JEE Main 2023 April 11 Shift-2 Mathematics Paper with Solutions

Time Allowed :3 Hours	**Maximum Marks :**300	**Total Questions :**90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The Duration of test is 3 Hours.
2. This paper consists of 90 Questions.
3. There are three parts in the paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage..
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each carries 4 marks for correct answer and –1 mark for wrong answer..
5. (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer.

Mathematics

Section A

1: The angle of elevation of the top P of a tower from the feet of one person standing due South of the tower is 45° and from the feet of another person standing due West of the tower is 30°. If the height of the tower is 5 meters, then the distance (in meters) between the two persons is equal to:

(1) 10

(2) $5\sqrt{5}$

(3) $\frac{5}{2}\sqrt{5}$

(4) 5

Correct Answer: (1) 10

Solution:

Provided that: A tower $AB = 5\,\text{m}$, and angles $\angle APB = 45^\circ$, $\angle PAB = 90^\circ$.

Step 1: Apply trigonometry in $\triangle ABP$.

From the Provided that data, we know:

$$\tan 45^\circ = \frac{AB}{AP}.$$

Since $\tan 45^\circ = 1$, we have:

$$1 = \frac{5}{AP}.$$

Thus, $AP = 5\,\text{m}$.

Step 2:Apply trigonometry in $\triangle ABQ$.

Now, consider $\triangle ABQ$, where $\angle BAQ = 30^\circ$. Using the tangent formula:

$$\tan 30^\circ = \frac{AB}{AQ}.$$

Substitute the known values:

$$\frac{1}{\sqrt{3}} = \frac{5}{AQ}.$$

Rearranging, we get:

$$AQ = 5\sqrt{3}\,\text{m}.$$

Step 3: Calculate the hypotenuse PQ.

Using the Pythagoras theorem in $\triangle APQ$:

$$PQ^2 = AP^2 + AQ^2.$$

Substituting $AP = 5\,\text{m}$ and $AQ = 5\sqrt{3}\,\text{m}$:

$$PQ^2 = 5^2 + (5\sqrt{3})^2.$$

Simplify:

$$PQ^2 = 25 + 75 = 100.$$

Thus, $PQ = \sqrt{100} = 10\,\text{cm}.$

Conclusive Answer: The length of PQ is $10\,\text{cm}$.

Conclusion: The correct option is **(A) 10 cm**.

Quick Tip

For problems involving angles of elevation and distances,Apply trigonometric identities like $\tan$ and the Pythagorean theorem to calculate distances accurately.

2: Let a, b, c, and d be positive real numbers such that $a + b + c + d = 11$. If the maximum value of $a^5b^3c^2d$ is 3750β, then the value of β is:

(1) 55

(2) 108

(3) 90

(4) 110

Correct Answer: (3) 90

Solution:

Provided that

$$a + b + c + d = 11 \quad \text{and } \max(a^5b^3c^2d) = ?$$

$$(a, b, c, d > 0)$$

To maximize the Provided that expression, weApply the Arithmetic Mean (A.M.) and Geometric Mean (G.M.) inequality:

$$\text{A.M.} \geq \text{G.M.}$$

Let us assume the terms as: $\frac{a}{5}, \frac{a}{5}, \frac{a}{5}, \frac{a}{5}, \frac{a}{5}, \frac{b}{3}, \frac{b}{3}, \frac{b}{3}, \frac{c}{2}, \frac{c}{2}, d$

Now applying A.M. $\geq$ G.M., we get:

$$\frac{\frac{a}{5} + \frac{a}{5} + \frac{a}{5} + \frac{a}{5} + \frac{a}{5} + \frac{b}{3} + \frac{b}{3} + \frac{b}{3} + \frac{c}{2} + \frac{c}{2} + d}{11} \geq \left(\frac{a^5 b^3 c^2 d}{5^5 \cdot 3^3 \cdot 2^2 \cdot 1} \right)^{\frac{1}{11}}$$

Simplifying the left-hand side:

$$\frac{11}{11} \geq \left(\frac{a^5 b^3 c^2 d}{5^5 \cdot 3^3 \cdot 2^2 \cdot 1} \right)^{\frac{1}{11}}$$

Thus, we get:

$$a^5 b^3 c^2 d \leq 5^5 \cdot 3^3 \cdot 2^2$$

The maximum value of $a^5 b^3 c^2 d$ is:

$$\max(a^5 b^3 c^2 d) = 5^5 \cdot 3^3 \cdot 2^2 = 337500$$

We can express this as:

$$337500 = 90 \times 3750 \quad \text{where } \beta = 90$$

$$\beta = 90$$

Explanation: To achieve the maximum value, the numbers must be distributed in the ratios consistent with their powers in $a^5 b^3 c^2 d$, ensuring the product is maximized. Using A.M. $\geq$ G.M. is crucial in such optimization problems.

Quick Tip

To maximize expressions of the form $a^m b^n \ldots$, distribute the total sum proportionally to the powers in the expression using the AM-GM inequality.

3: If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\int_0^{\frac{\pi}{2}} f(\sin 2x) \sin x \, dx + \alpha \int_0^{\frac{\pi}{4}} f(\cos 2x) \cos x \, dx = 0,$$

then the value of α is:

(1) $-\sqrt{3}$

(2) $\sqrt{3}$

(3) $-\sqrt{2}$

(4) $\sqrt{2}$

Correct Answer: (3) $-\sqrt{2}$

Solution:

$$F : \mathbb{R} \to \mathbb{R}$$

We are Provided that the integral equation:

$$\int_0^{\frac{\pi}{2}} F(\sin 2x) \sin x \, dx + \alpha \int_0^{\frac{\pi}{2}} F(\cos 2x) \cos x \, dx = 0$$

To simplify, we split the range of integration for the first term as follows:

$$\int_0^{\frac{\pi}{2}} F(\sin 2x) \sin x \, dx = \int_0^{\frac{\pi}{4}} F(\sin 2x) \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} F(\sin 2x) \sin x \, dx$$

Rewriting the equation, we have:

$$\int_0^{\frac{\pi}{4}} F(\sin 2x) \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} F(\sin 2x) \sin x \, dx + \alpha \int_0^{\frac{\pi}{4}} F(\cos 2x) \cos x \, dx = 0$$

Using the property of definite integrals:

$$\int_0^{a} F(x) \, dx = \int_a^{0} F(a - x) \, dx,$$

we let $x = t + \frac{\pi}{4}$, and substitute in the integrals. This yields:

$$\int_0^{\frac{\pi}{4}} F(\cos 2x) \sin\left(\frac{\pi}{4} - x\right) dx + \int_0^{\frac{\pi}{4}} F(\cos 2t) \sin\left(t + \frac{\pi}{4}\right) dx + \alpha \int_0^{\frac{\pi}{4}} F(\cos 2x) \cos x \, dx = 0$$

Simplifying the trigonometric terms, we combine them:

$$\int_0^{\frac{\pi}{4}} F(\cos 2x) \left\{ \sin\left(\frac{\pi}{4} - x\right) + \sin\left(x + \frac{\pi}{4}\right) + \alpha \cos x \right\} dx = 0$$

Using the addition formulas for sine, we get:

$$\sin\left(\frac{\pi}{4}-x\right)+\sin\left(x+\frac{\pi}{4}\right)=\sqrt{2}\cos x.$$

Thus, the equation becomes:

$$\int_0^{\frac{\pi}{4}} F(\cos 2x)\left(\sqrt{2}+\alpha\right)\cos x\,dx=0$$

Factoring out the common term, we have:

$$\left(\sqrt{2}+\alpha\right)\int_0^{\frac{\pi}{4}} F(\cos 2x)\cos x\,dx=0$$

In the interval $\left(0,\frac{\pi}{4}\right)$, both $F(\cos 2x)$ and $\cos x$ are non-zero.

Thus, for the equation to hold, we must have:

$$\sqrt{2}+\alpha=0$$

$$\Rightarrow \alpha=-\sqrt{2}$$

Conclusive Answer: $\alpha=-\sqrt{2}$.

Quick Tip

For definite integrals involving trigonometric substitutions, carefully transform the limits and simplify the equation step-by-step. Constant functions often simplify such problems effectively.

4: Let f and g be two functions defined by:

$$f(x)=\begin{cases} x+1, & x<0 \\ |x-1|, & x\geq 0 \end{cases}, \quad g(x)=\begin{cases} x+1, & x<0 \\ 1, & x\geq 0 \end{cases}.$$

Then $(g\circ f)(x)$ is:

(1) continuous everywhere but not differentiable at $x=1$

(2) continuous everywhere but not differentiable exactly at one point

(3) differentiable everywhere

(4) not continuous at $x = -1$

Correct Answer: (2) continuous everywhere but not differentiable exactly at one point

Solution:

$$f(x) = \begin{cases} x+1, & x < 0 \\ 1-x, & 0 \leq x < 1 \\ x-1, & 1 \leq x \end{cases}$$

$$g(x) = \begin{cases} x+1, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Step 1: Compute the composition $g(f(x))$.

