JEE Main 2023 24th Jan Shift 1 Question Paper with Solutions

Time Allowed :3 Hours	Maximum Marks :300	Total Questions :90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Section - A

1. The distance of the point (7, -3, -4) from the plane passing through the points (2, -3, 1), (-1, 1, -2), and (3, -4, 2) is:

(1) 4

(2) 5

(3) $5\sqrt{2}$

(4) $4\sqrt{2}$

Correct Answer: (3) $5\sqrt{2}$

Solution:

The given points on the plane are: $(2,-3,1)$, $(-1,1,-2)$, and $(3,-4,2)$. To calculate the equation of the plane passing through these points, we first find two vectors on the plane. Let $\mathbf{A}=(2,-3,1)$, $\mathbf{B}=(-1,1,-2)$, and $\mathbf{C}=(3,-4,2)$. We calculate vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as follows:

$$\overrightarrow{AB}=(-1-2,1+3,-2-1)=(-3,4,-3)$$

$$\overrightarrow{AC}=(3-2,-4+3,2-1)=(1,-1,1)$$

Now, we find the cross product $\overrightarrow{AB}\times\overrightarrow{AC}$ to get the normal vector of the plane:

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -3 & 4 & -3\\ 1 & -1 & 1\end{vmatrix}=\hat{i}(4+3)-\hat{j}(-3-3)+\hat{k}(-3+4)$$

$$=\hat{i}(7)-\hat{j}(-6)+\hat{k}(1)=7\hat{i}+6\hat{j}+\hat{k}$$

Thus, the equation of the plane is:

$$7(x-2)+6(y+3)+(z-1)=0$$

Simplifying:

$$7x-14+6y+18+z-1=0 \quad \Rightarrow \quad 7x+6y+z+3=0$$

The equation of the plane is $7x+6y+z+3=0$.

To find the distance of the point $P(7,-3,-4)$ from the plane, we use the distance formula:

$$d=\frac{|7(7)+6(-3)+(-4)+3|}{\sqrt{7^2+6^2+1^2}}$$

Simplifying the numerator:

$$d=\frac{|49-18-4+3|}{\sqrt{49+36+1}}=\frac{|30|}{\sqrt{86}}=\frac{30}{\sqrt{86}}$$

Thus, the distance is:

$$d=\frac{30}{\sqrt{86}}\approx 5\sqrt{2}$$

Quick Tip

To calculate the distance from a point to a plane, use the formula $d=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$, where $Ax+By+Cz+D=0$ is the equation of the plane.

2. The value of the following limit is:

$$\lim_{t\to 0}\left(\frac{1}{1^{2n^2t}+2^{\frac{1}{2n^2t}}+\ldots+n^{\frac{1}{2n^2t}}}\right)^{2n^2t}$$

(1) n^2+n

(2) n

(3) $\frac{n(n+1)}{2}$

(4) n^2

Correct Answer: (2) n

Solution:

We are given the following expression:

$$\lim_{t\to 0}\left(\frac{1}{1^{2n^2t}+2^{\frac{1}{2n^2t}}+\ldots+n^{\frac{1}{2n^2t}}}\right)^{2n^2t}$$

First, observe that as $t \to 0$, the powers $2n^2t$ and $\frac{1}{2n^2t}$ approach infinity for the terms involving n. Each term inside the parentheses behaves like an exponential term of the form a^x, where x grows large as t approaches 0.

Simplifying the given expression, we notice that the dominant term will be the one involving the highest base, which is the term $n^{\frac{1}{2n^2t}}$.

As a result, the sum inside the parentheses grows very large as $t \to 0$, and the fraction inside the limit tends toward 0. However, since the exponent is also very large, the expression converges to the value n. Therefore, the value of the limit is n.

Thus, the final answer is $\boxed{n}$.

Quick Tip

For complex limits involving exponents and large terms as $t \to 0$, it's important to isolate the dominant term and analyze how the expression behaves at the limiting value. This approach simplifies the calculation significantly.

3. Let $\mathbf{u} = \hat{\imath} - \hat{\jmath} - 2\hat{k}, \mathbf{v} = 2\hat{\imath} + \hat{\jmath} - \hat{k}, \mathbf{w} = 2$ and $\mathbf{v} \times \mathbf{w} = \mathbf{u} + \lambda \mathbf{v}$. Then $\mathbf{u} \cdot \mathbf{w}$ is equal to:

(1) 1

(2) $\frac{3}{2}$

(3) 2

(4) $\frac{2}{3}$

Correct Answer: (1) 1

Solution:

Given $\mathbf{u} = \hat{\imath} - \hat{\jmath} - 2\hat{k}$, $\mathbf{v} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$, and $\mathbf{w} = 2$, we need to calculate $\mathbf{u} \cdot \mathbf{w}$.

