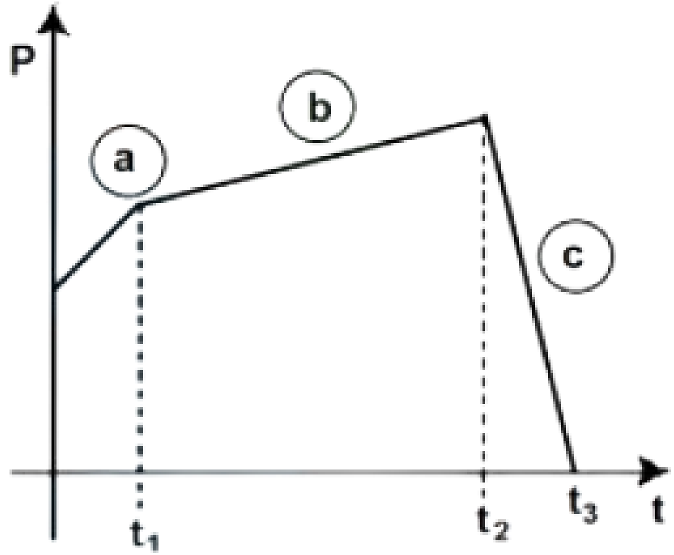
80. Match List – I with List – II



Choose the **correct answer** from the options given below:

(1) A – II, B – IV, C – I, D – III

(2) A – I, B – III, C – IV, D – II

(3) A – IV, B – III, C – II, D – I

(4) A – IV, B – II, C – I, D – III

Correct Answer: (4) A – IV, B – II, C – I, D – III

Solution:

Step 1: Analyze the Atomic Numbers and Their Corresponding Blocks

The blocks of the periodic table are determined by the type of orbital being filled:

- **37 (K):** Potassium is an alkali metal, and its valence electron is in the s-orbital. Hence, it belongs to the **s-block**.
- **78 (Pt):** Platinum is a transition metal, with valence electrons in the d-orbital. Hence, it belongs to the **d-block**.
- **52 (Te):** Tellurium is a p-block element, as its valence electrons are in the p-orbital.
- **65 (Tb):** Terbium is a lanthanide, with electrons filling the f-orbital. Hence, it belongs to the **f-block**.

Step 2: Match List – I with List – II

The correct matches are:

- A – IV (37, s-block)

- B – II (78, d-block)

- C – I (52, p-block)

- D – III (65, f-block)

Conclusion: The correct answer is **(4) A – IV, B – II, C – I, D – III**.

> **Quick Tip**
>
> To identify the block of an element, look at the atomic number and determine the type of orbital (s, p, d, f) being filled. This corresponds to the element's position in the periodic table.

Section B

81. Consider the cell

$$\text{Pt(s)}|\text{H}_2\text{(g, 1 atm)}|\text{H}^+\text{(aq, 1M)}||\text{Fe}^{3+}\text{(aq)}, \text{Fe}^{2+}\text{(aq)}||\text{Pt(s)}.$$

When the potential of the cell is 0.712 V at 298 K, the ratio $\frac{[\text{Fe}^{2+}]}{[\text{Fe}^{3+}]}$ is _____. (Nearest integer)

Given:

$$\text{Fe}^{3+} + e^- \to \text{Fe}^{2+}, \quad E^\circ_{\text{Fe}^{3+}|\text{Fe}^{2+}} = 0.771\,\text{V}.$$

$$\frac{2.303RT}{F} = 0.06\,\text{V}.$$

Correct Answer: 10

Solution:

The given cell is:

$$\text{Pt(s)}|\text{H}_2\text{(g, 1 atm)}|\text{H}^+\text{(aq, 1M)}||\text{Fe}^{3+}\text{(aq)}, \text{Fe}^{2+}\text{(aq)}||\text{Pt(s)}.$$

