JEE Main 2023 31 Jan Shift 1 Question Paper with Solutions

Time Allowed :3 Hours	**Maximum Marks :**300	**Total Questions :**90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

PHYSICS

Section-A

1. A bar magnet with a magnetic moment 5.0 Am^2 is placed in parallel position relative to a magnetic field of 0.4 T. The amount of required work done in turning the magnet from parallel to antiparallel position relative to the field direction is:

(1) 4 J

(2) 1 J

(3) 2 J

(4) Zero

Correct Answer: (1) 4 J

Solution: The potential energy of a magnetic moment in a magnetic field is given by:

$$u = -MB\cos\theta$$

where M is the magnetic moment, B is the magnetic field, and θ is the angle between the magnetic moment and the magnetic field.

The work done in changing the position is:

$$W = \Delta u = -MB\cos 180^\circ - (-MB\cos 0^\circ) = 2MB \times (0.4) = 4\,\text{J}$$

Quick Tip

The work done to rotate a magnet in a magnetic field is proportional to the product of the magnetic moment, field strength, and the change in cosine of the angle between the moment and the field.

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2. If a source of electromagnetic radiation having power 15 kW produces 10^{16} photons per second, the radiation belongs to a part of spectrum is:

(1) Micro waves

(2) Ultraviolet rays

(3) Gamma rays

(4) Radio waves

Correct Answer: (3) Gamma rays

Solution: The energy of one photon is given by the formula:

$$E = \frac{\text{Power}}{\text{Photon frequency}}$$

We know that:

$$E = h\nu \quad (\text{where } h = 6 \times 10^{-34}\,\text{Js})$$

Given:

$$\text{Power} = 15\,\text{kW} = 15 \times 10^3\,\text{W}, \quad \text{Photon frequency} = 10^{16}\,\text{photons/second}$$

So, the energy of one photon is:

$$E = \frac{15 \times 10^3}{10^{16}} = 15 \times 10^{-13}\,\text{J}$$

Now, using $E = h\nu$, we calculate the frequency:

$$\nu = \frac{E}{h} = \frac{15 \times 10^{-13}}{6 \times 10^{-34}} = 2.5 \times 10^{21}\,\text{Hz}$$

Since this frequency lies in the range of gamma rays, the correct answer is gamma rays.

Quick Tip

Gamma rays have frequencies greater than 10^{19} Hz, which are much higher than visible light or radio waves.

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3. The amplitude of $15\sin(1000\pi t)$ is modulated by $10\sin(4\pi t)$ signal. The amplitude modulated signal contains frequency(ies) of:

(A) 500 Hz

(B) 2 Hz

(C) 250 Hz

(D) 498 Hz

(E) 502 Hz

Choose the correct answer from the options given below:

(1) A only

(2) A,D and E only

(3) B only

(4) A and B only

Correct Answer: (2) A,D and E only

Solution: The modulated signal consists of a carrier frequency and sideband frequencies.

The carrier wave frequency is given by:

$$V_c = \frac{100\pi}{2\pi} = 500\,\text{Hz}$$

The modulating wave frequency is:

$$V_m = \frac{4\pi}{2\pi} = 2\,\text{Hz}$$

The modulated signal contains the frequencies:

$$V_c - V_m = 500 - 2 = 498\,\text{Hz}, \quad V_c + V_m = 500 + 2 = 502\,\text{Hz}$$

Thus, the frequencies are 498 Hz and 502 Hz.

Quick Tip

In amplitude modulation, the modulated signal contains the carrier frequency and two sidebands: one at $f_c - f_m$ and the other at $f_c + f_m$, where f_c is the carrier frequency and f_m is the modulating frequency.

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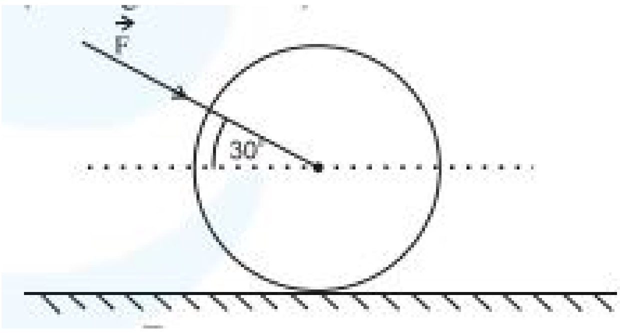
4. As shown in figure, a 70 kg garden roller is pushed with a force of $F = 200$ N at an angle of 30° with horizontal. The normal reaction on the roller is: (Given $g = 10$ m/s^2)



(1) $800\sqrt{2}\,\text{N}$

(2) 600 N

(3) 800 N

(4) $200\sqrt{3}\,\text{N}$

Correct Answer: (3) 800 N

Solution: The normal reaction N is given by the formula:

$$N = mg + F\sin 30^\circ$$

where: $m = 70\,\text{kg}$, $g = 10\,\text{m/s}^2$, $F = 200\,\text{N}$.

Now,

$$N = 700 + 200 \times \sin 30^\circ = 700 + 200 \times \frac{1}{2} = 700 + 100 = 800\,\text{N}$$

Thus, the normal reaction is 800 N.

Quick Tip

The normal reaction is the force exerted by a surface perpendicular to the object. In problems involving inclined planes or angles, use the component of the applied force in the direction of the normal to calculate the total normal reaction.

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5. The initial speed of a projectile fired from ground is u. At the highest point during its motion, the speed of the projectile is $\frac{\sqrt{5}}{2}u$. The time of flight of the projectile is:

(1) $\frac{u}{2g}$

(2) $\frac{u}{g}$

(3) $\frac{2u}{g}$

(4) $\frac{\sqrt{5}u}{g}$

Correct Answer: (2) $\frac{u}{g}$

Solution: At the highest point of the projectile, the vertical component of the velocity becomes zero, and the horizontal component remains unchanged. Given that the speed at the highest point is $\frac{\sqrt{5}}{2}u$, we use the relation:

$$u\cos\theta = \frac{\sqrt{5}}{2}u \quad \Rightarrow \quad \cos\theta = \frac{\sqrt{5}}{2}$$

This implies:

$$\theta = 30^\circ$$

Now, using the formula for the time of flight T, which is given by:

$$T = \frac{2u\sin\theta}{g} = \frac{2u\sin 30^\circ}{g} = \frac{u}{g}$$

Thus, the time of flight is $\frac{u}{g}$.

