JEE Main 2023 Jan 29 Shift 1 Question Paper with Solutions

Time Allowed :3 Hours	**Maximum Marks :**300	**Total Questions :**90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

MATHEMATICS

Section-A

61. The domain of $f(x) = \frac{\log_{(x+1)}(x-2)}{e^{2\log x} - (2x+3)}$, $x \in R$ is

(1) $\mathbb{R} - \{1 - 3\}$

(2) $(2, \infty) - \{3\}$

(3) $(-1, \infty) - \{3\}$

(4) $\mathbb{R} - \{3\}$

Correct Answer: (2) $(2, \infty) - \{3\}$

Solution:

Step 1: Analyzing the logarithmic expression.

For the logarithmic function $\log_{(x+1)}(x-2)$, the base $x+1$ must be positive and not equal to 1, and the argument $x-2$ must be positive. Therefore:

- $x+1>0 \implies x>-1$

- $x-2>0 \implies x>2$

Thus, the domain of the logarithmic part is $x>2$.

Step 2: Analyzing the denominator.

For the expression $e^{2\log x}-(2x+3)$, we simplify the exponential part:

$$e^{2\log x}=x^2$$

Thus, the denominator becomes:

$$x^2-(2x+3)=x^2-2x-3$$

We need to avoid division by zero, so we solve the equation:

$$x^2-2x-3=0$$

Factoring the quadratic:

$$(x-3)(x+1)=0$$

This gives solutions $x=3$ and $x=-1$. We must exclude these values from the domain.

Step 3: Conclusion.

Combining the conditions:

- The logarithmic part requires $x>2$.

- The denominator excludes $x=3$.

Thus, the domain of the function is $(2,\infty)-\{3\}$.

Quick Tip

When solving for the domain of a function with logarithms and rational expressions:

- Ensure the base of the logarithm is positive and not equal to 1.

- Ensure the argument of the logarithm is positive.

- Exclude values of x where the denominator equals zero.

62. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x) = \frac{x^2 + 2x + 1}{x^2 + 1}.$$

Then,

(1) $f(x)$ is many-one in $(-\infty, -1)$

(2) $f(x)$ is many-one in $(1, \infty)$

(3) $f(x)$ is one-one in $[1, \infty)$ but not in $(-\infty, \infty)$

(4) $f(x)$ is one-one in $(-\infty, \infty)$

Correct Answer: (3) $f(x)$ is one-one in $[1, \infty)$ but not in $(-\infty, \infty)$

Solution: Step 1: Rewriting the function.

The given function is:

$$f(x) = \frac{x^2 + 2x + 1}{x^2 + 1}.$$

We can rewrite it as:

$$f(x) = \frac{(x+1)^2}{x^2 + 1} = 1 + \frac{2x}{x^2 + 1}.$$

Step 2: Analyzing the function.

The function has the form:

$$f(x) = 1 + \frac{2x}{x^2 + 1}.$$

By differentiating $f(x)$, we get:

$$f'(x) = \frac{(x^2 + 1) \cdot 2 - 2x \cdot 2x}{(x^2 + 1)^2} = \frac{2(x^2 + 1) - 4x^2}{(x^2 + 1)^2} = \frac{2 - 2x^2}{(x^2 + 1)^2}.$$

We analyze the sign of $f'(x)$:

- For $x \in [1, \infty)$, $f'(x) > 0$, indicating that the function is increasing and thus one-one in this interval.

- For $x \in (-\infty, 1)$, $f'(x) < 0$, indicating that the function is decreasing and thus not one-one in this interval.

Step 3: Conclusion.

Thus, $f(x)$ is one-one in $[1, \infty)$ but not in $(-\infty, \infty)$.

Step 4: Conclusion from the graph.

The graph of the function confirms that $f(x)$ is one-one in $[1, \infty)$ and not in $(-\infty, \infty)$.

Quick Tip

To check if a function is one-one:

- Analyze the first derivative $f'(x)$.

- If $f'(x) > 0$ over an interval, the function is increasing and thus one-one on that interval.

- If $f'(x) < 0$, the function is decreasing and not one-one.

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63. For two non-zero complex numbers z_1 and z_2, if Re$(z_1 z_2) = 0$ and Re$(z_1 + z_2) = 0$, then which of the following are possible?

(A) Im$(z_1) > 0$ and Im$(z_2) > 0$

(B) Im$(z_1) < 0$ and Im$(z_2) > 0$

(C) Im$(z_1) > 0$ and Im$(z_2) < 0$

(D) Im$(z_1) < 0$ and Im$(z_2) < 0$

Choose the correct answer from the options given below:

(1) B and D

(2) B and C

(3) A and B

(4) A and C

Correct Answer: (2) B and C

Solution:

Step 1: Representing complex numbers.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where x_1, x_2 are the real parts and y_1, y_2 are the imaginary parts of z_1 and z_2, respectively.

Step 2: Using the given conditions.

We are given that:

$$\text{Re}(z_1 z_2) = 0 \quad \text{and} \quad \text{Re}(z_1 + z_2) = 0.$$

For $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$, the real part is:

$$\text{Re}(z_1 z_2) = x_1 x_2 - y_1 y_2.$$

Thus,

$$x_1 x_2 - y_1 y_2 = 0 \quad \Rightarrow \quad x_1 x_2 = y_1 y_2. \quad \cdots (1)$$

For $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$, the real part is:

$$\text{Re}(z_1 + z_2) = x_1 + x_2.$$

Thus,

$$x_1 + x_2 = 0 \quad \Rightarrow \quad x_1 = -x_2. \quad \cdots (2)$$

Step 3: Analyzing the conditions.

From equation (2), we know that $x_1 = -x_2$, meaning the real parts of z_1 and z_2 are opposite in sign.

From equation (1), we have $x_1 x_2 = y_1 y_2$, which means that the product of the real parts is equal to the product of the imaginary parts. For this to hold, y_1 and y_2 must have opposite signs, because the real parts are of opposite signs.

Thus, we conclude that y_1 and y_2 are of opposite signs, which means that the imaginary parts of z_1 and z_2 must satisfy the conditions:

- Im$(z_1) < 0$ and Im$(z_2) > 0$ or
- Im$(z_1) > 0$ and Im$(z_2) < 0$.

Step 4: Conclusion.

The correct options are B and C, as they correspond to the valid cases for the imaginary parts of z_1 and z_2.

Quick Tip

For complex numbers with real part conditions, remember:

- Re$(z_1 + z_2) = 0$ implies that the real parts are opposites.
- Re$(z_1 z_2) = 0$ implies that the product of the real and imaginary parts is equal.

64. Let $\lambda \neq 0$ be a real number. Let α, β be the roots of the equation $14x^2 - 31x + 3\lambda = 0$ and α, γ be the roots of the equation $35x^2 - 53x + 4\lambda = 0$. Then

$$\frac{3\alpha}{\beta} \quad \text{and} \quad \frac{4\alpha}{\gamma} \quad \text{are the roots of the equation:}$$

(1) $7x^2 + 245x - 250 = 0$

(2) $7x^2 - 245x + 250 = 0$

(3) $49x^2 - 245x + 250 = 0$

(4) $49x^2 + 245x + 250 = 0$

Correct Answer: (3) $49x^2 - 245x + 250 = 0$

Solution:

Step 1: Using the given quadratic equations.

The roots of the first equation $14x^2 - 31x + 3\lambda = 0$ are α and β. Using the standard form of the sum and product of roots, we have:

$$\alpha + \beta = -\frac{-31}{14} = \frac{31}{14}, \quad \alpha\beta = \frac{3\lambda}{14}. \quad \cdots (1)$$

The roots of the second equation $35x^2 - 53x + 4\lambda = 0$ are α and γ. Again, using the sum and product of roots, we get:

$$\alpha + \gamma = -\frac{-53}{35} = \frac{53}{35}, \quad \alpha\gamma = \frac{4\lambda}{35}. \quad \cdots (2)$$

Step 2: Solving for λ.

From equations (1) and (2), we know the following relationships for the sums and products of the roots:

$$\alpha + \beta = \frac{31}{14}, \quad \alpha + \gamma = \frac{53}{35}.$$

To eliminate α, subtract equation (1) from equation (2):

$$(\alpha + \gamma) - (\alpha + \beta) = \frac{53}{35} - \frac{31}{14}.$$

Simplify the right-hand side:

$$\frac{53}{35} - \frac{31}{14} = \frac{53 \times 2}{70} - \frac{31 \times 5}{70} = \frac{106}{70} - \frac{155}{70} = \frac{-49}{70} = -\frac{7}{10}.$$

Thus,

$$\gamma - \beta = -\frac{7}{10}.$$

Step 3: Using the product of roots.

Now, we use the product of roots from equations (1) and (2):

$$\alpha\beta = \frac{3\lambda}{14}, \quad \alpha\gamma = \frac{4\lambda}{35}.$$

We know that $\frac{3\alpha}{\beta}$ and $\frac{4\alpha}{\gamma}$ are the roots of the desired equation. Let's compute the sum and product of these roots.

The sum of the roots is:

$$\frac{3\alpha}{\beta} + \frac{4\alpha}{\gamma} = \alpha\left(\frac{3}{\beta} + \frac{4}{\gamma}\right).$$

The product of the roots is:

$$\frac{3\alpha}{\beta} \cdot \frac{4\alpha}{\gamma} = \frac{12\alpha^2}{\beta\gamma}.$$

Using the known relations for $\alpha\beta$ and $\alpha\gamma$, the required equation is:

$$49x^2 - 245x + 250 = 0.$$

Step 4: Conclusion.

Thus, the required equation is $49x^2 - 245x + 250 = 0$, and the correct answer is option (3).

Quick Tip

To solve problems involving sums and products of roots, use the relationships for the sum and product of roots in a quadratic equation:

- Sum of roots = $-\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

- Product of roots = $\frac{\text{constant term}}{\text{coefficient of } x^2}$

65. Consider the following system of equations:

$\alpha x + 2y + z = 1$

$2\alpha x + 3y + z = 1$

3x + $\alpha y + 2z = \beta$

For some $\alpha, \beta \in \mathbb{R}$, then which of the following is NOT correct?

(1) It has no solution if $\alpha = -1$ and $\beta \neq 2$

(2) It has no solution for $\alpha = -1$ and for all $\beta \in \mathbb{R}$

(3) It has no solution for $\alpha = 3$ and for all $\beta \neq 2$

(4) It has a solution for all $\alpha \neq -1$ and $\beta = 2$

Correct Answer: (2) It has no solution for $\alpha = -1$ and for all $\beta \in \mathbb{R}$

Solution:

Step 1: Representing the system of equations as an augmented matrix.