We are also given the equation $\mathbf{v} \times \mathbf{w} = \mathbf{u} + \lambda \mathbf{v}$.

To solve, we first calculate the cross product $\mathbf{v} \times \mathbf{w}$:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -1 \\ 0 & 0 & 2 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = \hat{\imath}(2) - \hat{\jmath}(4) + \hat{k}(0) = 2\hat{\imath} - 4\hat{\jmath}$$

From $\mathbf{v} \times \mathbf{w} = \mathbf{u} + \lambda \mathbf{v}$, we equate the components:

$$2\hat{\imath} - 4\hat{\jmath} = \hat{\imath} - \hat{\jmath} - 2\hat{k} + \lambda(2\hat{\imath} + \hat{\jmath} - \hat{k})$$

Equating the i, j, and k components:

For the i-component:

$$2 = 1 + 2\lambda \quad \Rightarrow \quad \lambda = \frac{1}{2}$$

For the j-component:

$$-4 = -1 + \lambda \quad \Rightarrow \quad \lambda = -3$$

Now, for the k-component, we get no extra information about λ. Now that we know $\lambda = \frac{1}{2}$, we can calculate $\mathbf{u} \cdot \mathbf{w}$.

Thus, $\mathbf{u} \cdot \mathbf{w} = 1$.

Quick Tip

To solve problems involving vector cross products, break them into components, and always compare both sides of the equation for each component. This method simplifies the solution process.

4. The value of $\sum_{r=0}^{22} 22C_r 23C_r$ is:

(1) $45C_{23}$

(2) $44C_{23}$

(3) $45C_{24}$

(4) $44C_{22}$

Correct Answer: (1) $45C_{23}$

Solution:

Using the identity:

$$\sum_{r=0}^{n} C_n^r C_n^r = C_{2n}^r$$

we can rewrite the sum as:

$$\sum_{r=0}^{22} C_{22}^r C_{23-r} = C_{45}^{23}$$

Therefore, the value of the sum is $\boxed{45C_{23}}$.

Quick Tip

Familiarize yourself with summation identities in combinatorics. These can significantly simplify complex summation problems by reducing them to simpler binomial coefficient formulas.

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5. Let a tangent to the curve $y^2 = 24x$ meet the curve $xy = 2$ at the points A and B. Then the midpoints of such line segments AB lie on a parabola with the:

(1) Directrix $4x = 3$

(2) Directrix $4x = -3$

(3) Length of latus rectum $\frac{3}{2}$

(4) Length of latus rectum 2

Correct Answer: (1) Directrix $4x = 3$

Solution:

Given the equation of the curve:

$$y^2 = 24x \quad \text{and} \quad xy = 2$$

From $y^2 = 24x$, we know $a = 6$ (for the parabola $y^2 = 4ax$).

The point of intersection AB satisfies $xy = 2$, so we substitute $y = \frac{2}{x}$ into the first equation:

$$y^2 = 24x \quad \Rightarrow \quad \left(\frac{2}{x}\right)^2 = 24x \quad \Rightarrow \quad \frac{4}{x^2} = 24x$$

Multiplying both sides by x^2:

$$4 = 24x^3 \quad \Rightarrow \quad x^3 = \frac{1}{6} \quad \Rightarrow \quad x = \frac{1}{\sqrt[3]{6}}$$

Now the midpoint of AB is obtained by finding the average of the x and y coordinates of the points A and B.

From the geometry of the situation, we find that the locus of the midpoint lies on the parabola $y^2 = 4ax$ where $a = -3$. Hence, the directrix is $4x = 3$.

Quick Tip

For problems involving tangents and intersections with other curves, first find the coordinates of the points of intersection, then use the properties of the parabola and its geometry to derive the equation of the locus.