Quick Tip

The time of flight of a projectile is only dependent on its initial speed and the acceleration due to gravity, and it is independent of the angle of projection if the speed remains the same.

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6. Spherical insulating ball and a spherical metallic ball of the same size and mass are dropped from the same height. Choose the correct statement out of the following (Assume negligible air friction):

(1) Time taken by them to reach the earth's surface will be independent of the properties of their materials

(2) Insulating ball will reach the earth's surface earlier than the metal ball

(3) Both will reach the earth's surface simultaneously

(4) Metal ball will reach the earth's surface earlier than the insulating ball

Correct Answer: (2) Insulating ball will reach the earth's surface earlier than the metal ball

Solution: When metal is passing through a magnetic field, the Eddy current will produce, and it will oppose the motion, so it will take more time.

> **Quick Tip**
>
> The metal ball experiences a force due to induced currents when passing through the magnetic field, which makes it slower compared to the insulating ball.

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7. A free neutron decays into a proton but a free proton does not decay into neutron. This is because:

(1) Neutron is an uncharged particle

(2) Proton is a charged particle

(3) Neutron is a composite particle made of a proton and an electron

(4) Neutron has larger rest mass than proton

Correct Answer: (4) Neutron has larger rest mass than proton

Solution: As neutron has more rest mass than proton it will require energy to decay proton into neutron.

> **Quick Tip**
>
> The difference in rest mass between neutrons and protons is why neutrons decay into protons, but protons cannot decay into neutrons.

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8. The effect of increase in temperature on the number of electrons in conduction band (n_e) and resistance of a semiconductor will be:

(1) Both n_e and resistance decrease

(2) Both n_e and resistance increase

(3) n_e increases, resistance decreases

(4) n_e decreases, resistance increases

Correct Answer: (3) n_e increases, resistance decreases

Solution: As temperature increases, more electrons excite to the conduction band and hence the conductivity increases, which reduces the resistance.

> **Quick Tip**
>
> In semiconductors, increasing the temperature causes an increase in the number of charge carriers, thus reducing resistance.

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9. The maximum potential energy of a block executing simple harmonic motion is 25 J. A is amplitude of oscillation. At $\frac{A}{2}$, the kinetic energy of the block is:

(1) 37.5 J

(2) 9.75 J

(3) 18.75 J

(4) 12.5 J

Correct Answer: (3) 18.75 J

Solution: The maximum potential energy u_{max} is given as:

$$u_{\text{max}} = \frac{1}{2}m\omega^2 A^2 = 25\,\text{J}$$

The total energy is constant and equal to the maximum potential energy. The kinetic energy at $x = \frac{A}{2}$ is:

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2\left(A^2 - \left(\frac{A}{2}\right)^2\right)$$

$$KE = \frac{1}{2}m\omega^2\frac{3A^2}{4} = \frac{3}{4} \times \frac{1}{2}m\omega^2 A^2 = \frac{3}{4} \times 25 = 18.75\,\text{J}$$

Thus, the kinetic energy is 18.75 J.

Quick Tip

In simple harmonic motion, the total energy is the sum of potential and kinetic energy. The kinetic energy at any point is the difference between the total energy and potential energy.

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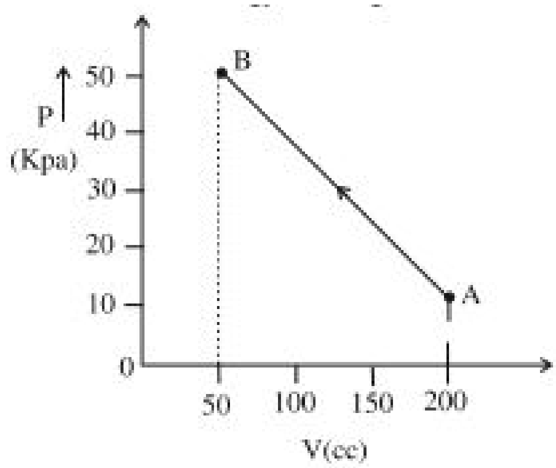
10. The pressure of a gas changes linearly with volume from A to B as shown in the figure. If no heat is supplied to or extracted from the gas, then change in the internal energy of the gas will be:



(1) 6 J

(2) Zero

(3) -4.5 J

(4) 4.5 J

Correct Answer: (4) 4.5 J

Solution: As there is no heat exchange ($\Delta q = 0$), the change in internal energy is equal to the work done by the gas:

$$\Delta u = -W$$

The work done by the gas is given by:

$$W = \int P\, dV$$

Since the pressure changes linearly, we can use the average pressure to calculate the work done:

$$W = P_{\text{avg}} \times \Delta V = 30 \times 10^3 \times 150 \times 10^{-6} = 4500 \times 10^{-3} = 4.5\,\text{J}$$

Thus, the change in internal energy is $4.5\,$J.

Quick Tip

For an ideal gas undergoing a process with no heat exchange, the change in internal energy is equal to the work done by or on the gas.