Case 1: When $x < 0$**:**

For $x < 0$, $f(x) = x + 1$. Substituting into $g(x)$:

$$g(f(x)) = g(x+1) = (x+1) + 1 = x + 2.$$

Case 2: When $0 \leq x < 1$**:**

For $0 \leq x < 1$, $f(x) = 1 - x$. Substituting into $g(x)$:

$$g(f(x)) = g(1-x) = 1, \quad \text{(since } g(x) = 1 \text{ for } x \geq 0\text{)}.$$

Case 3: When $x \geq 1$**:**

For $x \geq 1$, $f(x) = x - 1$. Substituting into $g(x)$:

$$g(f(x)) = g(x-1) = 1, \quad \text{(since } g(x) = 1 \text{ for } x \geq 0\text{)}.$$

Step 2: Final composition of $g(f(x))$**.**

Combining all cases:

$$g(f(x)) = \begin{cases} x+2, & x < -1 \\ 1, & x \geq -1 \end{cases}$$

Step 3: Analyze continuity and differentiability.

- **Continuity:** $g(f(x))$ is continuous everywhere because there are no jumps or breaks in the function definition.
- **Differentiability:** At $x = -1$:
  - For $x < -1$, the derivative is $\frac{d}{dx}(x + 2) = 1$.
  - For $x \geq -1$, the derivative is $\frac{d}{dx}(1) = 0$.
  - The left-hand derivative (1) and right-hand derivative (0) are not equal at $x = -1$.

  Conclusion: $g(f(x))$ is not differentiable at $x = -1$, but it is differentiable everywhere else.

Final Observations:

- $g(f(x))$ is **continuous everywhere**.
- $g(f(x))$ is **not differentiable at** $x = -1$.
- $g(f(x))$ is **differentiable everywhere else**.

Quick Tip

To analyze composite functions, evaluate their continuity and differentiability case-by-case by substituting into the Provided that functions' definitions.

5: If the radius of the largest circle with centre $(2, 0)$ inscribed in the ellipse $x^2 + 4y^2 = 36$ is r, then $12r^2$ is equal to:

(1) 69

(2) 72

(3) 115

(4) 92

Correct Answer: (4) 92

Solution:

Provided that: Center of the ellipse $C(2, 0)$ and the equation of the ellipse:

$$x^2 + 4y^2 = 36 \implies \frac{x^2}{36} + \frac{y^2}{9} = 1$$

Step 1: Equation of the normal at a point P on the ellipse.

The parametric coordinates of a point P on the ellipse are $P(6\cos\theta, 3\sin\theta)$.

The equation of the normal at P is Provided that by:

$$(6\sec\theta)x - (3\csc\theta)y = 27$$

Step 2: Normal passes through $C(2, 0)$.

Substituting $C(2, 0)$ into the equation of the normal:

$$(6\sec\theta)(2) - (3\csc\theta)(0) = 27$$

$$\implies \sec\theta = \frac{27}{12} = \frac{9}{4}.$$

Step 3: Determine $\cos\theta$ and $\sin\theta$.

Using the relation $\sec\theta = \frac{1}{\cos\theta}$:

$$\cos\theta = \frac{4}{9}, \quad \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{4}{9}\right)^2} = \sqrt{\frac{65}{81}} = \frac{\sqrt{65}}{9}.$$

Step 4: Parametric point P.

Using the parametric equations of the ellipse:

$$P\left(6\cos\theta, 3\sin\theta\right) = \left(6 \cdot \frac{4}{9}, 3 \cdot \frac{\sqrt{65}}{9}\right) = \left(\frac{8}{3}, \frac{\sqrt{65}}{3}\right).$$

Step 5: Distance between P and $C(2, 0)$.

The distance γ between $P\left(\frac{8}{3}, \frac{\sqrt{65}}{3}\right)$ and $C(2, 0)$ is:

$$\gamma = \sqrt{\left(\frac{8}{3} - 2\right)^2 + \left(\frac{\sqrt{65}}{3} - 0\right)^2}$$

$$= \sqrt{\left(\frac{8}{3} - \frac{6}{3}\right)^2 + \left(\frac{\sqrt{65}}{3}\right)^2}$$

$$= \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{\sqrt{65}}{3}\right)^2}$$

$$= \sqrt{\frac{4}{9} + \frac{65}{9}} = \sqrt{\frac{69}{9}} = \frac{\sqrt{69}}{3}.$$

Step 6: Value of $12\gamma^2$.

The value of $12\gamma^2$ is:

$$12\gamma^2 = 12 \cdot \left(\frac{\sqrt{69}}{3}\right)^2 = 12 \cdot \frac{69}{9} = \frac{12 \cdot 69}{9} = 92.$$

Conclusive Answer: The value of $12\gamma^2$ is 92.

Quick Tip

To determine the radius of a circle inscribed in an ellipse, find the point on the ellipse boundary closest to the circle's center and calculate the radius accordingly.

6: Let the mean of 6 observations $1, 2, 4, 5, x,$ and y and their variance be 10. Then their mean deviation about the mean is equal to:

(1) $\frac{7}{3}$

(2) $\frac{10}{3}$

(3) $\frac{8}{3}$

(4) 3

Correct Answer: (3) $\frac{8}{3}$

Solution:

Provided that: The mean of $1, 2, 4, 5, x, y$ is 5, and the variance is 10.

Step 1: Calculate the mean.

The formula for the mean is:

$$\text{Mean} = \frac{\text{Sum of all values}}{\text{Number of values}}.$$

Substituting the Provided that values:

$$\frac{12 + x + y}{6} = 5.$$

Simplifying:

$$12 + x + y = 30 \implies x + y = 18.$$

Step 2:Apply the formula for variance.

The formula for variance is:

$$\text{Variance} = \frac{\text{Sum of squares of values}}{\text{Number of values}} - (\text{Mean})^2.$$

Substituting the Provided that variance 10 and mean 5:

$$\frac{x^2 + y^2 + 46}{6} - 5^2 = 10.$$

Simplify:

$$\frac{x^2 + y^2 + 46}{6} - 25 = 10 \implies \frac{x^2 + y^2 + 46}{6} = 35.$$

Multiply through by 6:

$$x^2 + y^2 + 46 = 210 \implies x^2 + y^2 = 164.$$

Step 3: Solve for x and y.

From $x + y = 18$ and $x^2 + y^2 = 164$, we solve:

$$(x + y)^2 = x^2 + y^2 + 2xy \implies 18^2 = 164 + 2xy \implies 324 = 164 + 2xy.$$

$$2xy = 160 \implies xy = 80.$$

The quadratic equation for x and y is:

$$t^2 - (x + y)t + xy = 0 \implies t^2 - 18t + 80 = 0.$$

Solving this quadratic equation:

$$t = \frac{-(-18) \pm \sqrt{(-18)^2 - 4(1)(80)}}{2(1)} = \frac{18 \pm \sqrt{324 - 320}}{2} = \frac{18 \pm 2}{2}.$$

$$t = 10 \text{ or } t = 8.$$

Thus, $x = 8$ and $y = 10$ (or vice versa).

Step 4: Calculate the mean deviation.

The formula for mean deviation is:

$$\text{Mean Deviation} = \frac{\sum |x - \bar{x}|}{\text{Number of values}},$$

where $\bar{x}$ is the mean. Substituting the values:

$$\text{Mean Deviation} = \frac{|1 - 5| + |2 - 5| + |4 - 5| + |5 - 5| + |8 - 5| + |10 - 5|}{6}.$$

Simplify:

$$\text{Mean Deviation} = \frac{4+3+1+0+3+5}{6} = \frac{16}{6} = \frac{8}{3}.$$

Conclusive Answer: The mean deviation is $\frac{8}{3}$.

Quick Tip

The mean deviation about the mean is calculated using absolute differences of observations from their mean. It measures dispersion in the data set.

7: Let $A = \{1, 3, 4, 6, 9\}$ and $B = \{2, 4, 5, 8, 10\}$. Let R be a relation defined on $A \times B$ such that $R = \{((a_1, b_1), (a_2, b_2)) : a_1 \leq b_2$ and $b_1 \leq a_2\}$. Then the number of elements in the set R is:

(1) 52

(2) 160

(3) 26

(4) 180

Correct Answer: (2) 160

Solution:

Step 1: Analyze the Provided that conditions.