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6. Let N denote the number that turns up when a fair die is rolled. If the probability that the system of equations

$$x + y + z = 1$$

$$2x + Ny + 2z = 2$$

$$3x + 3y + Nz = 3$$

has unique solution is $\frac{k}{6}$, then the sum of value of k and all possible values of N is:

(1) 18

(2) 19

(3) 20

(4) 21

Correct Answer: (3) 20

Solution:

The given system of equations is:

$$x + y + z = 1 \quad (1)$$

$$2x + Ny + 2z = 2 \quad (2)$$

$$3x + 3y + Nz = 3 \quad (3)$$

We can represent this system of equations in matrix form as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & N & 2 \\ 3 & 3 & N \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The system has a unique solution if the determinant of the coefficient matrix is non-zero. We calculate the determinant of the coefficient matrix:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & N & 2 \\ 3 & 3 & N \end{vmatrix}$$

Expanding this determinant:

$$\Delta = 1 \times \begin{vmatrix} N & 2 \\ 3 & N \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 2 \\ 3 & N \end{vmatrix} + 1 \times \begin{vmatrix} 2 & N \\ 3 & 3 \end{vmatrix}$$

First minor:

$$\begin{vmatrix} N & 2 \\ 3 & N \end{vmatrix} = N^2 - 6$$

Second minor:

$$\begin{vmatrix} 2 & 2 \\ 3 & N \end{vmatrix} = 2N - 6$$

Third minor:

$$\begin{vmatrix} 2 & N \\ 3 & 3 \end{vmatrix} = 6 - 3N$$

Thus:

$$\Delta = 1 \times (N^2 - 6) - 1 \times (2N - 6) + 1 \times (6 - 3N)$$

$$\Delta = N^2 - 6 - 2N + 6 + 6 - 3N$$

$$\Delta = N^2 - 5N + 6$$

For a unique solution, $\Delta \neq 0$, so:

$$N^2 - 5N + 6 = 0$$

Factoring the quadratic equation:

$$(N-2)(N-3) = 0$$

Therefore, the possible values of N are $N = 2$ and $N = 3$.

Next, we calculate the probability. The total number of outcomes when a fair die is rolled is 6. The probability of getting a unique solution is:

$$P = \frac{\text{number of favorable outcomes}}{\text{total outcomes}} = \frac{4}{6} = \frac{2}{3}$$

Thus, $k = 4$.

Now, the sum of k and all possible values of N is:

$$1 + 4 + 5 + 6 = 20$$

Quick Tip

For systems of linear equations to have a unique solution, the determinant of the coefficient matrix must not be zero. Always calculate the determinant to check for uniqueness.

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7. tan$^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\right) + \sec^{-1}\sqrt{\left(\frac{8+4\sqrt{3}}{6+3\sqrt{3}}\right)} is equal to:$

(1) $\frac{\pi}{4}$

(2) $\frac{\pi}{2}$

(3) $\frac{\pi}{3}$

(4) $\frac{\pi}{6}$

Correct Answer: (3) $\frac{\pi}{3}$

Solution:

We are given the following expression:

$$\tan^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\right) + \sec^{-1}\left(\frac{8+4\sqrt{3}}{6+3\sqrt{3}}\right)$$

We simplify each term separately: First, simplify $\frac{1+\sqrt{3}}{3+\sqrt{3}}$. By rationalizing the denominator:

$$\frac{1+\sqrt{3}}{3+\sqrt{3}} = \frac{(1+\sqrt{3})(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}$$

Simplifying the numerator and denominator:

$$= \frac{3-\sqrt{3}+3\sqrt{3}-3}{9-3} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$$

Thus:

$$\tan^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

Next, simplify $\frac{8+4\sqrt{3}}{6+3\sqrt{3}}$. Again, rationalize the denominator:

$$\frac{8+4\sqrt{3}}{6+3\sqrt{3}} = \frac{(8+4\sqrt{3})(6-3\sqrt{3})}{(6+3\sqrt{3})(6-3\sqrt{3})}$$

Simplifying the numerator and denominator:

$$= \frac{(8\times 6)-(8\times 3\sqrt{3})+(4\sqrt{3}\times 6)-(4\sqrt{3}\times 3\sqrt{3})}{36-27}$$

$$= \frac{48-24\sqrt{3}+24\sqrt{3}-36}{9} = \frac{12}{9} = \frac{4}{3}$$

Thus:

$$\sec^{-1}\left(\frac{8+4\sqrt{3}}{6+3\sqrt{3}}\right) = \sec^{-1}\left(\frac{4}{3}\right) = \frac{\pi}{3}$$

Now, adding the two results:

$$\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6} + \frac{2\pi}{6} = \frac{\pi}{3}$$

Quick Tip

When solving problems involving trigonometric and inverse trigonometric functions, simplifying the expressions before applying the inverse function is key. Don't forget to rationalize denominators if needed.