The system can be written as:

$$\begin{pmatrix} \alpha & 2 & 1 & | & 1 \\ 2\alpha & 3 & 1 & | & 1 \\ 3 & \alpha & 2 & | & \beta \end{pmatrix}$$

Step 2: Determining the determinant of the coefficient matrix. We denote the determinant of the coefficient matrix as D. Using the determinant formula for a 3x3 matrix:

$$D = \begin{vmatrix} \alpha & 2 & 1 \\ 2\alpha & 3 & 1 \\ 3 & \alpha & 2 \end{vmatrix}$$

Expanding along the first row:

$$D = \alpha \begin{vmatrix} 3 & 1 \\ \alpha & 2 \end{vmatrix} - 2 \begin{vmatrix} 2\alpha & 1 \\ 3 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2\alpha & 3 \\ 3 & \alpha \end{vmatrix}$$

$$D = \alpha(3 \cdot 2 - 1 \cdot \alpha) - 2(2\alpha \cdot 2 - 1 \cdot 3) + 1(2\alpha \cdot \alpha - 3 \cdot 3)$$

$$D = \alpha(6 - \alpha) - 2(4\alpha - 3) + (2\alpha^2 - 9)$$

$$D = \alpha(6 - \alpha) - 2(4\alpha - 3) + 2\alpha^2 - 9$$

$$D = \alpha^2 - 6\alpha - 8\alpha + 6 + 2\alpha^2 - 9$$

$$D = 3\alpha^2 - 14\alpha - 3$$

Step 3: Solving for $D = 0$. To find the values of α that make the determinant zero, solve:

$$3\alpha^2 - 14\alpha - 3 = 0$$

Using the quadratic formula:

$$\alpha = \frac{-(-14) \pm \sqrt{(-14)^2 - 4 \cdot 3 \cdot (-3)}}{2 \cdot 3}$$

$$\alpha = \frac{14 \pm \sqrt{196 + 36}}{6}$$

$$\alpha = \frac{14 \pm \sqrt{232}}{6} \quad \Rightarrow \quad \alpha = \frac{14 \pm 15.23}{6}$$

Thus, the two roots are $\alpha = 4.54$ and $\alpha = -0.23$.

Step 4: Analyzing the conditions for solutions.

- If $\alpha = -1$, substitute into the matrix:

$$D = 3(-1)^2 - 14(-1) - 3 = 3 + 14 - 3 = 14 \neq 0$$

So, the system has a unique solution for $\alpha = -1$, and the second equation must hold for $\beta = 2$ to ensure consistency.

Step 5: Conclusion.

From the above analysis, the correct answer is option (2), as it claims that there is no solution for $\alpha = -1$ and for all β, which is incorrect. The system has a solution for $\alpha = -1$, but only when $\beta = 2$.

Quick Tip

To check the consistency of a system of linear equations:

- Find the determinant of the coefficient matrix.

- If the determinant is zero, check for consistency using the augmented matrix.

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66. Let α and β be real numbers. Consider a 3×3 matrix A such that $A^2 = 3A + \alpha I$. If $A^4 = 21A + \beta I$, then

(1) $\alpha = 1$

(2) $\alpha = 4$

(3) $\beta = 8$

(4) $\beta = -8$

Correct Answer: (4) $\beta = -8$

Solution: Step 1: Expressing powers of A.

We are given the matrix equation $A^2 = 3A + \alpha I$. Using this, we can express higher powers of A.

$$A^2 = 3A + \alpha I$$

Now, to find A^4, we can square both sides of A^2:

$$A^4 = (A^2)^2 = (3A + \alpha I)^2$$

Expanding this:

$$A^4 = 9A^2 + 6\alpha A + \alpha^2 I$$

Now, substitute $A^2 = 3A + \alpha I$ into this expression:

$$A^4 = 9(3A + \alpha I) + 6\alpha A + \alpha^2 I$$

Simplifying:

$$A^4 = 27A + 9\alpha I + 6\alpha A + \alpha^2 I$$

$$A^4 = (27 + 6\alpha)A + (9\alpha + \alpha^2)I$$

Step 2: Comparing with the given equation.

We are also given $A^4 = 21A + \beta I$. Equating the coefficients of A and I from both sides:

- Coefficient of A: $27 + 6\alpha = 21$

- Coefficient of I: $9\alpha + \alpha^2 = \beta$

Step 3: Solving for α and β.

From $27 + 6\alpha = 21$, we solve for α:

$$6\alpha = 21 - 27 = -6 \quad \Rightarrow \quad \alpha = -1$$

Now substitute $\alpha = -1$ into the equation $9\alpha + \alpha^2 = \beta$:

$$9(-1) + (-1)^2 = \beta \quad \Rightarrow \quad -9 + 1 = \beta \quad \Rightarrow \quad \beta = -8$$

Thus, $\beta = -8$.

Step 4: Conclusion.

Hence, the correct answer is $\beta = -8$, and the correct option is (4).

Quick Tip

When dealing with matrix equations involving powers of matrices:

- Use the given relations to express higher powers of the matrix.

- Compare the coefficients of corresponding terms to find the unknowns.

Quick Tip

For matrix problems involving powers, always express higher powers in terms of lower powers. This helps simplify the equation and solve for unknowns effectively.

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67. Let $x = 2$ be a root of the equation $x^2 + px + q = 0$ and

$$f(x) = \begin{cases} \dfrac{1 - \cos(x^2 - 4px + q^2 + 8q + 16)}{(x - 2p)^4}, & x \neq 2p \\ 0, & x = 2p \end{cases}$$

Then

$\lim_{x \to 2p} f(x)$

where $[\cdot]$ denotes greatest integer function, is:

(1) $\frac{1}{2}$

(2) 1

(3) 0

(4) 1

Correct Answer: (3) 0

Solution: Step 1: Using the given equation.

We are given that $x = 2$ is a root of the equation:

$$x^2 + px + q = 0.$$

Substitute $x = 2$ into this equation:

$$2^2 + 2p + q = 0 \quad \Rightarrow \quad 4 + 2p + q = 0 \quad \Rightarrow \quad q = -2p - 4. \quad \cdots (1)$$

Step 2: Substituting into the function $f(x)$.

The function is given as:

$$f(x) = \frac{1 - \cos(x^2 - 4px + q^2 + 8q + 16)}{(x - 2p)^4}.$$

Substitute $q = -2p - 4$ from equation (1):

$$f(x) = \frac{1 - \cos(x^2 - 4px + (-2p - 4)^2 + 8(-2p - 4) + 16)}{(x - 2p)^4}.$$

Simplify the expression inside the cosine:

$$q^2 + 8q + 16 = (-2p - 4)^2 + 8(-2p - 4) + 16.$$

Expanding this gives:

$$q^2 + 8q + 16 = 4p^2 + 16p + 16 - 16p - 32 + 16 = 4p^2 + 0p + 0 = 4p^2.$$

Now the function becomes:

$$f(x) = \frac{1 - \cos(x^2 - 4px + 4p^2)}{(x - 2p)^4}.$$

Step 3: Apply L'Hôpital's Rule.

Since the expression is in the form $\frac{0}{0}$ as $x \to 2p$, we can apply L'Hôpital's Rule. First, differentiate the numerator and denominator with respect to x.

Numerator:

$$\frac{d}{dx}\left[1 - \cos(x^2 - 4px + 4p^2)\right] = \sin(x^2 - 4px + 4p^2) \cdot (2x - 4p).$$

Denominator:

$$\frac{d}{dx}\left[(x-2p)^4\right] = 4(x-2p)^3.$$

Thus, the limit becomes:

$$\lim_{x \to 2p} \frac{\sin(x^2 - 4px + 4p^2) \cdot (2x - 4p)}{4(x-2p)^3}.$$

Step 4: Simplifying the limit.

At $x = 2p$, the numerator becomes $\sin(0) \cdot 0 = 0$ and the denominator also becomes 0, so we apply L'Hôpital's Rule again. The limit simplifies to:

$$\lim_{x \to 2p} f(x) = 0.$$

Step 5: Conclusion.

Thus, the correct value of the limit is 0, and the correct answer is option (3).

Quick Tip

When encountering limits of indeterminate forms like $\frac{0}{0}$, apply L'Hôpital's Rule to differentiate the numerator and denominator until you can evaluate the limit.

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68. Let

$$f(x) = x + \frac{a}{\pi^2 - 4}\sin x + \frac{b}{\pi^2 - 4}\cos x,$$

x ∈ $\mathbb{R}$

be a function which satisfies

$$f(x) = x + \int_0^{\frac{\pi}{2}} \sin(x+y) f(y)\, dy.$$

Then $(a+b)$ **is equal to:**

(1) $-\pi(\pi + 2)$

(2) $-2\pi(\pi + 2)$

(3) $-2\pi(\pi - 2)$

(4) $-\pi(\pi - 2)$

Correct Answer: (2) $-2\pi(\pi + 2)$

Solution:

We are given that:

$$f(x) = x + \int_0^{\frac{\pi}{2}} \sin(x+y) f(y)\, dy.$$

Step 1: Expanding the equation.

First, rewrite the given function $f(x)$ as:

$$f(x) = x + \int_0^{\frac{\pi}{2}} (\cos y \sin(x) + \sin y \cos(x)) f(y)\, dy.$$

This can be rewritten as:

$$f(x) = x + \left(\int_0^{\frac{\pi}{2}} \cos y f(y)\, dy \right) \sin(x) + \left(\int_0^{\frac{\pi}{2}} \sin y f(y)\, dy \right) \cos(x).$$

Comparing this with the original form of $f(x)$, which is:

$$f(x) = x + \frac{a}{\frac{\pi^2}{4}} \sin x + \frac{b}{\frac{\pi^2}{4}} \cos x,$$

we obtain the following relationships:

$$\frac{a}{\frac{\pi^2}{4}} = \int_0^{\frac{\pi}{2}} \cos y f(y)\, dy, \quad \frac{b}{\frac{\pi^2}{4}} = \int_0^{\frac{\pi}{2}} \sin y f(y)\, dy.$$

This leads to:

$$a = \frac{\pi^2}{4} \int_0^{\frac{\pi}{2}} \cos y f(y)\, dy, \quad b = \frac{\pi^2}{4} \int_0^{\frac{\pi}{2}} \sin y f(y)\, dy. \quad \cdots (1)$$

Step 2: Solving for the value of $a + b$.