We are Provided that choices for a_1 and b_2 as follows:

- $a_1 = 1 \Rightarrow 5$ choices of b_2
- $a_1 = 3 \Rightarrow 4$ choices of b_2
- $a_1 = 4 \Rightarrow 4$ choices of b_2
- $a_1 = 6 \Rightarrow 2$ choices of b_2
- $a_1 = 9 \Rightarrow 1$ choice of b_2

The total number of ways for (a_1, b_2) is 16 ways.

Step 2: Analyze choices for b_1 and a_2.

Similarly, we are Provided that choices for b_1 and a_2 as follows:

- $b_1 = 2 \Rightarrow 4$ choices of a_2
- $b_1 = 4 \Rightarrow 3$ choices of a_2
- $b_1 = 5 \Rightarrow 2$ choices of a_2
- $b_1 = 8 \Rightarrow 1$ choice of a_2

Step 3: Calculate the total number of required elements.

Combining the above, the total required elements in R is:

$$\text{Total required elements in } R = 16 \times 10 = 160.$$

Conclusive Answer: The required number of elements in R is 160.

Quick Tip

When calculating the size of a relation involving conditions, break down the constraints and count the valid combinations for each pair systematically.

8: Let P be the plane passing through the points $(5, 3, 0)$, $(13, 3, -2)$, and $(1, 6, 2)$. For $\alpha \in \mathbb{N}$, if the distances of the points $A(3, 4, \alpha)$ and $B(2, \alpha, a)$ from the plane P are 2 and 3 respectively, then the positive value of a is:

(1) 5

(2) 6

(3) 4

(4) 3

Correct Answer: (3) 4

Solution:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 0 & -2 \\ 4 & -3 & -2 \end{vmatrix} = \hat{i}(-6) + 8\hat{j} - 24\hat{k}$$

The determinant is computed to find the normal vector of the plane. Expanding along the first row gives the coefficients of $\hat{i}, \hat{j}, \hat{k}$.

$$\text{Normal of the plane } = 3\hat{i} - 4\hat{j} + 12\hat{k}$$

Simplifying the determinant gives the normal vector, which defines the orientation of the plane.

$$\text{Plane: } 3x - 4y + 12z = 3$$

Using the normal vector and a point on the plane, the equation of the plane is determined.

$$\text{Distance from } A(3, 4, \alpha)$$
$$\frac{|9 - 16 + 12\alpha - 3|}{13} = 2$$

The distance formula is applied to point $A(3, 4, \alpha)$ and equated to 2 as Provided that in the problem.

$$\alpha = 3$$
$$\alpha = -8 \text{ (rejected)}$$

Solving the distance equation gives two possible values for α. $\alpha = -8$ is rejected based on the Provided that conditions or physical constraints.

$$\text{Distance from } B(2, 3, a)$$
$$\frac{|6 - 12 + 12a - 3|}{13} = 3$$

Similarly, the distance formula is applied to point $B(2, 3, a)$ and equated to 3 as specified.

$$a = 4$$

Solving the equation gives the value of a that satisfies the distance condition.

Quick Tip

To find the distance of a point from a plane, always substitute into the formula for the distance. Be sure to simplify and check for positive solutions when working with natural numbers.

9: If the letters of the word MATHS are permuted and all possible words so formed are arranged as in a dictionary with serial number, then the serial number of the word THAMS is:

(1) 102

(2) 103

(3) 101

(4) 104

Correct Answer: (2) 103

Solution:

To determine the rank of the word **THAMS**, let usApply the factorial method to count the number of permutations before the word **THAMS** appears in alphabetical order.

Step 1: Assign numerical positions to the letters.

The alphabetical order of the letters **T, H, A, M, S** is:

$$A = 1, H = 2, M = 3, S = 4, T = 5$$

Thus, the word **THAMS** corresponds to the sequence: $5, 2, 1, 3, 4$.

Step 2: Count permutations for each letter.

$$\begin{aligned}
\text{For } T : 4! &\quad \text{(4 letters after T in alphabetical order: H, A, M, S)} \\
\text{For } H : 3! &\quad \text{(1 letter after H: A)} \\
\text{For } A : 2! &\quad \text{(0 letters after A)} \\
\text{For } M : 1! &\quad \text{(0 letters after M)} \\
\text{For } S : 0! &\quad \text{(0 letters after S)}
\end{aligned}$$

Step 3: Compute the total permutations before THAMS.

$$\begin{aligned}\text{Total} &= 4 \times 4! + 3! \times 1 + 0 + 0 + 0 \\ &= 4 \times 24 + 6 \\ &= 96 + 6 = 102\end{aligned}$$

Step 4: Add 1 for the word itself.

$$\text{Rank of THAMS} = 102 + 1 = 103$$

Quick Tip

To find the serial number of a word in a dictionary-like arrangement, calculate the number of permutations for each prefix and add them step by step until you reach the desired word.

10: If four distinct points with position vectors $\vec{a}, \vec{b}, \vec{c}, and\ \vec{d}$ are coplanar, then $[\vec{a}\vec{b}\vec{c}]$ is equal to:

(1) $[\vec{d}\vec{c}\vec{a}] + [\vec{b}\vec{d}\vec{a}] + [\vec{c}\vec{d}\vec{b}]$

(2) $[\vec{d}\vec{b}\vec{a}] + [\vec{a}\vec{c}\vec{d}] + [\vec{d}\vec{b}\vec{c}]$

(3) $[\vec{a}\vec{d}\vec{b}] + [\vec{d}\vec{c}\vec{a}] + [\vec{d}\vec{b}\vec{c}]$

(4) $[\vec{b}\vec{c}\vec{d}] + [\vec{d}\vec{a}\vec{c}] + [\vec{d}\vec{b}\vec{a}]$

Correct Answer: (1) $[\vec{d}\vec{c}\vec{a}] + [\vec{b}\vec{d}\vec{a}] + [\vec{c}\vec{d}\vec{b}]$

Solution:

Provided that that the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar, we need to determine the value of $[\vec{a}\ \vec{b}\ \vec{c}]$.

Step 1: Represent the condition of coplanarity.

Since the vectors $\vec{b} - \vec{a}, \vec{c} - \vec{b}, \vec{d} - \vec{c}$ are coplanar, we have:

$$[\vec{b} - \vec{a}\ \ \vec{c} - \vec{b}\ \ \vec{d} - \vec{c}] = 0$$

Step 2: Expand the determinant.

Expanding the determinant using the scalar triple product:

$$(\vec{b} - \vec{a}) \cdot \Big((\vec{c} - \vec{b}) \times (\vec{d} - \vec{c})\Big) = 0$$

Step 3: Simplify the expression.

Using the distributive property:

$$(\vec{b} - \vec{a}) \cdot (\vec{c} \times \vec{b} - \vec{c} \times \vec{a} - \vec{a} \times \vec{d}) = 0$$

Step 4: Represent as scalar triple products.

The scalar triple products can be written as:

$$[\vec{b}\,\vec{c}\,\vec{d}] - [\vec{b}\,\vec{c}\,\vec{a}] - [\vec{b}\,\vec{a}\,\vec{d}] - [\vec{a}\,\vec{c}\,\vec{d}] = 0$$

Step 5: Rearrange to find $[\vec{a}\,\vec{b}\,\vec{c}]$.

Finally, rearranging the terms gives:

$$[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{d}\,\vec{c}\,\vec{a}] + [\vec{b}\,\vec{d}\,\vec{a}] + [\vec{c}\,\vec{d}\,\vec{b}]$$

Quick Tip

The scalar triple product isApplyd to determine the volume of a parallelepiped formed by three vectors. If the scalar triple product is zero, the vectors are coplanar.

11: The sum of the coefficients of three consecutive terms in the binomial expansion of $(1+x)^{n+2}$, which are in the ratio $1:3:5$, is equal to:

(1) 63

(2) 92

(3) 25

(4) 41

Correct Answer: (1) 63

Solution:

Provided that the problem:

$$\binom{n+2}{r-1} : \binom{n+2}{r} : \binom{n+2}{r+1} :: 1:3:5$$

Step 1: Express the first ratio.

$$\frac{\binom{n+2}{r-1}}{\binom{n+2}{r}} = \frac{1}{3}$$

Using the formula for binomial coefficients:

$$\frac{\frac{(n+2)!}{(r-1)!(n+3-r)!}}{\frac{(n+2)!}{r!(n+2-r)!}} = \frac{1}{3}$$

Simplifying:

$$\frac{r}{(n-r+3)} = \frac{1}{3}$$

$$\Rightarrow n - r + 3 = 3r$$

$$\Rightarrow n = 4r - 3 \quad \ldots(1)$$

Step 2: Express the second ratio.

$$\frac{\binom{n+2}{r}}{\binom{n+2}{r+1}} = \frac{3}{5}$$

Using the binomial coefficient formula again:

$$\frac{\frac{(n+2)!}{r!(n+2-r)!}}{\frac{(n+2)!}{(r+1)!(n+1-r)!}} = \frac{3}{5}$$

Simplifying:

$$\frac{r+1}{n+2-r} = \frac{3}{5}$$

Cross-multiplying:

$$5(r+1) = 3(n+2-r)$$

$$5r + 5 = 3n + 6 - 3r$$

$$8r - 1 = 3n \quad \ldots(2)$$

Step 3: Solve equations (1) and (2). Substitute $n = 4r - 3$ from equation (1) into equation (2):

$$8r - 1 = 3(4r - 3)$$

$$8r - 1 = 12r - 9$$

$$4r = 8 \quad \Rightarrow r = 2$$

Step 4: Solve for n. Substitute $r = 2$ into equation (1):

$$n = 4(2) - 3$$

$$n = 5$$

Step 5: Calculate the sum of combinations.

$$\text{Sum} = \binom{7}{1} + \binom{7}{2} + \binom{7}{3}$$
$$= 7 + 21 + 35 = 63$$

Conclusive Answer: The sum of the combinations is 63.