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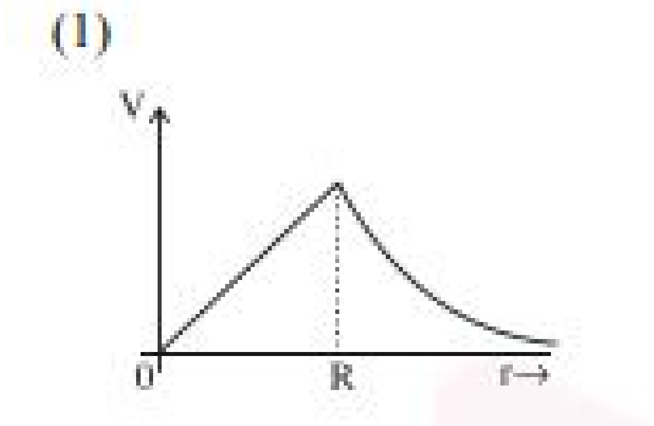
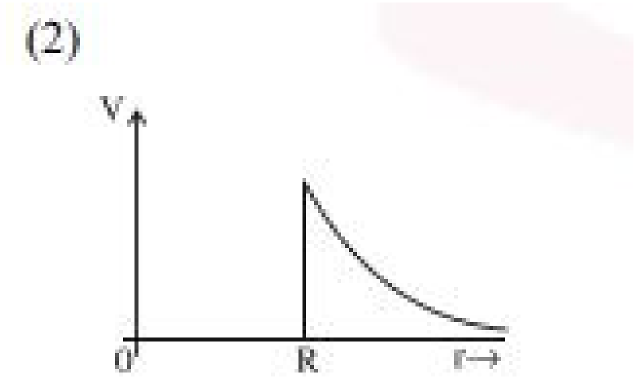
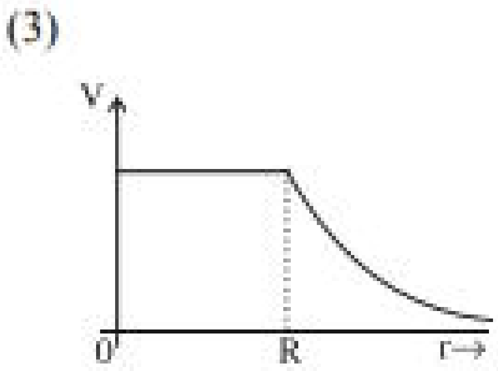
11. Which of the following correctly represents the variation of electric potential (V) of a charged spherical conductor of radius (R) with radial distance (r) from the centre?



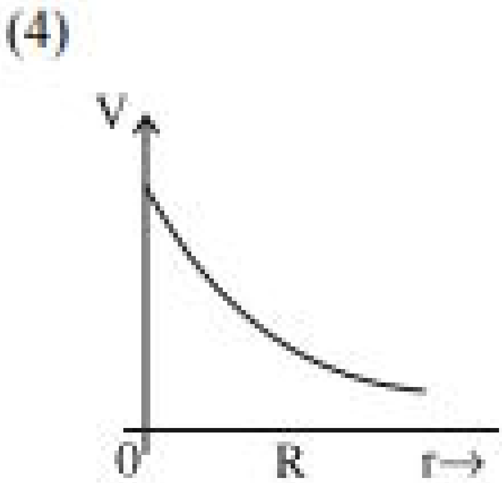
(1)



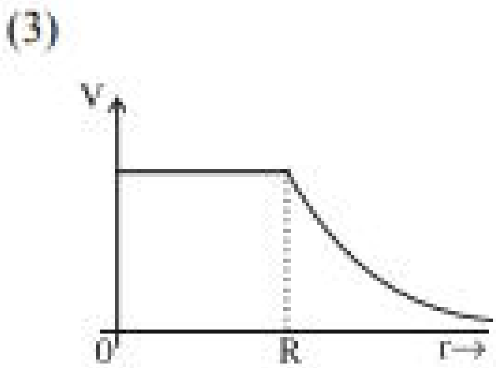
(2)



(3)



(4)



Correct Answer: (3)

Solution: For a spherical conductor, the electric potential inside the conductor is constant and equal to the potential at the surface, while outside, it decreases as the radial distance increases. Therefore, the variation of electric potential with radial distance should show a constant value up to the surface (radius R) and then decrease beyond it. The correct graph is option (3).

Quick Tip

The electric potential inside a spherical conductor is uniform and equal to the surface potential. Outside, the potential decreases with distance from the center.

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12. Given below are two statements: One is labelled as Assertion A and the other is labelled as Reason R Assertion A: The beam of electrons shows wave nature and exhibits interference and diffraction. Reason R: Davisson Germer experimentally verified the wave nature of electrons. In the light of the above statements, choose the most appropriate answer from the options given below:

(1) A is correct but R is not correct

(2) A is not correct but R is correct

(3) Both A and R are correct but R is not the correct explanation of A

(4) Both A and R are correct and R is the correct explanation of A

Correct Answer: (4) Both A and R are correct and R is the correct explanation of A

Solution: Assertion A is correct, as the beam of electrons exhibits wave-like properties, such as interference and diffraction, which is a fundamental concept in quantum mechanics. Reason R is also correct, as Davisson and Germer's experiment confirmed the wave nature of electrons. Furthermore, Reason R correctly explains Assertion A, as their experimental results provided evidence of the wave nature of electrons.

Quick Tip

Davisson Germer's experiment on electron diffraction was key in proving the wave nature of particles, supporting the theory of wave-particle duality.

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13. The drift velocity of electrons for a conductor connected in an electrical circuit is V_d. The conductor is now replaced by another conductor with the same material and same length but double the area of cross-section. The applied voltage remains the same. The new drift velocity of electrons will be:

(1) V_d

(2) $\frac{V_d}{2}$

(3) $\frac{V_d}{4}$

(4) $2V_d$

Correct Answer: (1) V_d

Solution: The drift velocity V_d is given by the formula:

$$V_d = \frac{eE\tau}{m}$$

where e is the charge of the electron, E is the electric field, τ is the relaxation time, and m is the mass of the electron.

The drift velocity is independent of the area of cross-section of the conductor. Therefore, the new drift velocity will remain the same as V_d.

Quick Tip

Drift velocity is independent of the cross-sectional area of the conductor. It depends on the electric field and the relaxation time.

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14. At a certain depth d below the surface of the Earth, the value of acceleration due to gravity becomes four times that of its value at a height 3R above the Earth's surface. Where R is the radius of Earth (Take $R = 6400$ km). The depth d is equal to:

(1) 5260 km

(2) 640 km

(3) 2560 km

(4) 4800 km

Correct Answer: (4) 4800 km

Solution: Using the formula for gravitational force, we have:

$$\frac{GM}{R^2}\left(1 - \frac{d}{R}\right) = \frac{4 \times GM}{(4R)^2}$$

Simplifying:

$$1 - \frac{d}{R} = \frac{1}{16}$$

$$\frac{d}{R} = 1 - \frac{1}{16} = \frac{15}{16}$$

$$d = \frac{15}{16} \times R = \frac{15}{16} \times 6400\,\text{km} = 4800\,\text{km}$$

Thus, the depth is 4800 km.