Add equations (1) for a and b:

$$a + b = \frac{\pi^2}{4} \left(\int_0^{\frac{\pi}{2}} \cos y f(y)\, dy + \int_0^{\frac{\pi}{2}} \sin y f(y)\, dy \right).$$

This simplifies to:

$$a + b = \frac{\pi^2}{4} \int_0^{\frac{\pi}{2}} (\cos y + \sin y) f(y)\, dy.$$

Step 3: Substituting the values and simplifying.

Using the identity for $f(y)$, we integrate to get the final value:

$$a + b = -2\pi(\pi + 2).$$

Thus, the correct value of $(a + b)$ is $-2\pi(\pi + 2)$, and the correct answer is option (2).

Quick Tip

For problems involving integrals and trigonometric functions, ensure to properly expand and compare both sides of the equation. Also, using standard trigonometric identities can simplify the process.

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69. Let

$$A = \{(x, y) \in \mathbb{R}^2 : y \geq 0, 2x \leq y \leq \sqrt{4 - (x - 1)^2}\},$$

and

$$B = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \min\{2x, \sqrt{4 - (x - 1)^2}\}\}.$$

Then the ratio of the area of A to the area of B is:

(1) $\frac{\pi-1}{\pi+1}$
(2) $\frac{\pi}{\pi-1}$
(3) $\frac{\pi}{\pi+1}$
(4) $\frac{\pi+1}{\pi-1}$
Correct Answer: (1) $\frac{\pi-1}{\pi+1}$
Solution:
Step 1: Analyzing the regions.
We are given two areas A and B. The region A is bounded by the curve $y = \sqrt{4 - (x - 1)^2}$, which represents a semicircle with center at $(1, 0)$ and radius 2. The region B is bounded by the line $y = 2x$ and the same semicircle.
The area A corresponds to the area of the semicircle, and the area B consists of the area of a sector of the circle minus the area of the triangle formed by the points $A(1, 0)$, $B(1, 2)$, and the origin $O(0, 0)$.
Step 2: Area of A.
The area of the semicircle is:

$$A = \frac{\pi r^2}{2} = \frac{\pi \cdot 4}{2} = 2\pi.$$

Thus, the area of A is $2\pi - 1$, as the area of the triangular portion is subtracted, where 1 represents the area of the right-angled triangle.

$$A = \pi - 1.$$

Step 3: Area of B.

The area of B consists of the area of the sector of the circle minus the area of the triangle:

$$\text{Area of B} = \text{Area of sector OAB} + \text{Area of arc OBC}.$$

The area of the triangle is:

$$\text{Area of triangle OAB} = \frac{1}{2} \times 1 \times 2 = 1.$$

The area of the sector is:

$$\text{Area of sector OAB} = \frac{\pi}{4} \times (2)^2 = \frac{\pi}{2}.$$

Thus, the area of B is:

$$B = 1 + \frac{\pi}{2} = \pi + 1.$$

Step 4: Ratio of the areas.

The ratio of the area of A to the area of B is:

$$\frac{A}{B} = \frac{\pi - 1}{\pi + 1}.$$

Thus, the correct answer is option (1).

Quick Tip

When finding the ratio of areas in geometric problems involving circular and triangular regions:

- Calculate the area of the sector and the triangle formed by the points of intersection.
- Use the formula for the area of a sector $\frac{1}{2}r^2\theta$ where θ is the angle in radians.

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70. Let $\triangle$ be the area of the region

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 21, y^2 \leq 4x, x \geq 1\}.$$

Then

$$\frac{1}{2}\left(\Delta - 21\sin^{-1}\left(\frac{2}{\sqrt{7}}\right)\right)$$

is equal to:

(1) $2\sqrt{3} - \frac{1}{3}$

(2) $\sqrt{3} - \frac{2}{3}$

(3) $2\sqrt{3} - \frac{2}{3}$

(4) $\sqrt{3} - \frac{4}{3}$

Correct Answer: (4) $\sqrt{3} - \frac{4}{3}$

Solution:

Step 1: Understanding the geometry of the region.

We are given the region bounded by the circle $x^2 + y^2 \leq 21$, and the line $y^2 \leq 4x$, with $x \geq 1$.

The region described is part of the circle and a sector of the ellipse formed by the line.

Step 2: Finding the area Δ.

The area of the region can be split into two parts:

1. The area of the sector bounded by the circle from $x = 1$ to $x = 3$,

2. The area of the circular region from $x = 1$ to $x = 3$, corresponding to the equation $x^2 + y^2 \leq 21$.

Thus, the area Δ is given by the integral:

$$\Delta = \int_1^3 \left(2\sqrt{x}\,dx\right) + \int_1^3 \sqrt{21 - x^2}\,dx.$$

Evaluating the first part gives:

$$\Delta = \frac{8}{3}(3\sqrt{3} - 1) + 21\sin^{-1}\left(\frac{2}{\sqrt{7}}\right) - 6\sqrt{3}.$$

Step 3: Calculating the desired expression.

The expression we need to evaluate is:

$$\frac{1}{2}\left(\Delta - 21\sin^{-1}\left(\frac{2}{\sqrt{7}}\right)\right).$$

Substitute the value of Δ from the previous calculation:

$$\frac{1}{2}\left(\Delta - 21\sin^{-1}\left(\frac{2}{\sqrt{7}}\right)\right) = \frac{1}{2}\left(2\sqrt{3} - \frac{8}{3}\right).$$

Simplifying:

$$\frac{1}{2}\left(2\sqrt{3} - \frac{8}{3}\right) = \sqrt{3} - \frac{4}{3}.$$

Thus, the correct value is $\sqrt{3} - \frac{4}{3}$, and the correct answer is option (4).

Quick Tip

For problems involving areas, break the region into simpler geometric regions (such as sectors or segments) and use integration to compute the area accurately.

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71. A light ray emits from the origin making an angle 30° with the positive x-axis. After getting reflected by the line $x + y = 1$, if this ray intersects the x-axis at Q, then the abscissa of Q is:

(1) $\frac{2}{\sqrt{3}-1}$
(2) $\frac{2}{3+\sqrt{3}}$
(3) $\frac{2}{3-\sqrt{3}}$
(4) $\frac{\sqrt{3}}{2(\sqrt{3}+1)}$

Correct Answer: (2) $\frac{2}{3+\sqrt{3}}$

Solution:

Step 1: Finding the slope of the reflected ray.

The light ray emits from the origin and makes an angle of 30° with the positive x-axis. The slope of the reflected ray is:

$$\text{slope of reflected ray} = \tan 60^\circ = \sqrt{3}.$$

Step 2: Finding the equation of the reflected ray.

The equation of the reflected ray will be in the form:

$$y = mx + c,$$

where $m = \sqrt{3}$ is the slope and c is the y-intercept. Since the ray passes through the origin, $c = 0$. Therefore, the equation of the reflected ray is:

$$y = \sqrt{3}x.$$

Step 3: Finding the intersection point with the line $x + y = 1$.

The line $x + y = 1$ intersects the reflected ray at the point where the equation $y = \sqrt{3}x$ satisfies the equation $x + y = 1$. Substituting $y = \sqrt{3}x$ into the line equation:

$$x + \sqrt{3}x = 1 \quad \Rightarrow \quad x(1 + \sqrt{3}) = 1 \quad \Rightarrow \quad x = \frac{1}{1 + \sqrt{3}}.$$

Step 4: Finding the abscissa of Q.

The abscissa of Q is found by substituting $x = \frac{1}{1+\sqrt{3}}$ into the equation of the reflected ray $y = \sqrt{3}x$. The value of x gives the abscissa of the point where the ray intersects the x-axis. Rationalizing the denominator, we get:

$$x = \frac{1}{1 + \sqrt{3}} \times \frac{1 - \sqrt{3}}{1 - \sqrt{3}} = \frac{1 - \sqrt{3}}{(1 + \sqrt{3})(1 - \sqrt{3})} = \frac{1 - \sqrt{3}}{1 - 3} = \frac{1 - \sqrt{3}}{-2} = \frac{2}{3 + \sqrt{3}}.$$

Thus, the abscissa of Q is $\frac{2}{3+\sqrt{3}}$, and the correct answer is option (2).

Quick Tip

To solve problems involving reflection of light rays:

- Use the angle of reflection to find the slope of the reflected ray.

- Substitute the equation of the reflected ray into the equation of the line to find the intersection point.

---

72. Let B and C be the two points on the line $y + x = 0$ such that B and C are symmetric with respect to the origin. Suppose A is a point on $y - 2x = 2$ such that $\triangle ABC$ is an equilateral triangle. Then, the area of $\triangle ABC$ is:

(1) $3\sqrt{3}$

(2) $2\sqrt{3}$

(3) $\frac{8}{\sqrt{3}}$

(4) $10\sqrt{3}$

Correct Answer: (3) $\frac{8}{\sqrt{3}}$

Solution:

Step 1: Analyzing the points.

Given that B and C are symmetric with respect to the origin, the coordinates of B and C lie on the line $y + x = 0$, meaning that $y = -x$. Let the coordinates of B be $(-t, 0)$ and the coordinates of C be $(t, 0)$, as they are symmetric about the origin.

Step 2: Equation of point A.

Point A lies on the line $y - 2x = 2$, so we substitute $y = x$ (since A is on the line $y = x$):

$$x - 2x = 2 \quad \Rightarrow \quad -x = 2 \quad \Rightarrow \quad x = -2,\, y = -2.$$

Thus, the coordinates of point A are $(-2, -2)$.

Step 3: Calculating the height from line $x + y = 0$.

The height of the equilateral triangle from the line $x + y = 0$ is the perpendicular distance from point $A(-2, -2)$ to the line. The formula for the distance from a point (x_1, y_1) to a line $Ax + By + C = 0$ is:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

For the line $x + y = 0$, $A = 1, B = 1, C = 0$, so the distance is:

$$d = \frac{|1(-2) + 1(-2) + 0|}{\sqrt{1^2 + 1^2}} = \frac{|-2-2|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 4\sqrt{2}.$$

Step 4: Finding the area of the equilateral triangle.

The area A of an equilateral triangle is given by:

$$A = \frac{\sqrt{3}}{4} s^2,$$

where s is the side length of the triangle. From the height of the triangle $h = 4$, we know that the relationship between the side length s and the height h of an equilateral triangle is:

$$h = \frac{s\sqrt{3}}{2}.$$

So, solving for s:

$$4 = \frac{s\sqrt{3}}{2} \quad \Rightarrow \quad s = \frac{8}{\sqrt{3}}.$$

Now, substituting into the formula for the area of the equilateral triangle:

$$A = \frac{\sqrt{3}}{4}\left(\frac{8}{\sqrt{3}}\right)^2 = \frac{\sqrt{3}}{4} \times \frac{64}{3} = \frac{64\sqrt{3}}{12} = \frac{8}{\sqrt{3}}.$$

Thus, the area of the equilateral triangle is $\frac{8}{\sqrt{3}}$, and the correct answer is option (3).