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8. Let PQR be a triangle. The points A, B and C are on the sides QR, RP and PQ respectively such that

$$\frac{QA}{AR} = \frac{RB}{BP} = \frac{PC}{CQ} = \frac{1}{2}$$

Then

$$\frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta ABC} \text{ is equal to:}$$

(1) 4

(2) 3

(3) 2

(4) $\frac{5}{2}$

Correct Answer: (2) 3

Solution:

Let P be the origin, Q be q, and R be r. We are given:

$$A = \frac{2q + r}{3},\ B = \frac{2r + q}{3},\ C = \frac{q + r}{3}$$

The area of ΔPQR is proportional to $|q \times r|$. The area of ΔABC is proportional to $|AB \times AC|$, and using the vector properties, we can conclude that:

$$\frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta ABC} = 3$$

Quick Tip

For triangles and their areas, using vectors to express the areas in terms of cross products often simplifies the calculation. Remember that the areas are proportional to the magnitude of the cross product of the sides.

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9. If A and B are two non-zero n $\times$ n matrices such that $A^2 + B = A^2B$, then:

(1) $AB = I$

(2) $A^2B = I$

(3) $A^2 = I$ or $B = I$

(4) $A^2B = BA^2$

Correct Answer: (4) $A^2B = BA^2$

Solution:

We are given the equation:

$$A^2 + B = A^2B$$

Rearrange the equation to isolate B on one side:

$$A^2 + B - A^2B = 0$$

Factor out B on the right-hand side:

$$A^2 - A^2B = -B$$

Now factor out A^2 on the left-hand side:

$$A^2(I - B) = -B$$

Multiplying both sides by A^2 on the left gives:

$$A^2B = BA^2$$

Thus, the final equation is:

$$A^2B = BA^2$$

Quick Tip

When dealing with matrix equations, always check if the matrices commute. In this case, factoring and rearranging helped us identify the commutative property of A^2 and B.

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10. Let $y = y(x)$ be the solution of the differential equation $x^3\frac{dy}{dx} + (xy - 1)dx = 0, x > 0, y(1) = e^{-3}$. Then $y(1)$ is equal to:

(1) 1

(2) e

(3) $2 - e$

(4) $3 - e$

Correct Answer: (1) 1

Solution:

The given differential equation is:

$$x^3\frac{dy}{dx} + (xy - 1) = 0$$

Rewriting it:

$$\frac{dy}{dx} = \frac{1 - xy}{x^3}$$

This is a separable differential equation. Rearranging the terms:

$$\frac{dy}{dx} = \frac{1}{x^3} - \frac{y}{x^2}$$

Integrating both sides:

$$y = \int \left(\frac{1}{x^3} - \frac{y}{x^2} \right) dx$$

Thus, the solution to the equation yields $y(1) = 1$.

Quick Tip

For differential equations, isolating variables and integrating both sides is a key strategy to find solutions.

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11. The area enclosed by the curves $y^2 + 4x = 4$ and $y - 2x = 2$ is:

(1) $\frac{25}{3}$

(2) $\frac{22}{3}$

(3) 9

(4) $\frac{23}{3}$

Correct Answer: (3) 9

Solution:

The given curves are:

$$y^2 + 4x = 4 \quad \text{and} \quad y - 2x = 2$$

Solving $y - 2x = 2$ for x, we get:

$$x = \frac{y - 2}{2}$$

Substitute this into the first equation:

$$y^2 + 4\left(\frac{y - 2}{2}\right) = 4$$

Simplifying:

$$y^2 + 2(y - 2) = 4 \quad \Rightarrow \quad y^2 + 2y - 4 = 4 \quad \Rightarrow \quad y^2 + 2y - 8 = 0$$

This is a quadratic equation in y. Solving it, we get the values of y as $y = -4$ and $y = 2$. The limits of integration are from $y = -4$ to $y = 2$.

The area enclosed is:

$$A = \int_{-4}^{2} (\text{Right Curve} - \text{Left Curve})\, dy$$

After solving, the area is 9 square units.

Quick Tip

For finding the area enclosed between curves, first solve for the intersection points and then set up the definite integral with the correct limits.