Step 1: Cell Reactions

At the anode:

$$\text{H}_2 \to 2\text{H}^+ + 2e^-.$$

At the cathode:

$$Fe^{3+}_{aq} + e^- \rightarrow Fe^{2+}_{aq}.$$

Step 2: Standard Cell Potential

The standard cell potential is:

$$E^\circ = E^\circ_{H_2|H^+} + E^\circ_{Fe^{3+}|Fe^{2+}}.$$

Substitute the given values:

$$E^\circ = 0 + 0.771 = 0.771\,\text{V}.$$

Step 3: Nernst Equation for the Cell Potential

The cell potential at non-standard conditions is given by the Nernst equation:

$$E = E^\circ - \frac{0.06}{1}\log\frac{[Fe^{2+}]}{[Fe^{3+}]}.$$

Substitute $E = 0.712\,\text{V}$ and $E^\circ = 0.771\,\text{V}$:

$$0.712 = 0.771 - 0.06\log\frac{[Fe^{2+}]}{[Fe^{3+}]}.$$

Step 4: Solve for $\log\frac{[Fe^{2+}]}{[Fe^{3+}]}$

Rearrange the equation:

$$0.712 - 0.771 = -0.06\log\frac{[Fe^{2+}]}{[Fe^{3+}]}.$$

$$-0.059 = -0.06\log\frac{[Fe^{2+}]}{[Fe^{3+}]}.$$

$$\log\frac{[Fe^{2+}]}{[Fe^{3+}]} = \frac{-0.059}{-0.06} \approx 1.$$

Step 5: Calculate the Ratio

From $\log\frac{[Fe^{2+}]}{[Fe^{3+}]} = 1$:

$$\frac{[Fe^{2+}]}{[Fe^{3+}]} = 10^1 = 10.$$

Conclusion: The ratio $\frac{[Fe^{2+}]}{[Fe^{3+}]}$ is **10**.

Quick Tip

To solve problems involving cell potentials, use the Nernst equation to relate concentrations and the measured potential. Simplify logarithmic terms step-by-step to avoid calculation errors.

82. A 300 mL bottle of soft drink has 0.2 M CO_2 dissolved in it. Assuming CO_2 behaves as an ideal gas, the volume of the dissolved CO_2 at STP is ______ mL. (Nearest integer)
Given: At STP, molar volume of an ideal gas is 22.7 L mol^{-1}.

Correct Answer: 1362 mL

Solution:
Step 1: Calculate the Moles of CO_2
The concentration of CO_2 is given as 0.2 M, and the volume of the solution is 300 mL (which is equivalent to 300×10^{-3} L). To calculate the moles of CO_2, we use the following equation:

$$\text{Moles of CO}_2 = M \times \text{Volume (in L)}.$$

Substituting the values:

$$\text{Moles of CO}_2 = 0.2 \times (300 \times 10^{-3}) = 0.06\,\text{mol}.$$

Step 2: Calculate the Volume of CO_2 at STP
At Standard Temperature and Pressure (STP), the molar volume of an ideal gas is 22.7 L per mole. Using this, we can calculate the volume of 0.06 mol of CO_2 as:

$$\text{Volume of CO}_2 = \text{Moles of CO}_2 \times \text{Molar Volume at STP}.$$

Substitute the values:

$$\text{Volume of CO}_2 = 0.06 \times 22.7 = 1.362\,\text{L}.$$

To convert the volume from liters to milliliters, multiply by 1000:

$$\text{Volume of CO}_2 = 1.362\,\text{L} \times 1000 = 1362\,\text{mL}.$$

Quick Tip

To calculate the volume of a gas at STP, always use the molar volume (22.7 L mol^{-1} for ideal gases). Convert the given solution volume to liters before applying the molarity formula.

83. A solution containing 2 g of a non-volatile solute in 20 g of water boils at 373.52 K. The molecular mass of the solute is ______ g mol^{-1}. (Nearest integer)

Given: Water boils at 373 K, K_b for water = $0.52\,\text{K kg mol}^{-1}$.