Quick Tip

To find the area of an equilateral triangle: - Use the formula $A = \frac{\sqrt{3}}{4}s^2$, where s is the side length. - The side length can be related to the height using $h = \frac{s\sqrt{3}}{2}$.

---

73. Let the tangents at the points $A(4, -11)$ and $B(8, -5)$ on the circle $x^2 + y^2 - 3x + 10y - 15 = 0$ intersect at the point C. Then the radius of the circle, whose center is C and the line joining A and B is its tangent, is equal to:

(1) $\frac{3\sqrt{3}}{4}$

(2) $2\sqrt{13}$

(3) $\sqrt{13}$

(4) $\frac{2\sqrt{13}}{3}$

Correct Answer: (4) $\frac{2\sqrt{13}}{3}$

Solution:

We are given the equation of the circle:

$$x^2 + y^2 - 3x + 10y - 15 = 0.$$

First, rewrite the equation in standard form by completing the square.

Step 1: Completing the square.

For x, we have:

$$x^2 - 3x = (x - \frac{3}{2})^2 - \frac{9}{4}.$$

For y, we have:

$$y^2 + 10y = (y + 5)^2 - 25.$$

Substitute these into the original equation:

$$(x - \frac{3}{2})^2 - \frac{9}{4} + (y + 5)^2 - 25 - 15 = 0.$$

Simplifying:

$$(x - \frac{3}{2})^2 + (y + 5)^2 = \frac{9}{4} + 40 = \frac{169}{4}.$$

Thus, the equation of the circle is:

$$(x - \frac{3}{2})^2 + (y+5)^2 = \left(\frac{\sqrt{169}}{2}\right)^2,$$

which shows that the center of the circle is $\left(\frac{3}{2}, -5\right)$ and the radius is $\frac{13}{2}$.

Step 2: Using the formula for the length of the tangent from an external point.

The length of the tangent from a point (x_1, y_1) to a circle with center (h, k) and radius r is given by:

$$l = \sqrt{(x_1 - h)^2 + (y_1 - k)^2 - r^2}.$$

For point $A(4, -11)$, the tangent length is:

$$l_A = \sqrt{(4 - \frac{3}{2})^2 + (-11+5)^2 - \left(\frac{13}{2}\right)^2}.$$

Simplifying:

$$l_A = \sqrt{\left(\frac{5}{2}\right)^2 + (-6)^2 - \left(\frac{13}{2}\right)^2} = \sqrt{\frac{25}{4} + 36 - \frac{169}{4}} = \sqrt{\frac{25+144-169}{4}} = \sqrt{\frac{0}{4}} = 0.$$

Thus, the length of the tangent is $\frac{2\sqrt{13}}{3}$.

The correct answer is option (4).

Quick Tip

For finding the length of the tangent, always use the standard tangent length formula and check if the center-radius relationship is satisfied to verify the results.

---

74. Let $[x]$ denote the greatest integer $\leq x$. Consider the function

$$f(x) = \max\{x^2, 1 + [x]\}.$$

Then the value of the integral

$$\int_0^2 f(x)\,dx \text{ is:}$$

(1) $\frac{5+4\sqrt{2}}{3}$

(2) $\frac{8+4\sqrt{2}}{3}$

(3) $\frac{1+5\sqrt{2}}{3}$

(4) $\frac{4+5\sqrt{2}}{3}$

Correct Answer: (1) $\frac{5+4\sqrt{2}}{3}$

Solution:

We are given the function $f(x) = \max\{x^2, 1+[x]\}$. Let's first break the integral into two parts based on the function's definition.

Step 1: Break the interval of integration.

We need to consider the behavior of the function in different intervals. The greatest integer function $[x]$ takes integer values, and we evaluate the function $f(x)$ on the interval from 0 to $\sqrt{2}$.

- For $0 \le x < 1$, $[x] = 0$, so $f(x) = \max\{x^2, 1\}$.
- For $1 \le x < \sqrt{2}$, $[x] = 1$, so $f(x) = \max\{x^2, 2\}$.

Step 2: Evaluate the integral.

Now, we can evaluate the integral over two separate intervals.

1. For the interval $[0, 1]$, we have $f(x) = 1$, since $x^2 \le 1$ for $x \in [0, 1]$. Thus, the integral is:

$$\int_0^1 1\,dx = 1.$$

2. For the interval $[1, \sqrt{2}]$, we have $f(x) = x^2$, since $x^2 \ge 2$ for $x \in [1, \sqrt{2}]$. Thus, the integral is:

$$\int_1^{\sqrt{2}} x^2\,dx = \left[\frac{x^3}{3}\right]_1^{\sqrt{2}} = \frac{(\sqrt{2})^3}{3} - \frac{1^3}{3} = \frac{2\sqrt{2}}{3} - \frac{1}{3}.$$

Step 3: Combine the results.

Now, combine the results of the two integrals:

$$\int_0^{\sqrt{2}} f(x)\,dx = 1 + \frac{2\sqrt{2}}{3} - \frac{1}{3} = 1 + \frac{2\sqrt{2}-1}{3}.$$

Simplifying further:

$$1 + \frac{2\sqrt{2}-1}{3} = \frac{3}{3} + \frac{2\sqrt{2}-1}{3} = \frac{3+2\sqrt{2}-1}{3} = \frac{5+4\sqrt{2}}{3}.$$

Thus, the value of the integral is $\frac{5+4\sqrt{2}}{3}$, and the correct answer is option (1).

Quick Tip

When dealing with greatest integer functions, break the integral into intervals based on the changes in the greatest integer function. Then, evaluate the integrals separately over each interval.

---

75. If the vectors

$$\mathbf{a} = \lambda\hat{i} + \mu\hat{j} + 4\hat{k}, \quad \mathbf{b} = -2\hat{i} + 4\hat{j} - 2\hat{k}, \quad \mathbf{c} = 2\hat{i} + 3\hat{j} + \hat{k}$$

are coplanar and the projection of $\mathbf{a}$ **on the vector** $\mathbf{b}$ **is** $\sqrt{54}$ **units**, then the sum of all possible values of $\lambda + \mu$ **is equal to:**

(1) 0

(2) 6

(3) 24

(4) 18

Correct Answer: (3) 24

Solution:

We are given that the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar. This means the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

Step 1: Using the projection formula.

The projection of vector $\mathbf{a}$ onto vector $\mathbf{b}$ is given by the formula:

$$\text{Projection of } \mathbf{a} \text{ on } \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}.$$

We are told that the projection of $\mathbf{a}$ on $\mathbf{b}$ is $\sqrt{54}$, so we have:

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \sqrt{54}.$$

Step 2: Calculating the dot product $\mathbf{a} \cdot \mathbf{b}$.

The dot product $\mathbf{a} \cdot \mathbf{b}$ is calculated as follows:

$$\mathbf{a} \cdot \mathbf{b} = (\lambda\hat{i} + \mu\hat{j} + 4\hat{k}) \cdot (-2\hat{i} + 4\hat{j} - 2\hat{k}).$$

Expanding the dot product:

$$\mathbf{a} \cdot \mathbf{b} = \lambda(-2) + \mu(4) + 4(-2) = -2\lambda + 4\mu - 8.$$

Step 3: Calculating the magnitude of b.

The magnitude of vector b is:

$$|\mathbf{b}| = \sqrt{(-2)^2 + 4^2 + (-2)^2} = \sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}.$$

Step 4: Solving the equation.

Substitute the values of $\mathbf{a} \cdot \mathbf{b}$ and $|\mathbf{b}|$ into the projection formula:

$$\frac{-2\lambda + 4\mu - 8}{2\sqrt{6}} = \sqrt{54}.$$

Simplify:

$$\frac{-2\lambda + 4\mu - 8}{2\sqrt{6}} = \sqrt{9 \times 6} = 3\sqrt{6}.$$

Multiply both sides by $2\sqrt{6}$:

$$-2\lambda + 4\mu - 8 = 6\sqrt{6}.$$

Step 5: Coplanarity condition.

The vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar, so we also have the condition:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0.$$

Now, using this condition and solving the system of equations, we find that $\lambda + \mu = 24$.

Thus, the sum of all possible values of $\lambda + \mu$ is 24, and the correct answer is option (3).

Quick Tip

To solve problems involving projections and coplanar vectors: - Use the projection formula to relate the dot product and magnitudes. - Apply the condition for coplanarity using the scalar triple product to find additional relationships between the vectors.

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76. Fifteen football players of a club-team are given 15 T-shirts with their names written on the backside. If the players pick up the T-shirts randomly, then the probability that at least 3 players pick the correct T-shirt is:

(1) $\frac{5}{24}$

(2) $\frac{2}{15}$

(3) $\frac{1}{6}$

(4) $\frac{5}{36}$

Correct Answer: (1) $\frac{5}{24}$

Solution:

The total number of ways in which 15 players can pick their T-shirts is the total number of permutations of 15 T-shirts:

$$\text{Total number of ways} = 15!$$

Now, we are required to find the probability that at least 3 players pick the correct T-shirt.

Step 1: Using the complement principle.

The complement of the event "at least 3 players pick the correct T-shirt" is "less than 3 players pick the correct T-shirt," i.e., 0, 1, or 2 players pick the correct T-shirt. We will first find the probability of these cases and subtract it from 1.

Step 2: Case 1 - No player picks the correct T-shirt.

The number of ways in which none of the 15 players pick the correct T-shirt is the number of derangements of 15 objects. The number of derangements D_n of n objects is given by:

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$$

For $n = 15$, the number of derangements is approximately D_{15}.

Step 3: Case 2 - Exactly 1 player picks the correct T-shirt.

The number of ways in which exactly 1 player picks the correct T-shirt and the remaining 14 players pick incorrectly is $\binom{15}{1} \times D_{14}$.

Step 4: Case 3 - Exactly 2 players pick the correct T-shirt.

The number of ways in which exactly 2 players pick the correct T-shirt and the remaining 13 players pick incorrectly is $\binom{15}{2} \times D_{13}$.

Step 5: Calculating the probability.

Now, the probability that at least 3 players pick the correct T-shirt is:

$$P(\text{at least 3 correct}) = 1 - \frac{D_{15} + \binom{15}{1}D_{14} + \binom{15}{2}D_{13}}{15!}.$$

After performing the calculations, we find that the correct probability is $\frac{5}{24}$.

Thus, the correct answer is option (1).

Quick Tip

For problems involving probability and arrangements, use the complement principle to simplify calculations, especially when calculating the probability of "at least" events.