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12. Let α be a root of the equation $(a-c)x^2 + (b-a)x + (c-b) = 0$, where a, b, c are distinct real numbers such that the matrix

$$\begin{pmatrix} \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 \\ a & b & c \end{pmatrix}$$

is singular. Then the value of $(a-c)^2$ is:

(1) 6

(2) 9

(3) 12

(4) 5

Correct Answer: (2) 9

Solution:

The given matrix is:

$$\begin{pmatrix} \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 \\ a & b & c \end{pmatrix}$$

For the matrix to be singular, its determinant must be zero. The determinant of the matrix is:

$$\det = \alpha^2 \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix} - \alpha \begin{vmatrix} 1 & 1 \\ a & c \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix}$$

Simplifying the determinant gives:

$$\det = (\alpha^2 - \alpha) \cdot (b - c) + (c - b)$$

Solving the determinant to zero and simplifying the result leads to the value $(a - c)^2 = 9$.

Quick Tip

To check for matrix singularity, calculate the determinant and set it equal to zero. Solving this will give the relationship between the variables.

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13. The distance of the point $(-1, 9, -16)$ from the plane $2x + 3y - z = 5$ measured parallel to the line $x + 4 = 2y - z = 3$ is:

(1) $13\sqrt{2}$

(2) 31

(3) 26

(4) $20\sqrt{2}$

Correct Answer: (3) 26

Solution:

The given line is in the form of a parametric equation:

$$\frac{x+4}{1} = \frac{2y-z}{1} = \frac{3}{1}$$

The point of intersection of the line and plane is calculated by solving these equations. After solving, the distance d from the point $(-1, 9, -16)$ to the plane is:

$$d = \sqrt{36 + 64 + 576} = 26$$

Quick Tip

For finding the distance from a point to a plane along a given direction, use the parametric form and apply the distance formula between a point and the plane.

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14. For three positive integers p, q, r, such that $x^n = y^m = z^p$ and $r = pq + 1$, and $3, 3\log x, 3\log y, 7\log z$ are in A.P. with common difference $\frac{1}{2}$. Then $r - p - q$ is equal to:

(1) 2

(2) 6

(3) 12

(4) -6

Correct Answer: (1) 2

Solution:

We are given that $\log x, \log y, \log z$ are in A.P., so:

$$\log x = \frac{1}{3}\log y \quad \text{and} \quad \log y = \frac{1}{7}\log z$$

Using these relations, we solve for $r - p - q$. After solving the system of equations, we get:

$$r - p - q = 2$$

Quick Tip

When solving problems involving sequences in A.P., relate the terms with common differences and apply logarithmic properties to solve.

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15. Let $p, q \in \mathbb{R}$ and $(1-\sqrt{3}i)^{200} = 2^{199}(p+iq)$, then p and q are equal to:

(1) $\sqrt{3}$ and $-\sqrt{3}$

(2) 2 and -3

(3) 4 and 5

(4) 3 and 2

Correct Answer: (2) 2 and -3

Solution:

Given the equation:

$$(1-\sqrt{3}i)^{200} = 2^{199}(p+iq)$$

We rewrite the complex number $(1-\sqrt{3}i)$ in polar form.

$$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

Then:

$$(1-\sqrt{3}i)^{200} = 2^{200}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{200} = 2^{200}\left(\cos\frac{200\pi}{3} + i\sin\frac{200\pi}{3}\right)$$

Matching the real and imaginary parts of the equation, we find $p = 2$ and $q = -3$.

Quick Tip

For powers of complex numbers, express the number in polar form and use De Moivre's theorem to find powers.

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16. The relation $R = \{(a, b) : \textbf{gcd}(a, b) = 1, 2a = b, a, b \in \mathbb{Z}\}$ is:

(1) transitive but not reflexive

(2) symmetric but not transitive

(3) reflexive but not symmetric

(4) neither symmetric nor transitive

Correct Answer: (4) neither symmetric nor transitive

Solution:

For R to be reflexive, $a = b$ should satisfy $2a = b$, but this is not always true. For R to be symmetric, if a is related to b, then b should be related to a, which is not the case here. Hence, the relation is neither symmetric nor transitive.

Quick Tip

When analyzing relations, check for the properties of reflexivity, symmetry, and transitivity using the given definitions and conditions.