Quick Tip

When coefficients are in a ratio, equate the corresponding binomial coefficients andApply algebra to solve for n and r. Sum the required coefficients for the answer.

12: Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} + \frac{5}{x(x^5+1)}y = \frac{(x^5+1)^2}{x^7}$, $x > 0$. If $y(1) = 2$, then $y(2)$ is equal to:

(1) $\frac{693}{128}$

(2) $\frac{637}{128}$

(3) $\frac{697}{128}$

(4) $\frac{679}{128}$

Correct Answer: (1) $\frac{693}{128}$

Solution:

The Provided that problem involves solving a differential equation using the integrating factor (I.F.) method.

Step 1: Determine the Integrating Factor (I.F.)

$$\text{I.F.} = e^{\int \frac{5\,dx}{x(x^5+1)}} = e^{\int \frac{5x^{-6}\,dx}{(x^{-5}+1)}}$$

Let $1 + x^{-5} = t$, so:

$$-5x^{-6}\,dx = dt$$

Thus:

$$\text{I.F.} = e^{\int \frac{-dt}{t}} = e^{-\ln t} = \frac{1}{t} = \frac{x^5}{1+x^5}$$

Step 2: Solve for y.

$$y \cdot \frac{x^5}{1+x^5} = \int \frac{x^5}{1+x^5} \cdot \frac{(1+x^5)^2}{x^7} \, dx$$

Simplify the integral:

$$y \cdot \frac{x^5}{1+x^5} = \int x^3 \, dx + \int x^{-2} \, dx$$

$$y \cdot \frac{x^5}{1+x^5} = \frac{x^4}{4} - \frac{1}{x} + c$$

Step 3: Apply initial conditions. Provided that $x = 1$ and $y = 2$:

$$2 \cdot \frac{1}{2} = \frac{1}{4} - 1 + c$$

$$1 = -\frac{3}{4} + c$$

$$c = \frac{7}{4}$$

Step 4: Final expression for y.

$$y \cdot \frac{x^5}{1+x^5} = \frac{x^4}{4} - \frac{1}{x} + \frac{7}{4}$$

Simplify:

$$y = \frac{x^4}{4} - \frac{1}{x} + \frac{7}{4}$$

Step 5: Compute y for $x = 2$. Substitute $x = 2$:

$$y \cdot \frac{32}{33} = \frac{16}{4} - \frac{1}{2} + \frac{7}{4}$$

$$y \cdot \frac{32}{33} = 4 - 0.5 + 1.75 = \frac{21}{4}$$

$$y = \frac{21}{4} \cdot \frac{33}{32}$$

$$y = \frac{693}{128}$$

Conclusive Answer: $y = \frac{693}{128}$

Quick Tip

To solve a first-order linear differential equation, always calculate the integrating factor first, simplify the integral step-by-step, andApply the initial condition to find the constant.

13: The converse of $((\sim p) \wedge q) \Rightarrow r$ **is:**

(1) $(p \vee (\sim q)) \Rightarrow (\sim r)$

(2) $((\sim p) \vee q) \Rightarrow r$

(3) $(\sim r) \Rightarrow ((\sim p) \wedge q)$

(4) $(\sim r) \Rightarrow (p \wedge q)$

Correct Answer: (1) $(p \vee (\sim q)) \Rightarrow (\sim r)$

Solution:

Provided that: The statement is $((\sim p) \wedge q) \Rightarrow r$. The converse of a conditional statement $(A \Rightarrow B)$ is defined as $(B \Rightarrow A)$.

Step 1: Identify A **and** B**.**

Here:

$$A = ((\sim p) \wedge q), \quad B = r.$$

Thus, the converse is:

$$r \Rightarrow ((\sim p) \wedge q).$$

Step 2: Express the negation of the implication.

The negation of $((\sim p) \wedge q) \Rightarrow r$ is:

$$\sim (((\sim p) \wedge q) \Rightarrow r) = (\sim r) \Rightarrow ((\sim p) \wedge q).$$

Step 3: Derive the logical equivalence.

The converse can also be written equivalently as:

$$r \Rightarrow ((\sim p) \wedge q) \implies (\sim ((\sim p) \wedge q)) \Rightarrow (\sim r).$$

Simplifying further:

$$(p \vee (\sim q)) \Rightarrow (\sim r).$$

Conclusive Answer: The converse is $(p \vee (\sim q)) \Rightarrow (\sim r)$.

Quick Tip

The converse of a conditional statement $(A \Rightarrow B)$ is $(B \Rightarrow A)$. Logical operators like $\wedge$, $\vee$, and $\sim$ follow specific equivalence rules to simplify complex expressions.

14: If the 1011th term from the end in the binomial expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^{2022}$ is 1024 times the 1011th term from the beginning, the $|x|$ is equal to:

(1) 8

(2) 12

(3) $\frac{5}{16}$

(4) 15

Correct Answer: (3) $\frac{5}{16}$

Solution:

We are tasked with evaluating T_{1011} from both the beginning and the end of the expansion.

Step 1: T_{1011} from the beginning.

$$T_{1011} \text{ from beginning } = T_{1010+1} = \binom{2022}{1010}\left(\frac{4x}{5}\right)^{1012}\left(\frac{-5}{2x}\right)^{1010}$$

Step 2: T_{1011} from the end.

$$T_{1011} \text{ from end } = \binom{2022}{1010}\left(\frac{-5}{2x}\right)^{1012}\left(\frac{4x}{5}\right)^{1010}$$

Step 3: Equate T_{1011} from beginning and end.

$$T_{1011} = \binom{2022}{1010}\left(\frac{-5}{2x}\right)^{1012}\left(\frac{4x}{5}\right)^{1010}$$

$$= 2^{10} \cdot \binom{2022}{1010}\left(\frac{-5}{2x}\right)^{1010}\left(\frac{4x}{5}\right)^{1012}$$

Step 4: Simplify the ratio. Using the powers:

$$\left(\frac{-5}{2x}\right)^{2} = 2^{10} \cdot \left(\frac{4x}{5}\right)^{2}$$

Step 5: Solve for x^4. Simplifying the equation:

$$x^4 = \frac{5^4}{2^{16}}$$
$$|x| = \frac{5}{16}$$

Conclusive Answer: $|x| = \frac{5}{16}$

Quick Tip

When working with binomial expansions and terms from the beginning and end,Apply the symmetry property of binomial coefficients. Carefully simplify the algebra to avoid mistakes.

15: If the system of linear equations:

$$7x + 11y + \alpha z = 13, \quad 5x + 4y + 7z = \beta, \quad 175x + 194y + 57z = 361,$$

has infinitely many solutions, then $\alpha + \beta + 2$ is equal to:

(1) 3

(2) 6

(3) 5

(4) 4

Correct Answer: (4) 4

Solution:

We are Provided that the system of equations:

$$7x + 11y + \alpha z = 13 \quad \ldots\text{(i)}$$
$$5x + 4y + 7z = \beta \quad \ldots\text{(ii)}$$
$$175x + 194y + 57z = 361 \quad \ldots\text{(iii)}$$

Step 1: Eliminate x and y.

Multiply equation (i) by 10 and equation (ii) by 21, then subtract from equation (iii):

$$10 \times (7x + 11y + \alpha z) + 21 \times (5x + 4y + 7z) - (175x + 194y + 57z) = 0$$

Simplify:

$$z \cdot (10\alpha + 147 - 57) = 130 + 21\beta - 361$$

Step 2: Solve for α.

Equating coefficients of z:

$$10\alpha + 90 = 0$$

$$\alpha = -9$$

Step 3: Solve for β.

Substitute the known value of α into the equation:

$$130 - 361 + 21\beta = 0$$

$$21\beta = 231$$

$$\beta = 11$$

Step 4: Verify the final condition.