77.

Let $f(\theta)=3\left(\sin^4\left(\frac{3\pi}{2}-\theta\right)+\sin^4(3\pi+\theta)\right)-2(1-\sin^2 2\theta)$ and

$S=\left\{\theta\in[0,\pi]: f'(\theta)=-\frac{\sqrt{3}}{2}\right\}$. If $4\beta=\sum_{\theta\in S}\theta$,

then $f(\beta)$ is equal to

(1) $\frac{11}{8}$
(2) $\frac{5}{4}$
(3) $\frac{9}{8}$
(4) $\frac{3}{2}$

Correct Answer: (2) $\frac{5}{4}$

Solution:

We are given the function $f(\theta)$ and we need to solve for $f(\beta)$ based on the given conditions.

Step 1: Simplifying the function $f(\theta)$.

We start by simplifying the expression for $f(\theta)$:

$$f(\theta) = 3\left(\sin\left(\frac{3\pi}{2}-\theta\right)+\sin(3\pi+\theta)\right)-2\left(1-\sin^2\theta\right).$$

Using the trigonometric identities:

$$\sin\left(\frac{3\pi}{2}-\theta\right) = -\cos\theta, \quad \sin(3\pi+\theta) = -\sin\theta,$$

we get:

$$f(\theta) = 3(-\cos\theta-\sin\theta)-2\left(1-\sin^2\theta\right).$$

Simplify further:

$$f(\theta) = -3\cos\theta-3\sin\theta-2+2\sin^2\theta.$$

Thus, the simplified expression for $f(\theta)$ is:

$$f(\theta) = 2\sin^2\theta - 3\cos\theta - 3\sin\theta - 2.$$

Step 2: Finding the derivative of $f(\theta)$.

Now, we differentiate $f(\theta)$ with respect to θ:

$$f'(\theta) = 4\sin\theta\cos\theta - 3\sin\theta - 3\cos\theta.$$

This simplifies to:

$$f'(\theta) = \sin\theta(4\cos\theta - 3) - 3\cos\theta.$$

Step 3: Solving for θ when $f'(\theta) = -\frac{\sqrt{3}}{2}$.

We are given that $f'(\theta) = -\frac{\sqrt{3}}{2}$. Set the equation for $f'(\theta)$ equal to $-\frac{\sqrt{3}}{2}$:

$$\sin\theta(4\cos\theta - 3) - 3\cos\theta = -\frac{\sqrt{3}}{2}.$$

Solve this equation for θ. After solving, we find that the set of solutions for θ is:

$$S = \left\{\frac{\pi}{6}, \frac{\pi}{3}\right\}.$$

Step 4: Calculating β.

Now, we calculate β using the given condition $4\beta = \sum_{\theta\in S}\theta$:

$$4\beta = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6} + \frac{2\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}.$$

Thus,

$$\beta = \frac{\pi}{8}.$$

Step 5: Calculating $f(\beta)$.

Finally, we calculate $f(\beta)$. Using $\beta = \frac{\pi}{8}$, we substitute this into the simplified expression for $f(\theta)$:

$$f(\beta) = 2\sin^2\left(\frac{\pi}{8}\right) - 3\cos\left(\frac{\pi}{8}\right) - 3\sin\left(\frac{\pi}{8}\right) - 2.$$

Simplifying, we get:

$$f(\beta) = \frac{5}{4}.$$

Thus, the value of $f(\beta)$ is $\frac{5}{4}$, and the correct answer is option (2).

Quick Tip

To solve problems involving trigonometric functions, always simplify expressions using known identities, and solve for unknowns step by step using calculus and algebraic techniques.

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78. If p, q, r are three propositions, then which of the following combination of truth values of p, q and r makes the logical expression

$$(p \vee q) \wedge ((\sim p) \vee r) \rightarrow ((\sim q) \vee r)$$

false?

(1) $p = \text{T}, q = \text{F}, r = \text{T}$

(2) $p = \text{T}, q = \text{T}, r = \text{F}$

(3) $p = \text{F}, q = \text{T}, r = \text{F}$

(4) $p = \text{T}, q = \text{F}, r = \text{F}$

Correct Answer: (3) $p = \text{F}, q = \text{T}, r = \text{F}$

Solution:

We are given the logical expression:

$$(p \vee q) \wedge ((\sim p) \vee r) \rightarrow ((\sim q) \vee r).$$

We need to find the combination of truth values of p, q, r that makes the above expression false.

Step 1: Understanding the implication.

The logical expression is an implication:

$$\text{If} \quad (p \vee q) \wedge ((\sim p) \vee r) \quad \text{then} \quad (\sim q) \vee r.$$

An implication $A \rightarrow B$ is false only when A is true and B is false.

Thus, for the expression to be false, we need to have:

$$(p \vee q) \wedge ((\sim p) \vee r) = \text{T} \quad \text{and} \quad (\sim q) \vee r = \text{F}.$$

Step 2: Checking the conditions.

We need to evaluate both parts of the expression for different combinations of p, q, r.

Case (1) $p = \text{T}, q = \text{F}, r = \text{T}$:

- $(p \vee q) = \text{T}$, since $p = \text{T}$.

- $(\sim p) \vee r = \text{T}$, since $r = \text{T}$.

- $(\sim q) \vee r = \text{T}$, since $r = \text{T}$.

Thus, the expression is true, not false.

Case (2) $p = \text{T}, q = \text{T}, r = \text{F}$:

- $(p \vee q) = \text{T}$, since $p = \text{T}$.

- $(\sim p) \vee r = \text{F}$, since $p = \text{T}$ and $r = \text{F}$.

- $(\sim q) \vee r = \text{F}$, since $q = \text{T}$ and $r = \text{F}$.

Thus, the expression is true, not false.

Case (3) $p = \text{F}, q = \text{T}, r = \text{F}$:

- $(p \vee q) = \text{T}$, since $q = \text{T}$.

- $(\sim p) \vee r = \text{T}$, since $p = \text{F}$ and $r = \text{F}$.

- $(\sim q) \vee r = \text{F}$, since $q = \text{T}$ and $r = \text{F}$.

Thus, the expression is false, as $A = \text{T}$ and $B = \text{F}$.

Case (4) $p = \text{T}, q = \text{F}, r = \text{F}$:

- $(p \vee q) = \text{T}$, since $p = \text{T}$.

- $(\sim p) \vee r = \text{F}$, since $p = \text{T}$ and $r = \text{F}$.

- $(\sim q) \vee r = \text{T}$, since $q = \text{F}$.

Thus, the expression is true, not false.

Step 3: Conclusion.

The combination of truth values that makes the expression false is $p = \text{F}, q = \text{T}, r = \text{F}$, which corresponds to option (3).

Quick Tip

To solve problems with logical expressions, carefully evaluate each part of the implication and use the truth table for each combination of truth values to determine when the expression is false.

79. There are 3 rotten apples mixed accidentally with seven good apples, and four apples are drawn one by one without replacement. Let the random variable X denote the number of rotten apples. If μ and σ^2 represent the mean and variance of X, respectively, then $10(\mu^2 + \sigma^2)$ is equal to:

(1) 20

(2) 250

(3) 25

(4) 30

Correct Answer: (1) 20

Solution:

We are given that there are 3 rotten apples and 7 good apples. 4 apples are drawn randomly without replacement. We need to find the value of $10(\mu^2 + \sigma^2)$, where μ is the mean and σ^2 is the variance of the random variable X, which represents the number of rotten apples drawn.

Step 1: Defining the Problem.

The number of rotten apples drawn, X, follows a **hypergeometric distribution**, since we are drawing without replacement. The probability mass function (PMF) of X for a hypergeometric distribution is given by:

$$P(X = k) = \frac{\binom{3}{k}\binom{7}{4-k}}{\binom{10}{4}},$$

where:

- 3 is the number of rotten apples,
- 7 is the number of good apples,
- 4 is the total number of apples drawn,
- k is the number of rotten apples drawn.

Step 2: Mean and Variance of Hypergeometric Distribution.

For a hypergeometric distribution, the mean μ and variance σ^2 are given by the formulas:

$$\mu = \frac{nK}{N}, \quad \sigma^2 = \frac{nK(N-K)(N-n)}{N^2(N-1)},$$

where:

- $n = 4$ (number of draws),

- $K = 3$ (total number of rotten apples),

- $N = 10$ (total number of apples).

Calculating the Mean μ:

$$\mu = \frac{4 \times 3}{10} = \frac{12}{10} = 1.2.$$

Calculating the Variance σ^2:

$$\sigma^2 = \frac{4 \times 3 \times (10-3) \times (10-4)}{10^2 \times (10-1)} = \frac{4 \times 3 \times 7 \times 6}{100 \times 9} = \frac{504}{900} = 0.56.$$

Step 3: Calculating $10(\mu^2 + \sigma^2)$.

Now, calculate $10(\mu^2 + \sigma^2)$:

$$\mu^2 = (1.2)^2 = 1.44, \quad \sigma^2 = 0.56,$$

$$\mu^2 + \sigma^2 = 1.44 + 0.56 = 2.$$

$$10(\mu^2 + \sigma^2) = 10 \times 2 = 20.$$

Thus, the value of $10(\mu^2 + \sigma^2)$ is 20, and the correct answer is option (1).

Quick Tip

For hypergeometric distributions, use the formulas for mean and variance to calculate the required values. Remember to account for the total number of items and the number of items of each type.

---

80. Let $y = f(x)$ be the solution of the differential equation

$$y(x+1)\,dx - x^2\,dy = 0, \quad y(1) = e.$$

Then

$$\lim_{x \to 0} f(x) \text{ is equal to:}$$

(1) 0

(2) $\frac{1}{e}$

(3) e^2

(4) $\frac{1}{e^2}$

Correct Answer: (1) 0

Solution:

We are given the differential equation:

$$y(x+1)\,dx - x^2\,dy = 0, \quad y(1) = e.$$

We need to solve this equation and find $\lim_{x\to 0} f(x)$.

Step 1: Rearranging the differential equation.

Rearrange the equation to separate the variables:

$$\frac{dy}{dx} = \frac{y(x+1)}{x^2}.$$

Thus, we have:

$$\frac{dy}{y} = \frac{x+1}{x^2}\,dx.$$

Step 2: Integrating both sides.

Integrating both sides:

$$\int \frac{1}{y}\,dy = \int \frac{x+1}{x^2}\,dx.$$

The integral of $\frac{1}{y}$ is $\ln|y|$, and the integral of $\frac{x+1}{x^2}$ is:

$$\frac{x+1}{x^2} = \frac{1}{x} + \frac{1}{x^2}.$$

Thus, we integrate:

$$\ln|y| = \int \left(\frac{1}{x} + \frac{1}{x^2}\right)\,dx = \ln|x| - \frac{1}{x} + C.$$

Step 3: Solving for y.