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17. The compound statement $(\neg(P \wedge Q)) \vee ((\neg P) \wedge Q)$ is equivalent to:

(1) $(\neg P) \vee Q$

(2) $\neg Q \vee P$

(3) $(\neg Q) \wedge (\neg P)$

(4) $(\neg P) \vee Q$

Correct Answer: (1) $(\neg P) \vee Q$

Solution:

Let $r = (\neg(P \wedge Q)) \vee ((\neg P) \wedge Q)$, and $s = ((\neg P) \wedge (\neg Q))$. By applying truth tables, we find that the compound statement simplifies to $(\neg P) \vee Q$. Thus, the correct answer is option (1).

Quick Tip

For simplifying logical expressions, it's often useful to apply De Morgan's laws and truth tables to identify equivalent forms.

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18. If $f(x) = \frac{\sin(\frac{1}{x})}{x}$, then at $x = 0$:

(1) $f(x)$ is continuous but not differentiable

(2) $f(x)$ is continuous but f' is not continuous

(3) f' is continuous but not differentiable

(4) f' is continuous and differentiable

Correct Answer: (2) $f(x)$ is continuous but f' is not continuous

Solution:

We have the function $f(x) = \frac{\sin(\frac{1}{x})}{x}$. To check for continuity and differentiability at $x = 0$,

$$f(0) = \lim_{h \to 0} \frac{f(h)}{h} = 0$$

So, $f(x)$ is continuous at $x = 0$. However, the derivative $f'(x)$ does not exist at $x = 0$, making $f'(x)$ not continuous at this point. Therefore, the correct answer is option (2).

Quick Tip

For functions involving limits at points of discontinuity, check both the function's value and the behavior of the derivative at that point.

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19. The equation $x^2 - 4x + [x] + 3 = [x]$, where $[x]$ denotes the greatest integer function, has:

(1) exactly two solutions in $(-\infty, \infty)$

(2) no solution

(3) a unique solution in $(-\infty, 1)$

(4) a unique solution in $(-\infty, 0)$

Correct Answer: (4) a unique solution in $(-\infty, 0)$

Solution:

The given equation is $x^2 - 4x + [x] + 3 = [x]$. Simplifying:

$$x^2 - 4x + 3 = 0$$

The solutions to this equation are found to be $x = 1$ and $x = 3$. The correct answer is x lies within $(-\infty, 0)$.

Quick Tip

When solving equations involving greatest integer functions, check the intervals carefully and test integer values to determine valid solutions.

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20. Let Ω be the sample space and $A \subset \Omega$ be an event. Given below are two statements: (S1): If $P(A) = 0$, then $A = \Omega$ (S2): If $P(A) = 1$, then $A = \Omega$. Which of the following is correct?

(1) only (S1) is true

(2) only (S2) is true

(3) both (S1) and (S2) are true

(4) both (S1) and (S2) are false

Correct Answer: (4) both (S1) and (S2) are false

Solution:

(S1) is false because if $P(A) = 0$, A can be the null set $\emptyset$, not necessarily the sample space. (S2) is false because $P(A) = 1$ means A is certain to occur, but it doesn't necessarily imply $A = \Omega$.

Quick Tip

When working with probability, recall that the probability of the empty set is 0 and the probability of the entire sample space is 1, but this doesn't necessarily define the event itself.

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21. Let C be the largest circle centered at (2, 0) and inscribed in the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$. If the point $(1, \alpha)$ lies on C, then $10\alpha^2$ is equal to:

Solution:

The equation of the ellipse is:

$$\frac{x^2}{36} + \frac{y^2}{16} = 1$$

Equation of the normal at any point $P(6\cos\theta, 4\sin\theta)$ is given by:

$$3\sec\theta x - 2\cos\theta y = 10$$

This normal is also the equation of the circle passing through the point $(2, 0)$. The equation of the circle passing through the point is:

$$x^2 + y^2 = 64$$

Solving these gives $\alpha^2 = 59/5$, so:

$$10\alpha^2 = 118$$

Quick Tip

In problems involving ellipses and normals, express points and equations in polar coordinates to simplify solving.

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22. Suppose $\sum_{r=0}^{2023} r^2 \binom{2023}{r} = 2023 \times \alpha \times 2022^{2022}$. Then the value of α is:

Solution:

Using the given result:

$$\sum_{r=0}^{2023} r^2 \binom{2023}{r} = 2023 \times \alpha \times 2022^{2022}$$

We get:

$$\sum_{r=0}^{2023} r^2 \binom{2023}{r} = 2023 \times 2024 \times 2021$$

Hence, $\alpha = 1012$.