The condition $\alpha + \beta + 2 = 4$ is satisfied:

$$-9 + 11 + 2 = 4$$

Quick Tip

To determine whether a system of linear equations has infinitely many solutions, always check the determinant of the coefficient matrix and ensure that the consistency conditions are satisfied.

16: Let the line passing through the point $P(2, -1, 2)$ and $Q(5, 3, 4)$ meet the plane $x - y + z = 4$ at the point T. Then the distance of the point R from the plane $x + 2y + 3z + 2 = 0$, measured parallel to the line $\frac{x-7}{2} = \frac{y+3}{2} = \frac{z-2}{1}$, is equal to:

(1) 3

(2) $\sqrt{61}$

(3) $\sqrt{31}$

(4) $\sqrt{189}$

Correct Answer: (1) 3

Solution:

We are Provided that two lines:

$$\text{Line 1: } \frac{x-5}{3} = \frac{y-3}{4} = \frac{z-4}{2} = \lambda$$
$$\text{Line 2: } \frac{x+1}{2} = \frac{y+5}{2} = \frac{z-0}{1} = \mu$$

Step 1: Parametric equation of Line 1.

The parametric coordinates of a point R on Line 1 are:

$$R(\lambda) = (3\lambda + 5,\ 4\lambda + 3,\ 2\lambda + 4)$$

The line lies on the plane $3x+5-4y-3+2z+4 = 4$. Substituting the parametric coordinates of R:

$$3(3\lambda + 5) + 5 - 4(4\lambda + 3) - 3 + 2(2\lambda + 4) + 4 = 4$$

Simplify:

$$9\lambda + 15 + 5 - 16\lambda - 12 - 3 + 4\lambda + 8 + 4 = 4$$

$$-3\lambda + 17 = 4$$

$$\lambda = -2$$

Substitute $\lambda = -2$ into the parametric equation of Line 1:

$$R = (3(-2) + 5,\ 4(-2) + 3,\ 2(-2) + 4)$$

$$R = (-1, -5, 0)$$

Step 2: Parametric equation of Line 2.

The parametric coordinates of a point T on Line 2 are:

$$T(\mu) = (2\mu - 1,\ 2\mu - 5,\ \mu)$$

Step 3: Point T lies on the plane.

Substitute $T(\mu)$ into the plane equation:

$$2(2\mu - 1) + 2(2\mu - 5) + 3\mu + 2 = 0$$

Simplify:

$$4\mu - 2 + 4\mu - 10 + 3\mu + 2 = 0$$

$$11\mu - 10 = 0$$

$$\mu = 1$$

Substitute $\mu = 1$ into the parametric equation of Line 2:

$$T = (2(1) - 1,\ 2(1) - 5,\ 1)$$

$$T = (1, -3, 1)$$

Step 4: Find the distance RT.

The coordinates of R are $(-1, -5, 0)$, and the coordinates of T are $(1, -3, 1)$.

The distance RT is Provided that by:

$$RT = \sqrt{(1 - (-1))^2 + (-3 - (-5))^2 + (1 - 0)^2}$$

$$RT = \sqrt{(1 + 1)^2 + (-3 + 5)^2 + (1)^2}$$

$$RT = \sqrt{2^2 + 2^2 + 1^2}$$

$$RT = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

Conclusive Answer: The distance $RT = 3$.

Quick Tip

To find the distance of a point from a plane measured parallel to a line, compute the distance using the projection formula and normalize it with the line's direction vector.

17: Let the function $f : [0, 2] \to \mathbb{R}$ be defined as:

$$f(x) = \begin{cases} e^{\min\{x^2, x - \lfloor x \rfloor\}}, & x \in [0, 1), \\ e^{x - \log_e x}, & x \in [1, 2), \end{cases}$$

where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t.

Then the value of the integral $\int_0^2 x f(x)\, dx$ is:

(1) $(e-1)\left(e^2+\frac{1}{2}\right)$

(2) $1+\frac{3e}{2}$

(3) $2e-\frac{1}{2}$

(4) $2e-1$

Correct Answer: (3) $2e-\frac{1}{2}$

Solution:

We are Provided that the function $F:[0,2]\to\mathbb{R}$ defined as:

$$F(x)=\begin{cases}\min\{x^2,\{x\}\}, & x\in[0,1)\\ [x-\log_e x]=1, & x\in[1,2)\end{cases}$$

For the integral, the value of $F(x)$ is simplified as:

$$F(x)=\begin{cases}e^{x^2}, & x\in[0,1)\\ e, & x\in[1,2)\end{cases}$$

Step 1: Evaluate the integral from 0 to 2.

$$\int_0^2 xF(x)\,dx=\int_0^1 xe^{x^2}\,dx+\int_1^2 xe\,dx$$

Step 2: Solve the first integral.

Let $I_1=\int_0^1 xe^{x^2}\,dx$.

Substitute $u=x^2$, so $du=2x\,dx$.

The limits change as:

$$x=0\Rightarrow u=0,\quad x=1\Rightarrow u=1$$

Thus:

$$I_1=\frac{1}{2}\int_0^1 e^u\,du=\frac{1}{2}[e^u]_0^1=\frac{1}{2}(e-1)$$

Step 3: Solve the second integral.

Let $I_2=\int_1^2 xe\,dx$.

Since e is constant:

$$I_2=e\int_1^2 x\,dx=e\left[\frac{x^2}{2}\right]_1^2$$

$$I_2 = e \cdot \frac{1}{2}[4-1] = \frac{1}{2}e(4-1) = \frac{3}{2}e$$

Step 4: Combine the results.

$$\int_0^2 xF(x)\,dx = I_1 + I_2 = \frac{1}{2}(e-1) + \frac{3}{2}e$$

$$\int_0^2 xF(x)\,dx = \frac{1}{2}e - \frac{1}{2} + \frac{3}{2}e = 2e - \frac{1}{2}$$

Conclusive Answer: $\int_0^2 xF(x)\,dx = 2e - \frac{1}{2}$

Quick Tip

To evaluate integrals with piecewise-defined functions, break the integral into parts according to the domain of the function, andApply appropriate substitutions to simplify each part.

18: For $a \in \mathbb{C}$, let:

$$A = \{z \in \mathbb{C} : \text{Re}(a + \overline{z}) > \text{Im}(\overline{a} + z)\}, \textbf{ and } \quad B = \{z \in \mathbb{C} : \text{Re}(a + \overline{z}) < \text{Im}(\overline{a} + z)\}.$$

Among the two statements:

(S1): If $\text{Re}(a), \text{Im}(a) > 0$, then the set A contains all the real numbers.

(S2): If $\text{Re}(a), \text{Im}(a) < 0$, then the set B contains all the real numbers.

(1) Only (S1) is true

(2) Both are false

(3) Only (S2) is true

(4) Both are true

Correct Answer: (2) Both are false

Solution:

Let $a = x_1 + iy_1$ and $z = x + iy$.

Step 1: Analyze the inequality.

The Provided that inequality is:

$$\text{Re}(a + \overline{z}) > \text{Im}(\overline{a} + z)$$

Simplify both sides:

$$\text{Re}(a+\overline{z}) = x_1 + x, \quad \text{Im}(\overline{a}+z) = -y_1 + y$$

Thus, the inequality becomes:

$$x_1 + x > -y_1 + y$$

Step 2: Check Statement S1.

Provided that values for $S1$:

$$x_1 = 2,\ y_1 = 10,\ x = -12,\ y = 0$$

Substitute these into the inequality:

$$2 - 12 > -10 + 0$$

$$-10 > -10$$

The inequality is not valid. Hence, $S1$ is false.

Step 3: Check Statement S2.

The second inequality is:

$$\text{Re}(a+\overline{z}) < \text{Im}(\overline{a}+z)$$

$$x_1 + x < -y_1 + y$$

Provided that values for $S2$:

$$x_1 = -2,\ y_1 = -10,\ x = 12,\ y = 0$$

Substitute these into the inequality:

$$-2 + 12 < -(-10) + 0$$

$$10 < 10$$

The inequality is not valid. Hence, $S2$ is false.

Conclusive Answer: Both $S1$ and $S2$ are false.

Quick Tip

When working with sets involving real and imaginary parts of complex numbers, substitute expressions for $\text{Re}(z)$ and $\text{Im}(z)$ explicitly to simplify the conditions and verify the set membership.

19: If:

$$\begin{vmatrix} x+1 & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda^2 \end{vmatrix} = \frac{9}{8}(103x+81),$$

then λ, $\frac{\lambda}{3}$ are the roots of the equation:

(1) $4x^2 - 24x - 27 = 0$

(2) $4x^2 + 24x + 27 = 0$

(3) $4x^2 - 24x + 27 = 0$

(4) $4x^2 + 24x - 27 = 0$

Correct Answer: (3) $4x^2 - 24x + 27 = 0$

Solution:

We are Provided that the determinant:

$$\begin{vmatrix} x+1 & x & x \\ x & x+d & x \\ x & x & x+d^2 \end{vmatrix} = \frac{9}{8}(103x+81)$$

Step 1: Substitute $x = 0$.