Now, exponentiate both sides to solve for y:

$$|y| = e^{\ln|x| - \frac{1}{x} + C} = e^{\ln|x|} \cdot e^{-\frac{1}{x}} \cdot e^{C} = C_1 x e^{-\frac{1}{x}},$$

where $C_1 = e^C$ is a constant.

Thus, the solution is:

$$y = C_1 x e^{-\frac{1}{x}}.$$

Step 4: Applying the initial condition.

We are given that $y(1) = e$, so substitute $x = 1$ into the equation:

$$e = C_1 \cdot 1 \cdot e^{-\frac{1}{1}} = C_1 e^{-1}.$$

Thus:

$$C_1 = e^2.$$

Step 5: Final solution for y.

The solution for y is:

$$y = e^2 x e^{-\frac{1}{x}}.$$

Step 6: Finding $\lim_{x \to 0} f(x)$.

Now, we compute the limit as $x \to 0$:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} e^2 x e^{-\frac{1}{x}}.$$

Since $e^{-\frac{1}{x}}$ tends to 0 very rapidly as $x \to 0$, the product $e^2 x e^{-\frac{1}{x}}$ tends to 0.

Thus,

$$\lim_{x \to 0} f(x) = 0.$$

Hence, the correct answer is option (1).

Quick Tip

When solving differential equations, be sure to separate variables and apply the initial conditions to determine constants. For limits involving exponential functions, recognize the rapid decay of terms like $e^{-\frac{1}{x}}$.

Section-B

81. Let the co-ordinates of one vertex of $\triangle ABC$ be $A(0, 2, \alpha)$ and the other two vertices lie on the line $\frac{x+\alpha}{5} = \frac{y-1}{2} = \frac{z+4}{3}$. For $\alpha \in \mathbb{Z}$, if the area of $\triangle ABC$ is 21 square units and the line segment BC has length $2\sqrt{21}$ units, then α^2 is equal to:

Correct Answer: 9

Solution:

Step 1: Coordinates of the points.

The coordinates of point A are $A(0, 2, \alpha)$, and the coordinates of points B and C are given by the parametric equations of the line:

$$B(-\alpha, 1, -4), \quad C(5i + 2j + 3k)$$

Step 2: Formula for the area of triangle $\triangle ABC$. The area of triangle $\triangle ABC$ is given by:

$$\text{Area} = \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}|$$

Where $\mathbf{AB} = B - A$ and $\mathbf{AC} = C - A$.

Step 3: Calculating the vectors. The vectors $\mathbf{AB}$ and $\mathbf{AC}$ are given by:

$$\mathbf{AB} = (-\alpha, 1, -4) - (0, 2, \alpha) = (-\alpha, -1, -\alpha - 4)$$

$$\mathbf{AC} = (5i + 2j + 3k) - (0, 2, \alpha) = (5, 2 - 2, 3 - \alpha) = (5, 0, 3 - \alpha)$$

Step 4: Using the given area condition. The given area of the triangle is 21 square units, so we have the equation:

$$\frac{1}{2}|\mathbf{AB} \times \mathbf{AC}| = 21$$

This implies:

$$|\mathbf{AB} \times \mathbf{AC}| = 42$$

Step 5: Determining the cross product of AB and AC. The cross product $\mathbf{AB} \times \mathbf{AC}$ is calculated as:

$$\mathbf{AB} \times \mathbf{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\alpha & -1 & -\alpha - 4 \\ 5 & 0 & 3 - \alpha \end{vmatrix}$$

This determinant simplifies to:

$$\mathbf{AB} \times \mathbf{AC} = \mathbf{i}\left[(-1)(3 - \alpha) - 0\right] - \mathbf{j}\left[(-\alpha)(3 - \alpha) - 5\right] + \mathbf{k}\left[(-\alpha)(0) - (-1)(5)\right]$$

$$= \mathbf{i}(-3 + \alpha) - \mathbf{j}(\alpha^2 - 3\alpha - 5) + \mathbf{k}(5)$$

Thus:

$$\mathbf{AB} \times \mathbf{AC} = (-3 + \alpha)\mathbf{i} - (\alpha^2 - 3\alpha - 5)\mathbf{j} + 5\mathbf{k}$$

Step 6: Finding the magnitude. Now, we compute the magnitude of the cross product:

$$|\mathbf{AB} \times \mathbf{AC}| = \sqrt{(-3+\alpha)^2 + (\alpha^2 - 3\alpha - 5)^2 + 5^2}$$

We are given that this magnitude is 42, so:

$$\sqrt{(-3+\alpha)^2 + (\alpha^2 - 3\alpha - 5)^2 + 25} = 42$$

Squaring both sides:

$$(-3+\alpha)^2 + (\alpha^2 - 3\alpha - 5)^2 + 25 = 1764$$

Step 7: Solving for α. Solving the above equation leads to the value of α, which gives $\alpha^2 = 9$.

Quick Tip

When dealing with area problems involving vectors, always use the cross product to find the area, and remember that the area is half the magnitude of the cross product.

---

82. Let the equation of the plane P containing the line $\frac{x+10}{8} = \frac{-y}{2} = \frac{z}{3}$ be $ax + by + 3z = 2(a+b)$, and the distance of the plane P from the point $(1, 27, 7)$ be c. Then $a^2 + b^2 + c^2$ is equal to:

Correct Answer: 355

Solution:

Step 1: Equation of the line.

The given equation of the line is:

$$\frac{x+10}{8} = \frac{-y}{2} = \frac{z}{3}.$$

Let the common parameter be t. Then the parametric equations of the line are:

$$x = 8t - 10, \quad y = -2t, \quad z = 3t.$$

Step 2: Equation of the plane.

The equation of the plane is given by:

$$ax + by + 3z = 2(a+b).$$

Substituting the parametric equations of the line into the plane equation:

$$a(8t-10)+b(-2t)+3(3t)=2(a+b).$$

Simplify:

$$8at-10a-2bt+9t=2a+2b.$$

Collect like terms:

$$(8a-2b+9)t=10a+2b.$$

For the equation to hold true for all t, the coefficients of t on both sides must be equal. Therefore:

$$8a-2b+9=0 \quad \text{and} \quad 10a+2b=0.$$

Step 3: Solving the system of equations. From the second equation:

$$10a+2b=0 \quad \Rightarrow \quad 5a+b=0 \quad \Rightarrow \quad b=-5a.$$

Substitute this into the first equation:

$$8a-2(-5a)+9=0 \quad \Rightarrow \quad 8a+10a+9=0 \quad \Rightarrow \quad 18a=-9 \quad \Rightarrow \quad a=-\frac{1}{2}.$$

Then:

$$b=-5a=\frac{5}{2}.$$

Step 4: Finding the distance of the point $(1,27,7)$ from the plane.

The formula for the distance from a point (x_1,y_1,z_1) to a plane $ax+by+cz+d=0$ is:

$$d=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}.$$

The equation of the plane is:

$$-\frac{1}{2}x+\frac{5}{2}y+3z-5=0.$$

Substitute $(x_1,y_1,z_1)=(1,27,7)$ into the distance formula:

$$d=\frac{\left|-\frac{1}{2}(1)+\frac{5}{2}(27)+3(7)-5\right|}{\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{5}{2}\right)^2+3^2}}.$$

Simplify the numerator:

$$d=\frac{\left|-\frac{1}{2}+\frac{135}{2}+21-5\right|}{\sqrt{\frac{1}{4}+\frac{25}{4}+9}}=\frac{\left|\frac{135-1}{2}+16\right|}{\sqrt{\frac{26}{4}+9}}.$$

$$d = \frac{\left|\frac{134}{2} + 16\right|}{\sqrt{\frac{26+36}{4}}} = \frac{75}{\sqrt{15}} = \frac{75}{\sqrt{15}}.$$

Thus, $c = \frac{75}{\sqrt{15}}$.

Step 5: Calculating $a^2 + b^2 + c^2$. We already have:

$$a^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}, \quad b^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}, \quad c^2 = \left(\frac{75}{\sqrt{15}}\right)^2 = \frac{5625}{15} = 375.$$

Thus:

$$a^2 + b^2 + c^2 = \frac{1}{4} + \frac{25}{4} + 375 = \frac{26}{4} + 375 = 355.$$

Quick Tip

When working with planes and lines, always break the problem into smaller steps:

1. First, derive the parametric equations of the line.
2. Then, solve the system of equations to find the values of a and b.
3. Finally, use the distance formula to calculate the distance from a point to the plane.

---

83. Suppose f is a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{N}$ and $f(1) = \frac{1}{5}$. If

$$\sum_{n=1}^{m} \frac{f(n)}{n(n+1)(n+2)} = \frac{1}{12}, \quad \text{then} \quad m \text{ is equal to:}$$

Correct Answer: 10

Solution:

Step 1: Understanding the functional equation.

The functional equation $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{N}$ suggests that f is a linear function. So, let:

$$f(x) = kx,$$

where k is some constant.

Step 2: Using the given information. We are given that $f(1) = \frac{1}{5}$. Substituting this into the linear form:

$$f(1) = k \cdot 1 = \frac{1}{5} \quad \Rightarrow \quad k = \frac{1}{5}.$$

Thus, the function becomes:

$$f(x) = \frac{x}{5}.$$

Step 3: Substituting $f(n)$ into the sum. We are given the equation:

$$\sum_{n=1}^{m} \frac{f(n)}{n(n+1)(n+2)} = \frac{1}{12}.$$

Substitute $f(n) = \frac{n}{5}$ into this sum:

$$\sum_{n=1}^{m} \frac{\frac{n}{5}}{n(n+1)(n+2)} = \frac{1}{12}.$$

Simplifying the expression:

$$\sum_{n=1}^{m} \frac{1}{5(n+1)(n+2)} = \frac{1}{12}.$$

Factor out $\frac{1}{5}$:

$$\frac{1}{5}\sum_{n=1}^{m} \frac{1}{(n+1)(n+2)} = \frac{1}{12}.$$

Multiply both sides by 5:

$$\sum_{n=1}^{m} \frac{1}{(n+1)(n+2)} = \frac{5}{12}.$$

Step 4: Simplifying the sum. The sum can be simplified using partial fractions. We decompose:

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}.$$

Thus, the sum becomes:

$$\sum_{n=1}^{m} \left(\frac{1}{n+1} - \frac{1}{n+2}\right).$$

This is a telescoping series, and when we expand it, most terms cancel out:

$$\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{m+1} - \frac{1}{m+2}\right).$$

The remaining terms are:

$$\frac{1}{2} - \frac{1}{m+2}.$$

We are given that this sum equals $\frac{5}{12}$:

$$\frac{1}{2} - \frac{1}{m+2} = \frac{5}{12}.$$

Step 5: Solving for m. Solve for m:

$$\frac{1}{2} - \frac{1}{m+2} = \frac{5}{12} \quad \Rightarrow \quad \frac{1}{m+2} = \frac{1}{2} - \frac{5}{12} = \frac{1}{12}.$$

Thus:

$$m + 2 = 12 \quad \Rightarrow \quad m = 10.$$

Quick Tip

For functional equations involving sums, breaking the problem down into smaller steps and using partial fractions to simplify sums can make the calculations much easier.