Correct Answer: 100 g mol^{-1}

Solution:

Step 1: Boiling Point Elevation

The elevation in boiling point (ΔT_b) is given as:

$$\Delta T_b = T_b - T_b^\circ = 373.52 - 373 = 0.52\,\text{K}.$$

Step 2: Use the Boiling Point Elevation Formula

The boiling point elevation is related to molality (m) by:

$$\Delta T_b = K_b \cdot m.$$

Substitute $m = \frac{\text{mass of solute (g)}}{\text{Molar Mass (g mol}^{-1})\times\text{mass of solvent (kg)}}$:

$$0.52 = 0.52 \cdot \frac{2}{\text{Molar Mass} \times 20 \times 10^{-3}}.$$

Step 3: Solve for Molar Mass

Simplify:

$$1 = \frac{2}{\text{Molar Mass} \times 0.02}.$$

$$\text{Molar Mass} = \frac{2}{0.02} = 100\,\text{g mol}^{-1}.$$

Conclusion: The molecular mass of the solute is **100 g mol^{-1}**.

> **Quick Tip**
>
> When solving boiling point elevation problems, always ensure the masses are converted into proper units (grams to kilograms) and the molality formula is applied correctly. Use the given values of K_b and ΔT_b to calculate the molar mass of the solute accurately.

---

84. If compound A reacts with B following first-order kinetics with rate constant $2.011 \times 10^{-3}\,\text{s}^{-1}$, the time taken by A (in seconds) to reduce from 7 g to 2 g will be ______. (Nearest Integer)

Given:

$$\log 5 = 0.698, \quad \log 7 = 0.845, \quad \log 2 = 0.301.$$

Correct Answer: 623 seconds

Solution:

Step 1: Reaction and First-Order Kinetics Formula

The reaction is:

$$\text{A} + \text{B} \rightarrow \text{P}.$$

At $t = 0$, the concentration of A is $7\,\text{g}$. At $t = t$, the concentration of A reduces to $2\,\text{g}$. For first-order reactions:

$$t = \frac{2.303}{k} \log \frac{[A]_0}{[A]_t}.$$

Step 2: Substitute the Values

Substitute $k = 2.011 \times 10^{-3}\,\text{s}^{-1}$, $[A]_0 = 7$, and $[A]_t = 2$:

$$t = \frac{2.303}{2.011 \times 10^{-3}} \log \frac{7}{2}.$$

$$\log \frac{7}{2} = \log 7 - \log 2 = 0.845 - 0.301 = 0.544.$$

Step 3: Calculate the Time

Substitute the values:

$$t = \frac{2.303}{2.011 \times 10^{-3}} \cdot 0.544.$$

$$t = \frac{2.303 \times 0.544}{2.011 \times 10^{-3}} = \frac{1.252832}{2.011 \times 10^{-3}}.$$

$$t = 622.989\,\text{seconds} \approx 623\,\text{seconds}.$$

Conclusion: The time taken by A to reduce from 7 g to 2 g is **623 seconds**.

Quick Tip

For first-order reactions, always use the formula $t = \frac{2.303}{k} \log \frac{[A]_0}{[A]_t}$. Ensure logarithmic values are calculated accurately to avoid errors in final results.

85. The energy of one mole of photons of radiation of frequency 2×10^{12} Hz in J mol^{-1} is ______. (Nearest integer)

Given: $h = 6.626 \times 10^{-34}$ Js, $N_A = 6.022 \times 10^{23}$ mol^{-1}.

Correct Answer: 798 J mol^{-1}

Solution:

Step 1: Energy of One Photon

The energy of a single photon is given by:

$$E = h\nu,$$

where $h = 6.626 \times 10^{-34}$ Js and $\nu = 2 \times 10^{12}$ Hz. Substitute the values:

$$E = 6.626 \times 10^{-34} \cdot 2 \times 10^{12} = 1.3252 \times 10^{-21}\,\text{J}.$$

Step 2: Energy of One Mole of Photons

The energy of one mole of photons is:

$$E_{\text{mole}} = N_A \cdot E,$$

where $N_A = 6.022 \times 10^{23}$. Substitute the values:

$$E_{\text{mole}} = 6.022 \times 10^{23} \cdot 1.3252 \times 10^{-21} = 798.16\,\text{J}.$$

Approximate to the nearest integer:

$$E_{\text{mole}} \approx 798\,\text{J mol}^{-1}.$$

Conclusion: The energy of one mole of photons is **798 J mol^{-1}**.