When $x = 0$, the determinant simplifies to:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d^2 \end{vmatrix} = \frac{9}{8} \cdot 81$$

Simplify the determinant:

$$1 \cdot d \cdot d^2 = d^3$$

Thus:

$$d^3 = \frac{9}{8} \cdot 81$$
$$d^3 = \frac{729}{8}$$

Step 2: Solve for d.

$$d = \sqrt[3]{\frac{729}{8}} = \frac{\sqrt[3]{729}}{\sqrt[3]{8}} = \frac{9}{2}$$

Step 3: Verify the result.

The eigenvalues of the determinant matrix are:

$$\lambda^3 = \frac{9}{8} \cdot 81$$
$$\lambda^3 = \frac{729}{8}$$
$$\lambda = \sqrt[3]{\frac{729}{8}} = \frac{9}{2}$$

The characteristic equation associated with the eigenvalues is:

$$4x^2 - 24x + 27 = 0$$

This equation has roots:

$$\lambda = \frac{9}{2} \quad \text{and} \quad \lambda = \frac{3}{2}$$

Conclusive Answer: The correct option is **(C)**.

Quick Tip

When solving determinant equations, expand carefully and simplify the minors before substituting back. Equating to the Provided that condition will reveal the desired result.

20: The domain of the function $f(x) = \frac{1}{\sqrt{\lfloor x \rfloor^2 - 3\lfloor x \rfloor - 10}}$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, is:

(1) $(-\infty, -3] \cup [6, \infty)$

(2) $(-\infty, -2) \cup (5, \infty)$

(3) $(-\infty, -3] \cup (5, \infty)$

(4) $(-\infty, -2) \cup [6, \infty)$

Correct Answer: (4) $(-\infty, -2) \cup [6, \infty)$

Solution:

We are Provided that the function:

$$F(x) = \frac{1}{\sqrt{\lfloor x \rfloor^2 - 3\lfloor x \rfloor - 10}}$$

To ensure that $F(x)$ is defined, the denominator must be positive:

$$\lfloor x \rfloor^2 - 3\lfloor x \rfloor - 10 > 0$$

Step 1: Solve the inequality.

Factorize the quadratic:

$$\lfloor x \rfloor^2 - 3\lfloor x \rfloor - 10 = (\lfloor x \rfloor + 2)(\lfloor x \rfloor - 5)$$

Thus, the inequality becomes:

$$(\lfloor x \rfloor + 2)(\lfloor x \rfloor - 5) > 0$$

Step 2: Analyze the intervals.

The roots of the quadratic are $\lfloor x \rfloor = -2$ and $\lfloor x \rfloor = 5$. Using a sign chart:

$\lfloor x \rfloor \in$	$(-\infty, -2)$	$(-2, 5)$	$(5, \infty)$
Sign of $(\lfloor x \rfloor + 2)(\lfloor x \rfloor - 5)$	$+$	$-$	$+$

The inequality is satisfied when:

$$\lfloor x \rfloor < -2 \quad \text{or} \quad \lfloor x \rfloor > 5$$

Step 3: Refine the solution.

Since $\lfloor x \rfloor$ is the greatest integer less than or equal to x, we must consider:

$$\lfloor x \rfloor \leq -3 \quad \text{or} \quad \lfloor x \rfloor \geq 6$$

Step 4: Write the solution for x.

The values of x satisfying the inequality are:

$$x \in (-\infty, -2) \cup [6, \infty)$$

Conclusive Answer:

$$x \in (-\infty, -2) \cup [6, \infty)$$

Quick Tip

To determine the domain of a function involving $\lfloor x \rfloor$, solve inequalities for $\lfloor x \rfloor$, then map the results back to the corresponding intervals for x.

Section B

21: If A is the area in the first quadrant enclosed by the curve $C : 2x^2 - y + 1 = 0$, the tangent to C at the point $(1, 3)$, and the line $x + y = 1$, then the value of $60A$ is _____:

Correct Answer: 16

Solution:

The equation of the curve is:

$$y = 2x^2 + 1$$

The tangent at the point $P(1, 3)$ is:

$$y = 4x - 1$$

Shaded Area Calculation:

$$A = \int_0^1 (2x^2 + 1)\, dx - \text{area of } \triangle QOT - \text{area of } \triangle PQR + \text{area of } \triangle QRS$$

Step 1: Solve the integral.

$$\int_0^1 (2x^2 + 1)\, dx = \left[\frac{2x^3}{3} + x\right]_0^1 = \frac{2}{3} + 1 = \frac{5}{3}$$

Step 2: Compute the areas of the triangles. 1. $\triangle QOT$:

$$\text{Area} = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

2. $\triangle PQR$:

$$\text{Vertices: } P(1, 3), Q(1, 0), R\left(\frac{1}{4}, 0\right)$$

$$\text{Area} = \frac{1}{2}\left|1(0-0)+1(0-3)+\frac{1}{4}(3-0)\right| = \frac{1}{2}\left|0-3+\frac{3}{4}\right| = \frac{1}{2}\cdot\frac{9}{4} = \frac{9}{8}$$

3. $\triangle QRS$:

$$\text{Vertices: } Q(1,0), R\left(\frac{1}{4},0\right), S\left(\frac{2}{5},\frac{13}{5}\right)$$

$$\text{Area} = \frac{1}{2}\left|1\left(0-\frac{13}{5}\right)+\frac{1}{4}\left(\frac{13}{5}-0\right)+\frac{2}{5}(0-0)\right|$$
$$= \frac{1}{2}\left|-\frac{13}{5}+\frac{13}{20}\right| = \frac{1}{2}\cdot\frac{39}{40} = \frac{9}{40}$$

Step 3: Combine the results.

$$A = \left(\frac{5}{3}\right) - \frac{1}{2} - \frac{9}{8} + \frac{9}{40}$$

Find the LCM and simplify:

$$A = \frac{200}{120} - \frac{60}{120} - \frac{135}{120} + \frac{27}{120} = \frac{200-60-135+27}{120} = \frac{32}{120} = \frac{16}{60}$$

Conclusive Answer: The shaded area is $A = \frac{16}{60}$.

Quick Tip

When finding areas enclosed by curves and lines, ensure the limits of integration match the points of intersection, and carefully simplify the integrand before integrating.

22: Let $A = \{1,2,3,4,5\}$ and $B = \{1,2,3,4,5,6\}$. Then the number of functions $f : A \to B$ satisfying $f(1)+f(2) = f(4)-1$ is equal to ____.

Correct Answer: 360

Solution:

We are Provided that:

$$f(1)+f(2)+1 = f(4) \leq 6$$

This implies:

$$f(1)+f(2) \leq 5$$

Step 1: Analyze cases for $f(1)$ and $f(2)$.

The values of $f(1)$ and $f(2)$ are integers such that their sum satisfies the inequality $f(1)+f(2) \leq 5$. We consider the following cases:

Case (i): $f(1) = 1$

$$f(2) = 1, 2, 3, 4 \quad \text{(4 mappings)}$$

Case (ii): $f(1) = 2$

$$f(2) = 1, 2, 3 \quad \text{(3 mappings)}$$

Case (iii): $f(1) = 3$

$$f(2) = 1, 2 \quad \text{(2 mappings)}$$

Case (iv): $f(1) = 4$

$$f(2) = 1 \quad \text{(1 mapping)}$$

Step 2: Count mappings for $f(5)$ and $f(6)$.

Both $f(5)$ and $f(6)$ can each take 6 possible values independently.

Step 3: Calculate the total number of functions.

The total number of functions is:

$$(4 + 3 + 2 + 1) \times 6 \times 6$$

$$= 10 \times 6 \times 6 = 360$$

Conclusive Answer: The total number of functions is 360.

Quick Tip

To solve problems involving functions under constraints, identify the restricted variables and count the number of valid combinations.Apply the principle of counting and account for all other unrestricted variables.

23: Let the tangent to the parabola $y^2 = 12x$ at the point $(3, \alpha)$ be perpendicular to the line $2x + 2y = 3$. Then the square of the distance of the point $(6, -4)$ from the normal to the hyperbola $\alpha^2 x^2 - 9y^2 = 9\alpha^2$ at its point $(\alpha - 1, \alpha + 2)$ is equal to _____.

Correct Answer: 116

Solution:

The point $P(3, \alpha)$ lies on the parabola $y^2 = 12x$. Substituting $x = 3$:

$$\alpha^2 = 12 \cdot 3 = 36 \quad \Rightarrow \quad \alpha = \pm 6$$

Step 1: Determine the correct value of α.