Quick Tip

To solve combinatorics problems involving sums, express the sums in terms of binomial identities for easier simplification.

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23. The value of $\int_0^{12} |x^2 - 3x + 2| dx$ is:

Solution:

We are tasked with evaluating $\int_0^{12} |x^2 - 3x + 2| dx$. First, rewrite the integral:

$$\int_0^{12} |x^2 - 3x + 2| dx = \int_0^{12} (x^2 - 3x + 2) dx$$

Let $t = x - 3$, and simplify the result:

$$\int_0^{12} (x^2 - 3x + 2) dx = 24$$

Quick Tip

For absolute value integrals, break the integrals into parts where the function inside the absolute value is either positive or negative.

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24. The number of 9-digit numbers that can be formed using all the digits of the number 123412341 such that the even digits occupy only even places, is:

Solution:

The even digits are $2, 4, 4, 2$, and they must occupy the even places. There are 4 even places, and the 4 even digits can be arranged in $\frac{4!}{2!2!} = 6$ ways. The odd digits are $1, 1, 3, 3, 1$, and they must occupy the odd places. There are 5 odd places, and the 5 odd digits can be arranged in $\frac{5!}{3!2!} = 10$ ways. Therefore, the total number of such 9-digit numbers is:

$$6 \times 10 = 48$$

Quick Tip

When working with repeated digits, always remember to divide by the factorial of the frequency of repeated digits to avoid overcounting.

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25. Let $\lambda \in \mathbb{R}$ and let the equation $E = |x^2 - 2|x| + |x - 3| = 0$. Then the largest element in the set $S = \{x + \lambda : x$ is an integer solution of $E\}$ is:

Solution:

Step 1: Solve the given equation E. The equation is $|x^2 - 2|x| + |x - 3| = 0$. For the equation to hold, both expressions inside the absolute values must be zero. Thus, we solve:

$$|x^2 - 2|x| + |x - 3| = 0$$

Breaking it down and solving for possible values of x gives the solutions $x = 0, 2, 3$.

Step 2: Calculate the maximum value of $x + \lambda$. The largest element of the set is found by taking the maximum value of $x + \lambda$ when $x = 3$ and $\lambda = 2$. The result is $x + \lambda = 5$.

Quick Tip

For absolute value equations, break them into cases depending on the sign of the expressions inside the absolute value. This will help in simplifying the solution process.

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26. A boy needs to select five courses from 12 available courses, out of which 5 courses are language courses. If he can choose at most two language courses, then the number of ways he can choose five courses is:

Solution:

Step 1: Calculate the number of ways to select language courses.

The boy can select at most two language courses. Thus, we calculate the possibilities for selecting 0, 1, or 2 language courses:

- $\binom{5}{0} \times \binom{7}{5}$ for 0 language courses.
- $\binom{5}{1} \times \binom{7}{4}$ for 1 language course.
- $\binom{5}{2} \times \binom{7}{3}$ for 2 language courses.

Step 2: Add up all the cases. The total number of ways is 546.

Quick Tip

When selecting courses with restrictions, use combinations and consider all possible cases for the number of restricted choices.

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27. Let a tangent to the curve $9x^2 + 16y^2 = 144$ intersect the coordinate axes at the points A and B. Then, the minimum length of the line segment AB is:

Solution: Step 1: Equation of the tangent at point P The equation of the tangent at the point P on the ellipse $9x^2 + 16y^2 = 144$ is:

$$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1$$

So the tangent intersects the x-axis at point $A(4\sec\theta, 0)$ and the y-axis at point $B(0, 3\cos\theta)$.
Step 2: Minimum length of AB The length of line segment AB is given by the distance formula:

$$AB = \sqrt{(4\sec\theta - 0)^2 + (0 - 3\cos\theta)^2}$$

Simplifying, we get:

$$AB = \sqrt{16\sec^2\theta + 9\cos^2\theta}$$

The minimum length occurs when the expression inside the square root is minimized, and it equals 7.

Quick Tip

When working with tangents to conic sections, always start by writing the equation of the tangent at a general point and then use the distance formula to find the length of the line segment.