Quick Tip

The acceleration due to gravity at a certain depth below the Earth's surface decreases, and at a height above the surface, it decreases as well, following the inverse square law.

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15. If 1000 droplets of water of surface tension 0.07 N/m, having same radius 1 mm each, combine to form a single drop. In the process, the released surface energy is: (Take $\pi = \frac{22}{7}$)

(1) 7.92×10^{-6} J

(2) 7.92×10^{-4} J

(3) 9.68×10^{-4} J

(4) 8.8×10^{-5} J

Correct Answer: (2) 7.92×10^{-4} J

Solution: The surface energy is given by the formula:

$$\text{Surface energy} = \Delta E = \text{Number of droplets} \times \left(\frac{4\pi}{3}\right) R^3$$

The radius of each droplet is 1 mm = 10 mm. Therefore,

$$T \times 1000 \times \left(\frac{4\pi}{3}\right) (10^{-3})^3 - T \times \left(\frac{4\pi}{3}\right) (10 \times 10^{-3})^3 = \Delta E$$

$$\Delta E = 4 \times 10^{-7}[1000 - 100] \times 10^{-6}$$

$$\Delta E = 7.92 \times 10^{-4}\,\text{J}$$

Quick Tip

When droplets combine to form a single larger droplet, the total surface area decreases, and the surface energy is released in the process.

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16. A rod with circular cross-section area 2 cm^2 and length 40 cm is wound uniformly with 400 turns of an insulated wire. If a current of 0.4 A flows in the wire windings, the total magnetic flux produced inside the windings is 4×10^{-6} Wb. The relative permeability of the rod is: (Given: Permeability of vacuum $\mu_0 = 4\pi \times 10^{-7}$ N/A^2)

(1) 12.5

(2) $\frac{32}{5}$

(3) 125

(4) $\frac{5}{16}$

Correct Answer: (3) 125

Solution: The magnetic flux ϕ is given by:

$$\phi = \mu_0 \frac{N}{\ell} IA$$

Where: $N = 400$ (number of turns), $\ell = 40\,\text{cm} = 0.4\,\text{m}$, $I = 0.4\,\text{A}$, $A = 2\,\text{cm}^2 = 2 \times 10^{-4}\,\text{m}^2$.

Substitute the values:

$$\phi = 4\pi \times 10^{-7} \times \frac{400}{0.4} \times 0.4 \times 2 \times 10^{-4} = 4 \times 10^{-6}\,\text{Wb}$$

The relative permeability μ_r is:

$$\mu_r = \frac{\phi}{\mu_0} = \frac{4 \times 10^{-6}}{4\pi \times 10^{-7}} = 125$$

Thus, the relative permeability is 125.

Quick Tip

The magnetic flux inside a coil is proportional to the current, the number of turns, the cross-sectional area, and the relative permeability of the material.

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17. The correct relation between $\gamma = \frac{C_p}{C_v}$ and temperature T is:

(1) $\gamma \propto \frac{1}{\sqrt{T}}$

(2) $\gamma \propto T^0$

(3) $\gamma \propto \frac{1}{T}$

(4) $\gamma \propto T$

Correct Answer: (2) $\gamma \propto T^0$

Solution: The value of γ is independent of temperature. It is a constant for a given gas.

Quick Tip

For an ideal gas, the ratio of specific heats γ does not change with temperature.

18. Two polaroide A and B are placed in such a way that the pass-axis of polaroids are perpendicular to each other. Now, another polaroid C is placed between A and B bisecting the angle between them. If intensity of unpolarised light is I_0, then intensity of transmitted light after passing through polaroid B will be:

(1) $\frac{I_0}{4}$

(2) $\frac{I_0}{2}$

(3) $\frac{I_0}{8}$

(4) Zero

Correct Answer: (3) $\frac{I_0}{8}$

Solution: The intensity after passing through polaroid A is:

$$I_A = \frac{I_0}{2}$$

The intensity after passing through polaroid C, with the angle between A and C being 45°, is:

$$I_C = I_A \cos^2 45^\circ = \frac{I_0}{2} \times \frac{1}{2} = \frac{I_0}{4}$$

The intensity after passing through polaroid B, with the angle between C and B being 45°, is:

$$I_B = I_C \cos^2 45^\circ = \frac{I_0}{4} \times \frac{1}{2} = \frac{I_0}{8}$$

Quick Tip

The intensity of light passing through successive polarising filters is given by the product of intensity and the cosine square of the angle between the polarising axes.

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19. If R, X_L, and X_C represent resistance, inductive reactance, and capacitive reactance, then which of the following is dimensionless:

(1) RX_LX_C

(2) $\frac{R}{\sqrt{X_LX_C}}$

(3) $\frac{R}{X_LX_C}$

(4) $\frac{X_L}{X_C}$

Correct Answer: (2) $\frac{R}{\sqrt{X_LX_C}}$

Solution: All three quantities have the same dimension, so $\frac{R}{\sqrt{X_LX_C}}$ is dimensionless.

Quick Tip

For an impedance problem, a ratio of resistance and reactance terms can be dimensionless if the dimensions of the terms cancel out.

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20. 100 balls each of mass m moving with speed v simultaneously strike a wall normally and are reflected back with the same speed, in time t. The total force exerted by the balls on the wall is:

(1) $\frac{100mv}{t}$

(2) $\frac{200mv}{t}$

(3) $\frac{200mvt}{t}$

(4) $\frac{mv}{100t}$

Correct Answer: (2) $\frac{200mv}{t}$

Solution: The momentum change for each ball is given by:

$$\Delta p = p_f - p_i = -Nm\hat{i} - Nm\hat{i} = -2Nm\hat{i}$$

Where: $N = 100$ (number of balls), $\hat{i}$ is the unit vector in the direction of motion.

The total force exerted by the balls on the wall is:

$$F_{\text{total}} = \frac{\Delta p}{\Delta t} = \frac{-200Nm}{t} = \frac{200mv}{t}$$

Thus, the total force is $\frac{200mv}{t}$.