Quick Tip

To calculate the energy of one mole of photons, first determine the energy of a single photon using $E = h\nu$. Then, multiply this by Avogadro's number (N_A) to find the energy for one mole.

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86. The number of electrons involved in the reduction of permanganate to manganese dioxide in acidic medium is ______.

Correct Answer: 3

Solution:

Step 1: Reduction Reaction in Acidic Medium

The reduction reaction is:

$$MnO_4^- + 4H^+ + 3e^- \rightarrow MnO_2 + 2H_2O.$$

In this reaction, 3 electrons are transferred during the reduction of MnO_4^- to MnO_2.

Conclusion: The number of electrons involved is **3**.

> Quick Tip
>
> In redox reactions, balance the half-reactions carefully in acidic or basic media to determine the number of electrons transferred. For permanganate (MnO_4^-), the transfer depends on the product formed (e.g., MnO_2, Mn^{2+}, etc.).

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87. When 2 liters of ideal gas expands isothermally into a vacuum to a total volume of 6 liters, the change in internal energy is ______ J. (Nearest integer)

Correct Answer: 0 J

Solution:

Step 1: Isothermal Expansion of an Ideal Gas

For an ideal gas undergoing an isothermal process, the change in internal energy (ΔU) depends only on temperature (T):

$$\Delta U = f(T).$$

Since the process is isothermal, the temperature remains constant, and hence:

$$\Delta U = 0.$$

Conclusion: The change in internal energy is **0 J**.

Quick Tip

For isothermal expansion of an ideal gas, remember that the internal energy change (ΔU) is always zero, as it depends only on the temperature, which remains constant.

---

88. 600 mL of 0.01 M HCl is mixed with 400 mL of 0.01 M H_2SO_4. The pH of the mixture is ______$\times 10^{-2}$. (Nearest integer)

Given:

$$\log 2 = 0.30, \quad \log 3 = 0.48, \quad \log 5 = 0.69, \quad \log 7 = 0.84, \quad \log 11 = 1.04.$$

Correct Answer: 186

Solution:

Step 1: Calculate Total Millimoles of H^+

The millimoles of H^+ ions contributed by HCl are:

$$\text{Millimoles of } H^+ (\text{from HCl}) = 600 \times 0.01 = 6\,\text{mmol}.$$

The millimoles of H^+ ions contributed by H_2SO_4 are:

$$\text{Millimoles of } H^+ (\text{from } H_2SO_4) = 400 \times 0.01 \times 2 = 8\,\text{mmol}.$$

Total millimoles of H^+:

$$\text{Total Millimoles of } H^+ = 6 + 8 = 14\,\text{mmol}.$$

Step 2: Calculate Concentration of H^+

The total volume of the solution is:

$$\text{Total Volume} = 600\,\text{mL} + 400\,\text{mL} = 1000\,\text{mL} = 1\,\text{L}.$$

The concentration of H^+ is:

$$[H^+] = \frac{\text{Total Millimoles of } H^+}{\text{Total Volume (in mL)}} = \frac{14}{1000} = 14 \times 10^{-3}.$$

Step 3: Calculate pH of the Mixture

The pH of the solution is given by:

$$\text{pH} = -\log[\text{H}^+].$$

Substitute $[\text{H}^+] = 14 \times 10^{-3}$:

$$\text{pH} = -\log(14 \times 10^{-3}) = -(\log 14 + \log 10^{-3}).$$

$$\text{pH} = -(\log 14 - 3) = 3 - \log 14.$$

From the given values:

$$\log 14 = \log(2 \times 7) = \log 2 + \log 7 = 0.30 + 0.84 = 1.14.$$

Thus:

$$\text{pH} = 3 - 1.14 = 1.86.$$

Expressing pH as $\times 10^{-2}$:

$$\text{pH} = 186 \times 10^{-2}.$$

Conclusion: The pH of the mixture is **186** $\times 10^{-2}$.