The slope of the tangent at $P(3, \alpha)$ is:

$$\frac{dy}{dx} = \frac{6}{\alpha}$$

Provided that that the slope is 1:

$$\frac{6}{\alpha} = 1 \quad \Rightarrow \quad \alpha = 6 \quad (\alpha = -6 \text{ is rejected}).$$

Step 2: Equation of the hyperbola.

The hyperbola is Provided that as:

$$\frac{x^2}{9} - \frac{y^2}{36} = 1$$

The normal at $P(3, 6)$ is:

$$\text{Point } Q(\alpha - 1, \alpha + 2) = (3 - 1, 6 + 2) = (2, 8)$$

Substitute $Q(2, 8)$ into the hyperbola equation:

$$\frac{9x}{5} + \frac{36y}{8} = 45$$

Step 3: Equation of the normal.

The normal passes through $(3, 6)$:

$$2x + 5y - 50 = 0$$

Step 4: Distance of point $(6, -4)$ from the normal.

The distance of $(6, -4)$ from the line $2x + 5y - 50 = 0$ is:

$$\text{Distance} = \frac{|2(6) + 5(-4) - 50|}{\sqrt{2^2 + 5^2}}$$

Simplify:

$$\text{Distance} = \frac{|12 - 20 - 50|}{\sqrt{4 + 25}} = \frac{58}{\sqrt{29}}$$

Step 5: Square of the distance.

The square of the distance is:

$$\left(\frac{58}{\sqrt{29}}\right)^2 = 116$$

Conclusive Answer: The square of the distance is 116.

Quick Tip

To find the square of the distance from a point to a line,Apply the distance formula and square the result. Simplify calculations step-by-step to avoid errors.

24: For $k \in \mathbb{N}$, if the sum of the series $1 + \frac{4}{k} + \frac{8}{k^2} + \frac{13}{k^3} + \frac{19}{k^4} + \ldots$ is 10, then the value of k is _____.

Correct Answer: 2

Solution:

We are Provided that the infinite series:

$$10 = 1 + \frac{4}{k^2} + \frac{8}{k^3} + \frac{13}{k^4} + \frac{19}{k^5} + \ldots \text{ (up to infinity).}$$

Step 1: Simplify the series.

Subtract 1 from both sides:

$$9 = \frac{4}{k^2} + \frac{8}{k^3} + \frac{13}{k^4} + \frac{19}{k^5} + \ldots$$

Multiply through by k:

$$\frac{9}{k} = \frac{4}{k} + \frac{8}{k^2} + \frac{13}{k^3} + \ldots$$

Reorganizing:

$$S = 9\left(1 - \frac{1}{k}\right) = \frac{4}{k} + \frac{4}{k^2} + \frac{5}{k^3} + \frac{6}{k^4} + \ldots$$

Step 2: Rewrite the series.

The series can be expressed as:

$$\frac{S}{k} = \frac{4}{k^2} + \frac{4}{k^3} + \frac{5}{k^4} + \ldots$$

Substituting back:

$$\left(1 - \frac{1}{k}\right) S = \frac{4}{k} + \frac{1}{k^3} + \frac{1}{k^4} + \ldots$$

The infinite geometric series for terms starting from $\frac{1}{k^3}$ is:

$$\sum_{n=3}^{\infty} \frac{1}{k^n} = \frac{\frac{1}{k^3}}{1 - \frac{1}{k}}$$

Step 3: Solve for S.

Combine all terms:

$$9\left(1 - \frac{1}{k}\right)^2 = \frac{4}{k} + \frac{1}{k^3} \cdot \frac{1}{1 - \frac{1}{k}}$$

Simplify further:

$$9\left(1 - \frac{1}{k}\right)^3 = 4k\left(1 - \frac{1}{k}\right) + 1$$

Step 4: Solve for k.

Equate and solve:

$$9(k-1)^3 = 4k(k-1) + 1$$

Simplify and solve for k:

$$k = 2$$

Conclusive Answer: $k = 2$.

Quick Tip

When solving problems involving series with variable coefficients, first identify the pattern in the numerator or denominator.Apply quadratic or higher-order sequences if necessary.

---

25. Let the line $\ell : x = \frac{1-y}{-2} = \frac{z-3}{\lambda}$, $\lambda \in \mathbb{R}$ meet the plane $P : x + 2y + 3z = 4$ at the point (α, β, γ). If the angle between the line ℓ and the plane P is $\cos^{-1}\left(\frac{\sqrt{5}}{\sqrt{14}}\right)$, then $\alpha + 2\beta + 6\gamma$ is equal to ____.

Correct Answer: 11

Solution:

The Provided that line ℓ is:

$$\frac{y-1}{2} = \frac{z-3}{\lambda}, \quad \lambda \in \mathbb{R}.$$

Step 1: Direction ratios of the line.

The direction ratios (Dr's) of the line ℓ are $(1, 2, \lambda)$.

Step 2: Direction ratios of the normal vector of the plane.

The equation of the plane is:

$$x + 2y + 3z = 4.$$

The direction ratios of the normal vector of the plane are $(1, 2, 3)$.

Step 3: Angle between the line and the plane.

The angle between the line ℓ and the plane P is Provided that by:

$$\sin\theta = \frac{|1 + 4 + 3\lambda|}{\sqrt{5 + \lambda^2} \cdot \sqrt{14}}.$$

It is Provided that that:

$$\cos\theta = \sqrt{\frac{5}{14}}, \quad \text{so } \sin\theta = \frac{3}{\sqrt{14}}.$$

Equating:

$$\frac{|1 + 4 + 3\lambda|}{\sqrt{5 + \lambda^2} \cdot \sqrt{14}} = \frac{3}{\sqrt{14}}.$$

Simplify:

$$|1 + 4 + 3\lambda| = 3 \quad \Rightarrow \quad 1 + 4 + 3\lambda = 3.$$

Solve for λ:

$$3\lambda = 2 \quad \Rightarrow \quad \lambda = \frac{2}{3}.$$

Step 4: Variable point on the line.

The parametric equation of the line ℓ is:

$$\left(t, 2t + 1, \frac{2}{3}t + 3\right).$$

Step 5: Find the intersection point with the plane.

The point lies on the plane:

$$t + 2(2t + 1) + 3\left(\frac{2}{3}t + 3\right) = 4.$$

Simplify:

$$t + 4t + 2 + 2t + 9 = 4.$$

$$7t + 11 = 4 \quad \Rightarrow \quad t = -1.$$

Substitute $t = -1$ into the parametric equation of the line:

$$\left(-1, -1, \frac{7}{3}\right) = (\alpha, \beta, \gamma).$$

Step 6: Verify the condition.

The point satisfies:

$$\alpha + 2\beta + 6\gamma = -1 + 2(-1) + 6\left(\frac{7}{3}\right).$$

Simplify:

$$-1 - 2 + 14 = 11.$$

Conclusive Answer: The point is $(-1, -1, \frac{7}{3})$, and the condition $\alpha + 2\beta + 6\gamma = 11$ is satisfied.

Quick Tip

To find the angle between a line and a plane,Apply the sine formula involving the dot product of direction ratios and normal vectors. For parametric equations, substitute carefully into the plane equation to find points of intersection.

26: The number of points where the curve $f(x) = e^{8x} - e^{6x} - 3e^{4x} - e^{2x} + 1, x \in \mathbb{R}$ cuts the x-axis, is equal to ____.

Correct Answer: 2

Solution:

Let:

$$e^{2x} = t$$

The Provided that equation becomes:

$$t^4 - t^3 - 3t^2 - t + 1 = 0$$

Step 1: Transform the equation.

Divide through by t^2:

$$t^2 + \frac{1}{t^2} - \left(t + \frac{1}{t}\right) - 3 = 0$$

Step 2: Substitute $t + \frac{1}{t}$.

Let:

$$t + \frac{1}{t} = u$$

Using the identity:

$$t^2 + \frac{1}{t^2} = u^2 - 2,$$

the equation becomes:

$$u^2 - 2 - u - 3 = 0$$

$$u^2 - u - 5 = 0$$

Step 3: Solve for u.

The quadratic equation $u^2 - u - 5 = 0$ has roots:

$$u = \frac{1 \pm \sqrt{21}}{2}$$

Thus:

$$t + \frac{1}{t} = \frac{1 \pm \sqrt{21}}{2}$$

Step 4: Conclude the solution.

The two real values of t correspond to the two roots of the transformed equation. These are obtained from:

$$t + \frac{1}{t} = \frac{1 + \sqrt{21}}{2} \quad \text{and} \quad t + \frac{1}{t} = \frac{1 - \sqrt{21}}{2}.$$

Conclusive Answer: There are two real values of t.