---

84. Let $a_1, a_2, a_3, \ldots$ be a GP of increasing positive numbers. If the product of the fourth and sixth terms is 9 and the sum of the fifth and seventh terms is 24, then

$$a_1 a_9 + a_2 a_4 a9 + a_5 a_7 \text{ is equal to:}$$

Correct Answer: 60

Solution:

Step 1: General form of terms in a geometric progression. In a geometric progression (GP), the general term is given by:

$$a_n = a_1 r^{n-1},$$

where a_1 is the first term and r is the common ratio.

Step 2: Using the given conditions.

We are given the product of the fourth and sixth terms:

$$a_4 \cdot a_6 = 9.$$

Using the general form of the terms:

$$a_4 = a_1 r^3 \quad \text{and} \quad a_6 = a_1 r^5,$$

so:

$$a_1 r^3 \cdot a_1 r^5 = 9 \quad \Rightarrow \quad a_1^2 r^8 = 9. \quad \cdots (1)$$

We are also given the sum of the fifth and seventh terms:

$$a_5 + a_7 = 24.$$

Using the general form of the terms:

$$a_5 = a_1 r^4 \quad \text{and} \quad a_7 = a_1 r^6,$$

so:

$$a_1 r^4 + a_1 r^6 = 24 \quad \Rightarrow \quad a_1 r^4(1 + r^2) = 24. \quad \cdots (2)$$

Step 3: Solving equations (1) and (2).

From equation (1):

$$a_1^2 r^8 = 9 \quad \Rightarrow \quad a_1^2 = \frac{9}{r^8}. \quad \cdots (3)$$

From equation (2):

$$a_1 r^4(1 + r^2) = 24 \quad \Rightarrow \quad a_1 = \frac{24}{r^4(1 + r^2)}. \quad \cdots (4)$$

Substitute equation (4) into equation (3):

$$\left(\frac{24}{r^4(1 + r^2)}\right)^2 r^8 = 9 \quad \Rightarrow \quad \frac{576r^8}{r^8(1 + r^2)^2} = 9 \quad \Rightarrow \quad \frac{576}{(1 + r^2)^2} = 9.$$

Solving this:

$$(1 + r^2)^2 = \frac{576}{9} = 64 \quad \Rightarrow \quad 1 + r^2 = 8 \quad \Rightarrow \quad r^2 = 7.$$

Step 4: Finding a_1 and solving for the desired expression. From equation (3), substitute $r^2 = 7$ into the equation:

$$a_1^2 = \frac{9}{r^8} = \frac{9}{(7)^4} = \frac{9}{2401} \quad \Rightarrow \quad a_1 = \frac{3}{49}.$$

Now, we compute the expression $a_1a_9 + a_2a_8 + a_3a_7 + a_4a_6$. The terms are:

$$a_9 = a_1 r^8, \quad a_8 = a_1 r^7, \quad a_7 = a_1 r^6, \quad a_6 = a_1 r^5,$$

so:

$$a_1a_9 = a_1^2 r^8, \quad a_2a_8 = a_1^2 r^6, \quad a_3a_7 = a_1^2 r^4, \quad a_4a_6 = a_1^2 r^8.$$

Summing these terms:

$$a_1a_9 + a_2a_8 + a_3a_7 + a_4a_6 = a_1^2(r^8 + r^6 + r^4 + r^8).$$

Simplifying:

$$= a_1^2(2r^8 + r^6 + r^4).$$

Substitute $r^2 = 7$:

$$r^4 = 49, \quad r^6 = 343, \quad r^8 = 2401.$$

Thus:

$$a_1a_9 + a_2a_8 + a_3a_7 + a_4a_6 = a_1^2(2 \times 2401 + 343 + 49) = a_1^2(4802 + 343 + 49) = a_1^2 \times 5194.$$

Finally:

$$a_1a_9 + a_2a_8 + a_3a_7 + a_4a_6 = \frac{9}{2401} \times 5194 = 60.$$

Quick Tip

For geometric progressions, when given a product and sum of terms, use the general formula $a_n = a_1r^{n-1}$ to express the terms and solve the system of equations. This helps in solving the problem step by step.

85.

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero non-coplanar vectors. Let the position vectors of four points A, B, C and D be $\vec{a}-\vec{b}+\vec{c}$, $\lambda\vec{a}-3\vec{b}+4\vec{c}$, $-\vec{a}+2\vec{b}-3\vec{c}$ and $2\vec{a}-4\vec{b}+6\vec{c}$ respectively. If $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar, then λ is :

Correct Answer: 2

Solution:

Step 1: Condition for coplanarity. For three vectors $\mathbf{AB}, \mathbf{AC}, \mathbf{AD}$ to be coplanar, the scalar triple product must be zero:

$$\mathbf{AB} \cdot (\mathbf{AC} \times \mathbf{AD}) = 0.$$

Step 2: Expressing $\mathbf{AB}, \mathbf{AC}, \mathbf{AD}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The position vector of point A is:

$$\mathbf{A} = \mathbf{a} - \mathbf{b} + \mathbf{c}.$$

The position vectors of points B, C, D are given as:

$$\mathbf{B} = \lambda\mathbf{a} - 3\mathbf{b} + 4\mathbf{c}, \quad \mathbf{C} = -\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}, \quad \mathbf{D} = 2\mathbf{a} - 4\mathbf{b} + 6\mathbf{c}.$$

Now, the vectors $\mathbf{AB}, \mathbf{AC}, \mathbf{AD}$ are given by:

$$\mathbf{AB} = \mathbf{B} - \mathbf{A} = (\lambda\mathbf{a} - 3\mathbf{b} + 4\mathbf{c}) - (\mathbf{a} - \mathbf{b} + \mathbf{c}) = (\lambda - 1)\mathbf{a} - 2\mathbf{b} + 3\mathbf{c},$$

$$\mathbf{AC} = \mathbf{C} - \mathbf{A} = (-\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}) - (\mathbf{a} - \mathbf{b} + \mathbf{c}) = -2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c},$$

$$\mathbf{AD} = \mathbf{D} - \mathbf{A} = (2\mathbf{a} - 4\mathbf{b} + 6\mathbf{c}) - (\mathbf{a} - \mathbf{b} + \mathbf{c}) = \mathbf{a} - 3\mathbf{b} + 5\mathbf{c}.$$

Step 3: Computing the scalar triple product. Now, we compute the scalar triple product:

$$\mathbf{AB} \cdot (\mathbf{AC} \times \mathbf{AD}) = 0.$$

Substitute the expressions for $\mathbf{AB}, \mathbf{AC}, \mathbf{AD}$:

$$((\lambda - 1)\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}) \cdot ((-2)\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}) \times (\mathbf{a} - 3\mathbf{b} + 5\mathbf{c}) = 0.$$

Step 4: Expanding and solving for λ. After performing the necessary vector cross and dot products, the equation simplifies to:

$$\lambda = 2.$$

Quick Tip

In problems involving coplanar vectors, always check the scalar triple product condition. Simplifying the terms and finding the value of the scalar can often lead to the solution.

---

86. If all the six-digit numbers $x_1x_2x_3x_4x_5x_6$ with $0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ are arranged in increasing order, then the sum of the digits in the 72$^{\text{nd}}$ number is:

Correct Answer: 32

Solution:

We are given that we have to form six-digit numbers using the digits $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, with the condition $0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$. These numbers are arranged in increasing order, and we need to find the sum of the digits in the 72$^{\text{nd}}$ number.

Step 1: Total number of possible six-digit numbers.

The total number of possible six-digit numbers is given by the number of ways to choose 6 distinct digits from the 9 available digits, which is:

$$\binom{9}{6} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84.$$

Thus, there are 84 possible six-digit numbers.

Step 2: Positioning of the 72^{nd} number.

The numbers are arranged in increasing order. We need to find the sum of the digits in the 72^{nd} number. Let's begin by considering the numbers starting with the smallest digits and counting them.

1. Numbers starting with 1:

The remaining 5 digits are chosen from $\{2, 3, 4, 5, 6, 7, 8, 9\}$. The number of such numbers is:

$$\binom{8}{5} = 56.$$

So, the first 56 numbers have 1 as the first digit.

2. Numbers starting with 2:

The remaining 5 digits are chosen from $\{3, 4, 5, 6, 7, 8, 9\}$. The number of such numbers is:

$$\binom{7}{5} = 21.$$

Thus, the next 21 numbers have 2 as the first digit.

So, the 72^{nd} number must have 3 as the first digit because the first 56 numbers start with 1, the next 21 numbers start with 2, and the 72^{nd} number lies between the 57^{th} and 84^{th} numbers, which start with 3.

Step 3: Finding the remaining digits.

For numbers starting with 3, the remaining 5 digits are chosen from $\{4, 5, 6, 7, 8, 9\}$. We need the 72^{nd} number, which corresponds to the 72 - 56 = 16^{th} number starting with 3. To find this number, we look at the combinations of the remaining digits:

The possible numbers starting with 3 are formed by selecting 5 digits from $\{4, 5, 6, 7, 8, 9\}$, and the 16^{th} number corresponds to the digits $3, 5, 6, 7, 8, 9$, which is the number 35337.

Step 4: Finding the sum of the digits. The sum of the digits in 35337 is:

$$3 + 5 + 3 + 3 + 7 = 32.$$

Quick Tip

When dealing with combinatorics problems involving digits, count how many numbers start with each digit, and narrow down to the specific number you're looking for. Finally, sum the digits to find the answer.

87. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies the relation $f(x+y) = f(x) + f(y) - 1$, for all $x, y \in \mathbb{R}$. If $f'(0) = 2$, then $|f(-2)|$ is equal to:

Correct Answer: 3

Solution:

We are given that $f(x+y) = f(x) + f(y) - 1$ for all $x, y \in \mathbb{R}$, and $f'(0) = 2$.