Quick Tip

The total force exerted by a group of objects striking and bouncing off a surface is proportional to the rate of change of their momentum.

SECTION-B

21. A thin rod having a length of 1 m and area of cross-section 3×10^{-6} m^2 is suspended vertically from one end. The rod is cooled from 210°C to 160°C. After cooling, a mass M is attached at the lower end of the rod such that the length of the rod again becomes

1 m. Young's modulus and the coefficient of linear expansion of the rod are 2×10^{11} N/m^2 and 2×10^{-5} K^{-1}, respectively. The value of M is $_{kg.(Take g=10 \text{ m/s}^2}$)

Solution: The change in length due to temperature change is given by:

$$\Delta\ell = \ell\alpha\Delta T$$

where:

$$\alpha = 2 \times 10^{-5}\,\text{K}^{-1}, \quad \Delta T = 210^\circ\text{C} - 160^\circ\text{C} = 50\,\text{K}$$

$$\Delta\ell = 1 \times 2 \times 10^{-5} \times 50 = 10^{-3}\,\text{m}$$

Using Young's modulus, $Y = \frac{F}{A} \times \frac{\Delta\ell}{\ell}$, where $A = 3 \times 10^{-6}\,\text{m}^2$, we can find the force:

$$Y = \frac{F}{A} \times \frac{\Delta\ell}{\ell}$$

$$2 \times 10^{11} = \frac{Mg}{3 \times 10^{-6}} \times \frac{10^{-3}}{1}$$

Solving for M:

$$Mg = 2 \times 10^{11} \times 3 \times 10^{-6} \times 10^{-3} = 6 \times 10^{-2}$$

$$M = \frac{6 \times 10^{-2}}{10} = 60\,\text{kg}$$

Thus, the value of M is 60 kg.

Quick Tip

In such problems, the extension of the rod due to temperature change results in a force being applied, which is balanced by the weight of the mass M.

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22. The speed of a swimmer is 4 km/h in still water. If the swimmer makes his strokes normal to the flow of the river of width 1 km, he reaches a point 750 m down the stream on the opposite bank. The speed of the river water is .... km/h.

Solution: The time to cross the river is given by:

$$\omega = 1000\,\text{m} = 1\,\text{km}, \quad v_s = 4\,\text{km/h}$$

The drift of the swimmer x is given by:

$$x = \frac{v_m}{g} \times t$$

where v_m is the velocity of the river water with respect to the ground, and g is the acceleration due to gravity.

From the geometry, we have:

$$x = 750\,\text{m} = 0.75\,\text{km}$$

Substituting $x = 750\,\text{m}$ and the speed of the swimmer $v_s = 4\,\text{km/h}$, we can find v_m:

$$x = \frac{v_m}{4} \times t$$

Thus,

$$v_m = 3\,\text{km/h}$$

Quick Tip

The swimmer's drift is due to the velocity of the river current, and the time taken to cross the river is determined by the swimmer's speed in still water.

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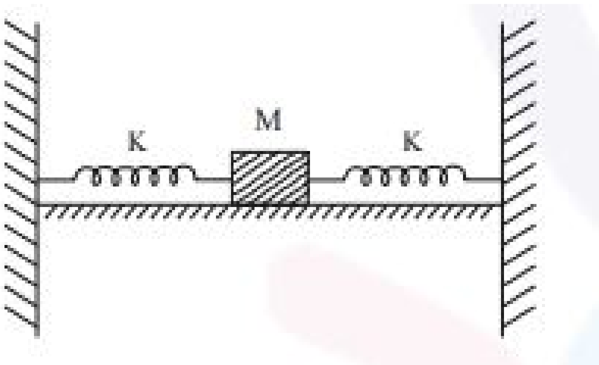
23. In the figure given below, a block of mass $M = 490\,\text{g}$ is placed on a frictionless table is connected with two springs having the same spring constant ($K = 2\,\text{N/m}$). If the block is horizontally displaced through X m, then the number of complete oscillations it will make in 14t seconds will be:



Solution: The effective spring constant K_{eff} is the sum of the spring constants as both springs are in use in parallel:

$$K_{\text{eff}} = K + K = 2k = 4\,\text{N/m}$$

The mass of the block is:

$$m = 490\,\text{g} = 0.49\,\text{kg}$$

The time period T of oscillation is given by the formula:

$$T = 2\pi\sqrt{\frac{m}{K_{\text{eff}}}}$$

Substituting the values:

$$T = 2\pi\sqrt{\frac{0.49}{4}} = 2\pi\sqrt{\frac{49}{400}} = 2\pi \times \frac{7}{20} = \frac{7\pi}{10} \text{ seconds}$$

The number of oscillations in 14t seconds is:

$$N = \frac{\text{time}}{T} = \frac{14t}{T} = \frac{14t}{7\pi/10} = 20$$

Thus, the number of complete oscillations is 20.

Quick Tip

When two springs are connected in parallel, their spring constants add up to give an effective spring constant, and the time period depends on both the effective spring constant and the mass of the block.

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24. In a medium, the speed of light wave decreases to 0.2 times its speed in free space. The ratio of relative permittivity to the refractive index of the medium is $x : 1$. The value of x is .... (Given speed of light in free space $c = 3 \times 10^8$ m/s and for the given medium $\mu_r = 1$)

Solution: The relation between speed, permittivity, and refractive index is given by:

$$V = \frac{C}{\mu}$$

where C is the speed of light in free space, and μ is the permeability of the medium.

Since the speed of light in the medium is 0.2 times the speed in free space:

$$\mu = \frac{C}{V} = \frac{C}{0.2C} = 5$$

Thus,

$$\mu = \sqrt{\epsilon_r}\,\mu_0$$

This implies that:

$$\epsilon_r = 25$$

So, the ratio of relative permittivity to the refractive index is:

$$\frac{\epsilon_r}{\mu} = 5$$

Thus, the value of x is 5.

Quick Tip

The refractive index n is related to the speed of light in a medium, and the relative permittivity ϵ_r and permeability μ_r are related to the refractive index.