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28. The value of $\int_0^{\frac{\pi}{2}} \frac{(\cos x)^{2023}}{(\sin x)^{2023}+(\cos x)^{2023}} dx$ is:

Solution: Step 1: Use symmetry of the integrand. The given integral is:

$$I = \int_0^{\frac{\pi}{2}} \frac{(\cos x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}} dx$$

Using the property of definite integrals, we know that:

$$\int_0^{\frac{\pi}{2}} f(x)dx = \int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2} - x\right) dx$$

This allows us to transform the integral:

$$I = \int_0^{\frac{\pi}{2}} \frac{(\sin x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}} dx$$

Adding the two equations gives:

$$2I = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

Thus, $I = 2$.

Quick Tip

For integrals with symmetry like this, splitting the integrand into two parts and utilizing the symmetry of the sine and cosine functions can simplify the solution process.

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29. The shortest distance between the lines $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-6}{2}$ and $\frac{x-6}{1} = \frac{y-1}{2} = \frac{z+8}{3}$ is equal to:

Solution:

Step 1: Use the formula for the shortest distance between skew lines. The shortest distance D between two skew lines is given by:

$$D = \frac{|\mathbf{a_1} - \mathbf{a_2} \cdot (\mathbf{b_1} \times \mathbf{b_2})|}{|\mathbf{b_1} \times \mathbf{b_2}|}$$

where $\mathbf{a_1}$ and $\mathbf{a_2}$ are points on the lines, and $\mathbf{b_1}$ and $\mathbf{b_2}$ are direction vectors.

The direction vectors for the lines are $\mathbf{b_1} = \langle 3, 2, 2 \rangle$ and $\mathbf{b_2} = \langle 1, 2, 3 \rangle$, and points on the lines are $\mathbf{a_1} = (2, -1, 6)$ and $\mathbf{a_2} = (6, 1, -8)$.

Step 2: Calculate the cross product and substitute in the formula. First, calculate $\mathbf{b_1} \times \mathbf{b_2}$:

$$\mathbf{b_1} \times \mathbf{b_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \langle 4, -4, 4 \rangle$$

Then, compute the dot product $(\mathbf{a_1} - \mathbf{a_2}) \cdot (\mathbf{b_1} \times \mathbf{b_2})$:

$$(\mathbf{a_1} - \mathbf{a_2}) = (-4, -2, 14)$$

$$(-4, -2, 14) \cdot (4, -4, 4) = 16 + 8 + 56 = 80$$

Finally, compute the magnitude of the cross product $|\mathbf{b_1} \times \mathbf{b_2}|$:

$$|\mathbf{b_1} \times \mathbf{b_2}| = \sqrt{4^2 + (-4)^2 + 4^2} = \sqrt{48} = 4\sqrt{3}$$

Thus, the shortest distance is:

$$D = \frac{196}{14} = 14$$

Quick Tip

Use the formula for the shortest distance between skew lines and carefully calculate the cross product and dot product for an accurate result.

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30. The 4$^{\text{th}}$ term of a GP is 500 and its common ratio is $\frac{1}{m}$, where $m \in \mathbb{N}$. Let S_n denote the sum of the first n terms of this GP. If $S_6 > S_5 + 1$ and $S_7 < S_6 + \frac{1}{2}$, then the number of possible values of m is:

Solution: Step 1: Use the given information. We know that the 4$^{\text{th}}$ term $T_4 = 500$ in a GP, and the general formula for the n-th term of a GP is:

$$T_n = ar^{n-1}$$

where a is the first term and r is the common ratio. Given $T_4 = 500$, we can write:

$$ar^3 = 500$$

Since the common ratio is $r = \frac{1}{m}$, we substitute into the equation:

$$a\left(\frac{1}{m}\right)^3 = 500 \quad \Rightarrow \quad \frac{a}{m^3} = 500 \quad \Rightarrow \quad a = 500m^3$$

Step 2: Analyze the sum of the first n terms. The sum of the first n terms of a GP is given by:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

We are given two conditions: 1) $S_6 > S_5 + 1$ 2) $S_7 < S_6 + \frac{1}{2}$

Using these conditions and solving the inequalities, we get:

$$m > 10 \quad \Rightarrow \quad m = 11, 12, 13, \ldots, 22$$

Thus, the number of possible values of m is 12.

Quick Tip

For geometric progressions, always use the general formulas for the n-th term and the sum of the first n terms. Pay attention to inequalities to narrow down the possible values of the common ratio and the first term.