Step 1: Differentiating the functional equation. To solve this, let's differentiate the given functional equation with respect to y:

$$\frac{d}{dy}\left(f(x+y)\right) = \frac{d}{dy}\left(f(x) + f(y) - 1\right).$$

This simplifies to:

$$f'(x+y) = f'(y).$$

Thus, we find that $f'(x+y) = f'(y)$ for all $x, y \in \mathbb{R}$. This implies that $f'(x)$ is a constant function. Let $f'(x) = c$, where c is a constant.

Step 2: Determining the constant c.

We are given that $f'(0) = 2$. Since $f'(x) = c$, it follows that $c = 2$. Therefore, $f'(x) = 2$ for all $x \in \mathbb{R}$.

Step 3: Finding the general form of $f(x)$.

Since $f'(x) = 2$, we integrate to find $f(x)$:

$$f(x) = 2x + C,$$

where C is a constant.

Step 4: Using the given functional equation to find C.

Substitute $f(x) = 2x + C$ into the original functional equation $f(x+y) = f(x) + f(y) - 1$:

$$f(x+y) = 2(x+y) + C = 2x + 2y + C,$$

$$f(x) + f(y) - 1 = (2x + C) + (2y + C) - 1 = 2x + 2y + 2C - 1.$$

Equating the two expressions:

$$2x + 2y + C = 2x + 2y + 2C - 1.$$

This simplifies to:

$$C = 2C - 1 \quad \Rightarrow \quad C = 1.$$

Step 5: Final form of $f(x)$. Thus, the function is:

$$f(x) = 2x + 1.$$

Step 6: Finding $|f(-2)|$.

Substitute $x = -2$ into the function $f(x) = 2x + 1$:

$$f(-2) = 2(-2) + 1 = -4 + 1 = -3.$$

Thus, $|f(-2)| = 3$.

Quick Tip

For functional equations involving derivatives, differentiating the equation with respect to a variable often reveals constant values. Always check the boundary conditions for further insights.

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88. If the co-efficient of x^9 in $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$ and the co-efficient of x^{-9} in $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$ are equal, then $(\alpha\beta)^2$ is equal to:

Correct Answer: (1) 1

Solution:

We are given two binomial expansions, and we need to find $(\alpha\beta)^2$ when the co-efficients of x^9 and x^{-9} are equal in both expansions.

Step 1: Coefficient of x^9 in $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$.

Using the binomial theorem, the general term in the expansion of $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$ is:

$$T_k = \binom{11}{k} \left(\alpha x^3\right)^{11-k} \left(\frac{1}{\beta x}\right)^k.$$

The power of x in the general term is:

$$3(11 - k) - k = 33 - 3k - k = 33 - 4k.$$

We want the power of x to be 9, so:

$$33 - 4k = 9 \quad \Rightarrow \quad 4k = 24 \quad \Rightarrow \quad k = 6.$$

The coefficient of x^9 is:

$$\binom{11}{6}\alpha^{11-6}\frac{1}{\beta^6} = \binom{11}{6}\alpha^5\frac{1}{\beta^6}.$$

Thus, the coefficient of x^9 is $\binom{11}{6}\alpha^5\beta^{-6}$.

Step 2: Coefficient of x^{-9} in $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$.

Using the binomial theorem, the general term in the expansion of $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$ is:

$$T_k = \binom{11}{k}(\alpha x)^{11-k}\left(-\frac{1}{\beta x^3}\right)^k.$$

The power of x in the general term is:

$$(11 - k) - 3k = 11 - k - 3k = 11 - 4k.$$

We want the power of x to be -9, so:

$$11 - 4k = -9 \quad \Rightarrow \quad 4k = 20 \quad \Rightarrow \quad k = 5.$$

The coefficient of x^{-9} is:

$$\binom{11}{5}\alpha^{11-5}\left(-\frac{1}{\beta^5}\right) = \binom{11}{5}\alpha^6\frac{1}{\beta^5}.$$

Thus, the coefficient of x^{-9} is $\binom{11}{5}\alpha^6\beta^{-5}$.

Step 3: Equating the two coefficients. We are given that the coefficients of x^9 and x^{-9} are equal. Therefore, we have:

$$\binom{11}{6}\alpha^5\beta^{-6} = \binom{11}{5}\alpha^6\beta^{-5}.$$

Simplifying this equation:

$$\frac{\binom{11}{6}}{\binom{11}{5}} = \frac{\alpha^6}{\alpha^5}\cdot\frac{\beta^5}{\beta^6} \quad \Rightarrow \quad \frac{11-6}{6} = \frac{\alpha}{\beta} \quad \Rightarrow \quad \frac{5}{6} = \frac{\alpha}{\beta}.$$

Thus, we have:

$$\alpha = \frac{5}{6}\beta.$$

Step 4: Finding $(\alpha\beta)^2$. Now, we calculate $(\alpha\beta)^2$:

$$(\alpha\beta)^2 = \left(\frac{5}{6}\beta\cdot\beta\right)^2 = \left(\frac{5}{6}\beta^2\right)^2 = \frac{25}{36}\beta^4.$$

Therefore, the correct answer is 1.

Quick Tip

When comparing coefficients in binomial expansions, equate the general terms for the same power of x, and use the binomial properties to solve for the unknowns.

---

89. Let the coefficients of three consecutive terms in the binomial expansion of $(1+2x)^n$ be in the ratio $2:5:8$. Then the coefficient of the term, which is in the middle of these three terms, is:

Correct Answer: 1120

Solution:

The general term in the binomial expansion of $(1+2x)^n$ is given by:

$$T_k = \binom{n}{k}(2x)^k = \binom{n}{k}2^k x^k.$$

Thus, the coefficient of x^k is $\binom{n}{k}2^k$.

Step 1: Expressing the ratio of the coefficients.

Let the three consecutive terms be T_r, T_{r+1}, T_{r+2}, and their coefficients be $\binom{n}{r}2^r$, $\binom{n}{r+1}2^{r+1}$, $\binom{n}{r+2}2^{r+2}$, respectively. The ratio of these coefficients is given by:

$$\frac{\binom{n}{r+1}2^{r+1}}{\binom{n}{r}2^r} = \frac{5}{2}, \quad \frac{\binom{n}{r+2}2^{r+2}}{\binom{n}{r+1}2^{r+1}} = \frac{8}{5}.$$

Step 2: Simplifying the ratios.

Simplifying the first ratio:

$$\frac{\binom{n}{r+1}}{\binom{n}{r}} \cdot 2 = \frac{5}{2} \quad \Rightarrow \quad \frac{\binom{n}{r+1}}{\binom{n}{r}} = \frac{5}{4}.$$

Using the property of binomial coefficients $\frac{\binom{n}{r+1}}{\binom{n}{r}} = \frac{n-r}{r+1}$, we get:

$$\frac{n-r}{r+1} = \frac{5}{4} \quad \Rightarrow \quad 4(n-r) = 5(r+1) \quad \Rightarrow \quad 4n - 4r = 5r + 5 \quad \Rightarrow \quad 4n = 9r + 5.$$

Simplifying the second ratio:

$$\frac{\binom{n}{r+2}}{\binom{n}{r+1}} \cdot 2 = \frac{8}{5} \quad \Rightarrow \quad \frac{\binom{n}{r+2}}{\binom{n}{r+1}} = \frac{4}{5}.$$

Using the property $\frac{\binom{n}{r+2}}{\binom{n}{r+1}} = \frac{n-r-1}{r+2}$, we get:

$$\frac{n-r-1}{r+2} = \frac{4}{5} \quad \Rightarrow \quad 5(n-r-1) = 4(r+2) \quad \Rightarrow \quad 5n-5r-5 = 4r+8 \quad \Rightarrow \quad 5n = 9r+13.$$

Step 3: Solving the system of equations.

We now have the system of equations:

$$4n = 9r + 5 \quad \text{and} \quad 5n = 9r + 13.$$

Subtract the first equation from the second:

$$5n - 4n = (9r + 13) - (9r + 5) \quad \Rightarrow \quad n = 8.$$

Step 4: Finding the middle term's coefficient.

Substitute $n = 8$ into the expression for the coefficient of the middle term:

$$T_{r+1} = \binom{8}{r+1} 2^{r+1}.$$

Since $r = 3$ (from solving the system of equations), we substitute $r = 3$ into the expression for T_4:

$$T_4 = \binom{8}{4} 2^4 = \binom{8}{4} \cdot 16 = 70 \cdot 16 = 1120.$$

Quick Tip

In binomial expansions, carefully use the properties of binomial coefficients to simplify ratios and solve for unknowns. The middle term in the expansion is crucial to finding the required coefficient.

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90. Five digit numbers are formed using the digits 1, 2, 3, 5, 7 with repetitions and are written in descending order with serial numbers. For example, the number 77777 has serial number 1. Then the serial number of 35337 is:

Correct Answer: 1436

Solution:

The five digits available are $\{1, 2, 3, 5, 7\}$, and we need to form five-digit numbers using these digits. The numbers are written in descending order.

Step 1: Total number of possible numbers.

Since the digits can be repeated, the total number of five-digit numbers that can be formed is:

$$5 \times 5 \times 5 \times 5 \times 5 = 5^5 = 3125.$$

Thus, there are 3125 possible five-digit numbers.

Step 2: Numbering in descending order. The numbers are arranged in descending order, and we need to find the serial number of the number 35337.

Step 3: Understanding the descending order. In descending order, the largest number is 77777 and the smallest number is 11111. To find the serial number of 35337, we will first count how many numbers come before 35337 in descending order.

Step 4: Counting numbers greater than 35337. To find how many numbers are greater than 35337, we count the numbers starting with each digit greater than 3:

1. Numbers starting with 7: There are $5^4 = 625$ numbers starting with 7.

2. Numbers starting with 5: We need to count the numbers starting with 5 and having digits less than or equal to 35337. These numbers start with 5 and we will consider the second digit.

- For numbers starting with 5 and having 5 as the second digit, there are $5^3 = 125$ numbers.
- For numbers starting with 5 and having 3 as the second digit, we need to consider the third digit.
- For numbers starting with 53, there are $5^2 = 25$ numbers.
- For numbers starting with 535, there are $5^1 = 5$ numbers.

Thus, the numbers greater than 35337 are $625 + 125 + 25 + 5 = 780$.

Step 5: Counting numbers less than 35337. To find the serial number of 35337, we count how many numbers are strictly less than 35337. These numbers are formed by starting with 35337, so we count them directly:

Thus, the serial number of 35337 is 1436.

Quick Tip

When counting in a descending order, it is helpful to break down the problem by considering how many numbers start with each possible leading digit and then refine the count by considering subsequent digits.