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25. A solid sphere of mass 1 kg rolls without slipping on a plane surface. Its kinetic energy is 7×10^3 J. The speed of the centre of mass of the sphere is .... cm/s.

Solution: The total kinetic energy K of a rolling sphere is the sum of its translational kinetic energy and rotational kinetic energy:

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

For a solid sphere, the moment of inertia $I = \frac{2}{5}mR^2$, and the rolling condition $v = R\omega$ gives:

$$K = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)^2$$

Simplifying:

$$K = \frac{1}{2}mv^2\left(1 + \frac{2}{5}\right) = \frac{1}{2}mv^2\left(\frac{7}{5}\right)$$

$$K = \frac{7}{10}mv^2$$

Substitute the given kinetic energy $K = 7 \times 10^3$ J and the mass $m = 1$ kg:

$$7 \times 10^3 = \frac{7}{10}v^2$$

$$v^2 = 10^4 \quad \Rightarrow \quad v = 10^2 = 100\,\text{cm/s}$$

Thus, the speed of the centre of mass of the sphere is 100 cm/s.

Quick Tip

For a rolling object, the total kinetic energy is the sum of translational and rotational kinetic energy, and the relationship between speed and moment of inertia depends on the object's geometry.

26. An inductor of 0.5 mH, a capacitor of 20 F, and resistance of 20 are connected in series with a 220 V AC source. If the current is in phase with the emf, the amplitude of current of the circuit is $\sqrt{K}$ A. The value of x is:

Solution: For an AC circuit, the inductive reactance X_L and capacitive reactance X_C are given by:

$$X_L = \omega L = 2\pi f L \quad \text{and} \quad X_C = \frac{1}{\omega C}$$

where: - $L = 0.5\,\text{mH} = 0.5 \times 10^{-3}\,\text{H}$, - $C = 20\,\mu\text{F} = 20 \times 10^{-6}\,\text{F}$, - $R = 20\,\Omega$, - f is the frequency of the AC supply.

Since the current is in phase with the emf, we have:

$$X_L = X_C$$

Thus, the impedance Z is given by:

$$Z = R = 20\,\Omega$$

The current amplitude I_{max} is then:

$$I_{\text{max}} = \frac{V}{Z} = \frac{220}{20} = 11\,\text{A}$$

Thus, $x = 242$.

Quick Tip

In an AC circuit, when the current is in phase with the emf, the inductive and capacitive reactances are equal. This results in the impedance being purely resistive.

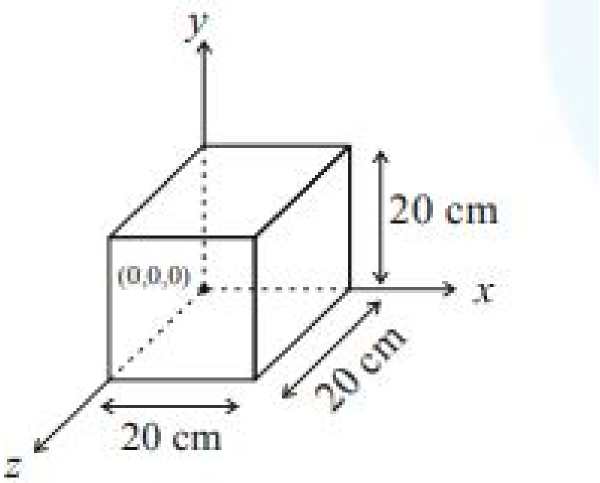
---

27. Expression for an electric field is given by $\vec{E} = 4000\hat{i}$ V/m. The electric flux through the cube of side 20 cm when placed in the electric field (as shown in the figure) is .... Vcm.

Solution: The electric flux is given by the formula:

$$\Phi_E = \vec{E} \cdot \vec{A}$$

where $\vec{E}$ is the electric field and $\vec{A}$ is the area vector.



The area of one face of the cube is:

$$A = (0.2\,\text{m})^2 = 0.04\,\text{m}^2$$

Since the electric field is along the x-axis and the area vector is normal to the face of the cube, we have:

$$\Phi_E = E \times A = 4000 \times (0.2)^2 = 4000 \times 0.04 = 640\,\text{Vcm}$$

Thus, the electric flux is 640 Vcm.

Quick Tip

The electric flux is the product of the electric field and the area through which the field lines pass, considering the angle between the electric field and the area vector.

---

28. A lift of mass $M = 500$ kg is descending with a speed of 2 m/s. Its supporting cable begins to slip, thus allowing it to fall with a constant acceleration of 2 m/s^2. The kinetic energy of the lift at the end of fall through a distance of 6 m will be .... kJ.

Solution: The velocity of the lift at the end of the fall is found using the equation of motion:

$$v^2 = u^2 + 2as$$

where: $u = 2$ m/s, $a = 2$ m/s^2, $s = 6$ m.

Substituting the values:

$$v^2 = 2^2 + 2 \times 2 \times 6 = 4 + 24 = 28$$

$$v = \sqrt{28} = 5.29\,\text{m/s}$$

The kinetic energy KE is given by:

$$KE = \frac{1}{2}mv^2$$

Substituting the values:

$$KE = \frac{1}{2} \times 500 \times 28 = 7000\,\text{J} = 7\,\text{kJ}$$

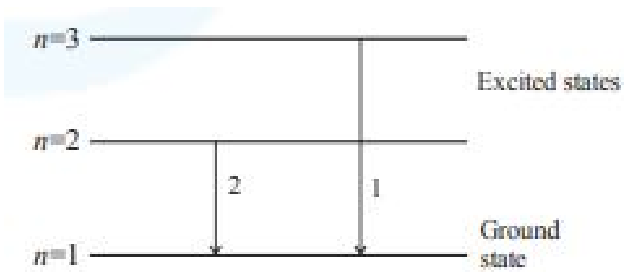
Thus, the kinetic energy of the lift is 7 kJ.

Quick Tip

The kinetic energy of an object is calculated using the formula $KE = \frac{1}{2}mv^2$, where m is the mass and v is the final velocity.