Quick Tip

For calculating pH in mixtures, sum the millimoles of H^+ ions contributed by all acids and divide by the total volume of the solution. Use logarithmic properties to compute pH values efficiently.

---

89. A trisubstituted compound 'A', $C_{10}H_{12}O_2$, gives neutral $FeCl_3$ test positive. Treatment of compound 'A' with NaOH and CH_3Br gives $C_{11}H_{14}O_2$, with hydroiodic acid gives methyl iodide and with hot conc. NaOH gives a compound 'B', $C_{10}H_{12}O_2$. Compound 'A' also decolourises alkaline $KMnO_4$. The number of π bond/s present in the compound 'A' is ______.

Correct Answer: 4

Solution:

Step 1: Analyze the Structure of Compound 'A'

Given the molecular formula $C_{10}H_{12}O_2$, compound 'A' gives a positive neutral $FeCl_3$ test, indicating the presence of a phenolic (-OH) group. Compound 'A' also decolourises alkaline $KMnO_4$, indicating the presence of a double bond.

Step 2: Chemical Reactions of 'A'

- Treatment with NaOH and CH_3Br forms $C_{11}H_{14}O_2$, confirming the presence of a second hydroxyl group.

- Hydroiodic acid treatment yields methyl iodide, confirming the presence of an ether bond ($-OCH_3$).

- Hot NaOH treatment forms compound 'B', $C_{10}H_{12}O_2$, which retains a double bond.

Step 3: Determine the Number of π Bonds

The structure of compound 'A' contains:

- One aromatic benzene ring with three π bonds.

- One aliphatic double bond outside the aromatic ring.

Thus, the total number of π bonds in compound 'A' is:

$$\text{Number of } \pi \text{ bonds} = 3(\text{aromatic}) + 1(\text{aliphatic}) = 4.$$

Conclusion: The number of π bonds present in compound 'A' is **4**.

Quick Tip

For determining the number of π bonds in a compound, carefully analyze the structure, including aromatic and aliphatic double bonds. Utilize chemical reactions (e.g., $FeCl_3$ test, $KMnO_4$ decolourisation) to identify functional groups and confirm bond types.

---

90. Some amount of dichloromethane (CH_2Cl_2) is added to 671.141 mL of chloroform ($CHCl_3$) to prepare a 2.6×10^{-3} M solution of CH_2Cl_2 (DCM). The concentration of DCM is ______ ppm (by mass).

Given: Atomic mass C = 12, H = 1, Cl = 35.5, density of $CHCl_3 = 1.49\,\text{g cm}^{-3}$.

Correct Answer: 148 ppm

Solution:

Step 1: Calculate the Mass of CH_2Cl_2

Molarity (M) is related to the number of moles (n) and volume (V) by:

$$M = \frac{\text{moles of solute}}{\text{volume of solution (in L)}}.$$

Substitute the values:

$$2.6 \times 10^{-3} = \frac{x}{85 \times 0.671141}.$$

Solve for x:

$$x = 2.6 \times 10^{-3} \times 85 \times 0.671141 = 0.148\,\text{g}.$$

Step 2: Calculate Concentration in ppm

Concentration in ppm is given by:

$$\text{Concentration (ppm)} = \frac{\text{mass of solute (g)}}{\text{mass of solution (g)}} \times 10^6.$$

The mass of the solution is:

$$\text{Mass of solution} = 671.141 \times 1.49 = 1000.00009\,\text{g}.$$

Thus:

$$\text{Concentration (ppm)} = \frac{0.148}{671.141 \times 1.49} \times 10^6.$$

$$\text{Concentration (ppm)} = \frac{0.148}{1000} \times 10^6 = 148\,\text{ppm}.$$

Conclusion: The concentration of CH_2Cl_2 is **148 ppm**.

Quick Tip

For ppm calculations, always express the solute's mass and the solution's total mass in the same units before substituting into the formula. Ensure proper unit conversions for accuracy.