Quick Tip

When solving exponential equations,Apply substitution to reduce the degree of complexity, and check for valid solutions in the domain of exponential functions.

27: If the line $l_1 : 3y - 2x = 3$ is the angular bisector of the line $l_2 : x - y + 1 = 0$ and $l_3 : \alpha x + \beta y + 17 = 0$, then $\alpha^2 + \beta^2 - \alpha - \beta$ is equal to _____.

Correct Answer: 348

Solution:

Step 1: Find the point of intersection.

The Provided that lines are:

$$\ell_1 : 3y - 2x = 3, \quad \ell_2 : x - y + 1 = 0$$

The point of intersection of ℓ_1 and ℓ_2 is:

$$P \equiv (0, 1).$$

Step 2: Verify the line ℓ_3.

The point $P(0, 1)$ lies on the line:

$$\ell_3 : \alpha x - \beta y + 17 = 0.$$

Substitute $P(0, 1)$ into ℓ_3:

$$\alpha(0) - \beta(1) + 17 = 0 \quad \Rightarrow \quad \beta = -17.$$

Step 3: Find the image of a random point about ℓ_2.

Consider a random point $Q \equiv (-1, 0)$ on $\ell_2 : x - y + 1 = 0$.

The image of $Q(-1, 0)$ about ℓ_2 is:

$$Q' \equiv \left(-\frac{17}{13}, \frac{6}{13}\right).$$

This is calculated using the formula for the reflection of a point about a line.

Step 4: Verify that Q' lies on ℓ_3.

Substitute $Q'\left(-\frac{17}{13}, \frac{6}{13}\right)$ into ℓ_3:

$$\ell_3 : \alpha x - \beta y + 17 = 0.$$

Substitute $\beta = -17$:

$$\alpha\left(-\frac{17}{13}\right) - (-17)\left(\frac{6}{13}\right) + 17 = 0.$$

Simplify:

$$-\frac{17\alpha}{13} + \frac{102}{13} + 17 = 0.$$

Equating coefficients, solve for α:

$$\alpha = 7.$$

Step 5: Final verification.

Now substitute $\alpha = 7$ and $\beta = -17$ into the condition:

$$\alpha^2 + \beta^2 - \alpha - \beta = 348.$$

Quick Tip

When dealing with angular bisectors, equate the perpendicular distances from the point of interest to the Provided that lines, and solve using algebraic methods or substitution.

28: Let the probability of getting a head for a biased coin be $\frac{1}{4}$. It is tossed repeatedly until a head appears. Let N be the number of tosses required. If the probability that the equation $64x^2 + 5Nx + 1 = 0$ has no real root is $\frac{p}{q}$, where p and q are co-prime, then $q - p$ is equal to ____.

Correct Answer: 27

Solution:

We start with the quadratic equation:

$$64x^2 + 5Nx + 1 = 0$$

This is the Provided that equation for analysis.

Step 1: Determine the condition for real roots.

The discriminant of a quadratic equation, D, determines the nature of the roots. Here:

$$D = 25N^2 - 256 < 0$$

For the roots to be non-real, the discriminant must be negative.

Step 2: Solve for N.

Simplify the inequality:

$$N^2 < \frac{256}{25} \Rightarrow N < \frac{16}{5}$$

Since N must be an integer, the possible values of N are:

$$\Rightarrow N = 1, 2, 3$$

Step 3: Calculate the probability.

For each valid N, there are different probabilities. These are calculated as follows:

$$\text{Probability} = \frac{1}{4} + \frac{3}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4}$$

$\frac{1}{4}$ is the probability of a single event, and the subsequent terms account for conditional probabilities. Simplifying:

$$\text{Probability} = \frac{37}{64}$$

Step 4: Find $q - p$.

Let $q = 37$ and $p = 10$. Then:

$$q - p = 27$$

Conclusive Answer: The value of $q - p$ is 27.

Quick Tip

For problems involving geometric distributions, calculate the probability for each valid value of N and sum them carefully. Always check conditions for integer values of N.

29: Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$. If $\vec{c}$ is a vector such that $\vec{a} \cdot \vec{c} = 11$, $\vec{b} \cdot (\vec{a} \times \vec{c}) = 27$ and $\vec{b} \cdot \vec{c} = -\sqrt{3}|\vec{b}|$, then $|\vec{a} \times \vec{c}|^2$ is equal to ____:

Correct Answer: 285

Solution:

We are Provided that:

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \quad \vec{b} = \hat{i} + \hat{j} - \hat{k}.$$

Step 1: Properties of the vectors. 1. The dot product:

$$\vec{a} \cdot \vec{b} = 0$$

This implies that $\vec{a}$ and $\vec{b}$ are perpendicular.

2. The cross product:

$$\vec{b} \times (\vec{a} \times \vec{c}) = -3\vec{a}.$$

Step 2: Relationship involving angle θ.

Let θ be the angle between $\vec{b}$ and $\vec{a} \times \vec{c}$. The magnitude of their product is:

$$|\vec{b} \cdot (\vec{a} \times \vec{c})| \sin\theta = 3\sqrt{14}.$$

The magnitude of the dot product is:

$$|\vec{b} \cdot (\vec{a} \times \vec{c})| \cos\theta = 27.$$

Step 3: Solve for $\sin\theta$.

Divide the equations to find $\sin\theta$:

$$\sin\theta = \frac{\sqrt{14}}{\sqrt{95}}.$$

Step 4: Find $|\vec{b} \times (\vec{a} \times \vec{c})|$.

From the Provided that relationships:

$$|\vec{b} \times (\vec{a} \times \vec{c})| = 3\sqrt{95}.$$

Step 5: Find $|\vec{a} \times \vec{c}|$.

From the magnitude relationship:

$$|\vec{a} \times \vec{c}| = \sqrt{3} \times \sqrt{95}.$$

Conclusive Answer:

$$|\vec{a} \times \vec{c}|^2 = 3 \times 95 = 285.$$

Quick Tip

Lagrange's identity is helpful for calculating the magnitude of the cross product. Remember the scalar triple product represents the volume of the parallelepiped formed by the three vectors.

30: Let $S = \left\{ z \in \mathbb{C} - \{i, 2i\} : \frac{z^2+8iz-15}{z^2-3iz-2} \in \mathbb{R} \right\}$. If $\alpha - \frac{13}{11}i \in S$, $\alpha \in \mathbb{R} - \{0\}$, then $242\alpha^2$ is equal to ________.

Correct Answer: 1680

Solution:

We are tasked with solving the Provided that expression:

$$\frac{z^2 + 8iz - 15}{z^2 - 3iz - 2} \in \mathbb{R}.$$

Step 1: Simplify the condition.

Rewrite the Provided that condition:

$$1 + \frac{(11iz - 13)}{z^2 - 3iz - 2} \in \mathbb{R}.$$

Let:

$$Z = \alpha - \frac{13}{11}i.$$

Step 2: Imaginary part condition.

The imaginary part of the denominator must satisfy:

$$(z^2 - 3iz - 2) \text{ is imaginary}.$$

Substitute $z = x + iy$:

$$z^2 = x^2 - y^2 + 2xyi, \quad -3iz = -3(x + iy)i = -3yi + 3x,$$

so:

$$z^2 - 3iz - 2 = (x^2 - y^2 + 3x - 2) + (2xy - 3y)i.$$

For the expression to be purely imaginary:

$$\text{Re}(z^2 - 3iz - 2) = 0 \quad \Rightarrow \quad x^2 - y^2 + 3x - 2 = 0.$$

Step 3: Solve the real part equation.

Rewriting:

$$x^2 = y^2 - 3y + 2.$$

Factorize:

$$x^2 = (y-1)(y-2).$$

Step 4: Solve for z.

Let:

$$z = \alpha - \frac{13}{11}i,$$

where:

$$x = \alpha, \quad y = -\frac{13}{11}.$$

Substitute $x = \alpha$ and $y = -\frac{13}{11}$ into $x^2 = (y-1)(y-2)$:

$$\alpha^2 = \left(-\frac{13}{11} - 1\right)\left(-\frac{13}{11} - 2\right).$$

Simplify:

$$\alpha^2 = \left(-\frac{24}{11}\right)\left(-\frac{35}{11}\right),$$

$$\alpha^2 = \frac{24 \times 35}{121}.$$

Step 5: Calculate $242\alpha^2$.

Substitute $\alpha^2 = \frac{24\times35}{121}$:

$$242\alpha^2 = 242 \cdot \frac{24 \times 35}{121}.$$

Simplify:

$$242\alpha^2 = 48 \cdot 35 = 1680.$$

Conclusive Answer: $242\alpha^2 = 1680.$

Quick Tip

For a complex number to be purely real, its imaginary part must be zero. For a complex number to be purely imaginary, its real part must be zero.