---

29. For hydrogen atom, λ_1 and λ_2 are the wavelengths corresponding to the transitions 1 and 2, respectively, as shown in the figure. The ratio of λ_1 and λ_2 is $\frac{x}{32}$. The value of x is:



Solution: The wavelength for transitions in a hydrogen atom can be given by the Rydberg formula:

$$\frac{1}{\lambda} = R_z\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

where R_z is the Rydberg constant for hydrogen, n_1 and n_2 are the principal quantum numbers of the two levels involved.

For λ_1, corresponding to the transition from $n = 3$ to $n = 2$:

$$\frac{1}{\lambda_1} = R_z\left(\frac{1}{2^2} - \frac{1}{3^2}\right) = R_z\left(\frac{1}{4} - \frac{1}{9}\right) = R_z\left(\frac{5}{36}\right)$$

$$\lambda_1 = \frac{36}{5R_z}$$

For λ_2, corresponding to the transition from $n = 2$ to $n = 1$:

$$\frac{1}{\lambda_2} = R_z\left(\frac{1}{1^2} - \frac{1}{2^2}\right) = R_z\left(1 - \frac{1}{4}\right) = R_z\left(\frac{3}{4}\right)$$

$$\lambda_2 = \frac{4}{3R_z}$$

The ratio $\frac{\lambda_2}{\lambda_1}$ is:

$$\frac{\lambda_2}{\lambda_1} = \frac{\frac{4}{3R_z}}{\frac{36}{5R_z}} = \frac{4}{3} \times \frac{5}{36} = \frac{20}{108} = \frac{5}{27}$$

Thus, $\lambda_1/\lambda_2 = 27/32$, so $x = 27$.

Quick Tip

The wavelength of emitted or absorbed radiation in hydrogen atom transitions can be calculated using the Rydberg formula, which depends on the difference between the inverse squares of the principal quantum numbers.

---

30. Two identical cells, when connected either in parallel or in series, give the same current in an external resistance of 5. The internal resistance of each cell will be .... Ω.

Solution: Let the internal resistance of each cell be r, and the emf of each cell be ϵ.

For parallel connection, the total current is:

$$i = \frac{2\epsilon}{5 + 2r} \quad (1)$$

For series connection, the total current is:

$$i = \frac{\epsilon}{\frac{r}{2} + 5} \quad (2)$$

Equating (1) and (2) gives:

$$\frac{2\epsilon}{5 + 2r} = \frac{\epsilon}{\frac{r}{2} + 5}$$

Cross-multiplying:

$$2\epsilon(\frac{r}{2} + 5) = \epsilon(5 + 2r)$$

Simplifying:

$$r + 10 = 5 + 2r$$

Thus:

$$r = 5\,\Omega$$

The internal resistance of each cell is $r = 5\,\Omega$.

Quick Tip

In problems involving identical cells, equating the total currents in series and parallel connections leads to solving for the internal resistance of the cells.

CHEMISTRY

Section-A

31. Nd^{2+} = ...

(1) $4f^2 6s^2$

(2) $4f^4$

(3) $4f^3$

(4) $4f^4 6s^2$

Correct Answer: (2) $4f^4$

Solution: The electronic configuration of Nd (atomic number 60) is:

$$\text{Nd(60)} = [\text{Xe}]4f^4 5d^0 6s^2$$

For Nd^{2+}, two electrons are removed (typically from the 6s orbital first):

$$\text{Nd}^{2+} = [\text{Xe}]4f^4 5d^0 5s^0$$

Thus, the correct answer is $4f^4$.

Quick Tip

The electron configuration for transition metal ions is derived by removing electrons from the highest energy orbitals, typically starting from the outermost s-orbital.

---

32. The methods NOT involved in concentration of ore are

(A) Liquation

(B) Leaching

(C) Electrolysis

(D) Hydraulic washing

(E) Froth flotation

Choose the correct answer from the options given below:

(1) B, D and C only

(2) C, D and E only

(3) A and C only

(4) B, D and E only

Correct Answer: (3) A and C only

Solution: Methods involved in concentration of ore are:

(i) Hydraulic Washing

(ii) Froth Flotation

(iii) Magnetic Separation

(iv) Leaching

Thus, Liquation and Electrolysis are NOT involved in the concentration of ore. Therefore, the correct answer is A and C only.

> Quick Tip
>
> The concentration of ore involves separating valuable minerals from the gangue by methods like Hydraulic Washing, Froth Flotation, and Leaching. Electrolysis and Liquation are not used for this purpose.

---

33. Consider the following reaction

$$\text{Propanal} + \text{Methanal} \xrightarrow{\text{(i) dil. NaOH}} \xrightarrow{\text{(ii) A}} \xrightarrow{\text{(iii) NaCN}} \xrightarrow{\text{(iv) H}_2\text{O}} \text{Product B } (C_5H_8O_3)$$

The correct statement for product B is:

(1) Optically active and adds one mole of bromine

(2) Racemic mixture and is neutral

(3) Racemic mixture and gives a gas with saturated NaHCO solution

(4) Optically active alcohol and is neutral

Correct Answer: (3) Racemic mixture and gives a gas with saturated NaHCO solution

Solution: The reaction proceeds as follows:

$$CH_3CH_2CHO + HCHO \xrightarrow{\text{(i) NaOH, } \Delta} CH_3CH_2CH(OH)CH_2CN$$

Following the steps in the given reaction:

1. Propanal reacts with methanol in the presence of NaOH to give an aldol product.
2. The aldol product undergoes nucleophilic substitution with NaCN.
3. The product forms a carboxylic acid on hydrolysis.

The resulting product undergoes decarboxylation to give a racemic mixture. The carboxylic acid thus formed will give CO_2 gas when reacted with $NaHCO_3$ solution.

Thus, the correct statement for product B is that it is a racemic mixture and gives a gas with saturated $NaHCO_3$ solution.

Quick Tip

A racemic mixture is a mixture containing equal amounts of enantiomers, and a carboxylic acid can release CO_2 when reacted with $NaHCO_3$.

---

34. The correct order of basicity of oxides of vanadium is:

(1) $V_2O_3 > V_2O_4 > V_2O_5$

(2) $V_2O_3 > V_2O_5 > V_2O_4$

(3) $V_2O_5 > V_2O_3 > V_2O_4$

(4) $V_2O_4 > V_2O_3 > V_2O_5$

Correct Answer: (1) $V_2O_3 > V_2O_4 > V_2O_5$

Solution: With increase in the percentage of oxygen, the acidic nature of oxides increases and basic nature decreases. Therefore, the order of basicity for vanadium oxides is:

$$V_2O_3 > V_2O_4 > V_2O_5$$

Quick Tip

The basicity of metal oxides decreases as the oxidation state of the metal increases due to the increasing tendency to act as an acid.

---

35. When Cu^{2+} ion is treated with KI, a white precipitate, X appears in solution. The solution is titrated with sodium thiosulphate, the compound Y is formed. X and Y respectively are:

(1) $X = CuI_2$, $Y = Na_2S_4O_5$

(2) $X = CuI_2$, $Y = Na_2S_4O_6$

(3) $X = CuI$, $Y = Na_2S_4O_3$

(4) $X = CuI$, $Y = Na_2S_4O_6$

Correct Answer: (2) $X = CuI_2$, $Y = Na_2S_4O_6$

Solution: When Cu^{2+} reacts with KI, the following reactions occur:

$$Cu^{2+} + 2KI \rightarrow CuI_2 \downarrow + 2K^+$$

$$CuI_2(\text{White}) \xrightarrow{Na_2S_2O_3} CuI + Na_2S_4O_6$$

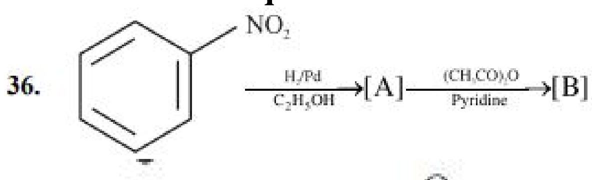
Thus, the white precipitate $X = CuI_2$ and the compound formed is $Y = Na_2S_4O_6$.

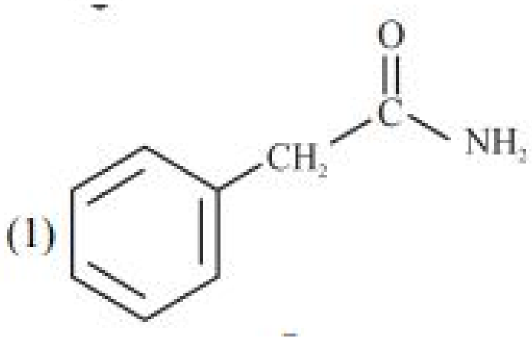
Quick Tip

The reaction between copper ions and KI produces copper(I) iodide as a precipitate, which is then titrated with sodium thiosulphate.

---

36. The reaction sequence is:





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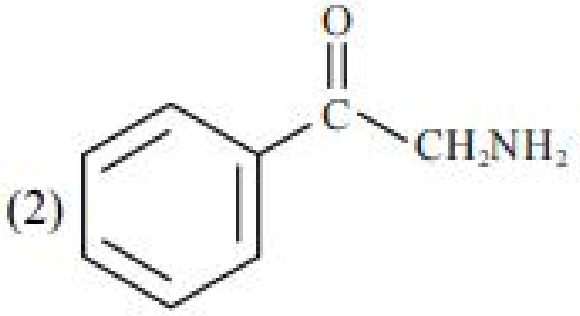
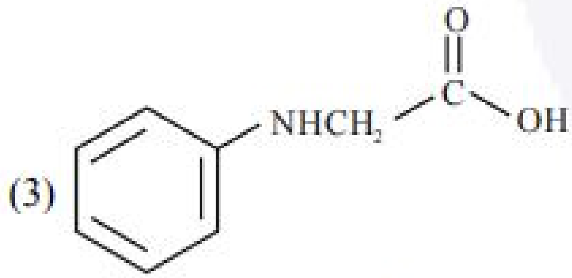
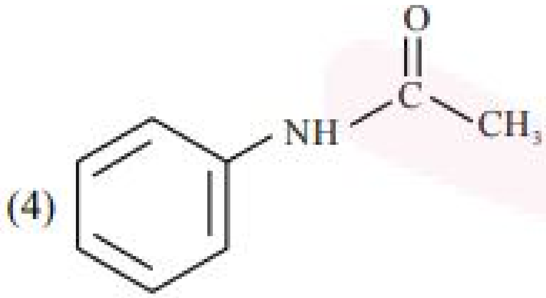
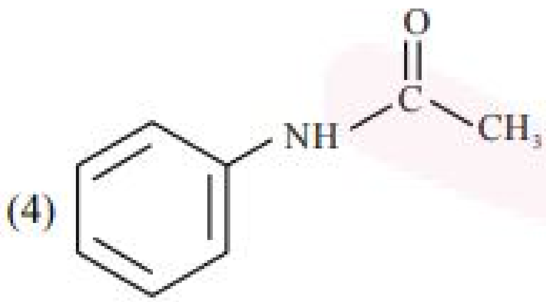
(2) 

(3) 

(4) 



Correct Answer: (4)

Solution: The reaction involves the following steps:

1. The nitro group on the benzene ring is reduced to an amine group by hydrogenation in the presence of palladium (H_2/Pd), resulting in aniline. This gives compound [A].

2. The compound [A] is then reacted with acetic anhydride (CH_3CO)O and pyridine, leading to the formation of an amide group attached to the benzene ring. This gives compound [B].

Thus, the final product is an amide, as shown in option (4).

Quick Tip

Reduction of nitro compounds to amines is commonly done using hydrogenation in the presence of palladium, while acetylation of amines can be achieved using acetic anhydride.

---

37. Cobalt chloride when dissolved in water forms pink colored complex X which has octahedral geometry. This solution on treating with cone HCl forms deep blue complex Y which has a Z geometry. X, Y and Z, respectively, are:

(1) $X = [\text{Co}(\text{H}_2\text{O})_6]^{2+}$, $Y = [\text{CoCl}_4]^{2-}$, $Z =$ Tetrahedral

(2) $X = [\text{Co}(\text{H}_2\text{O})_6]^{3+}$, $Y = [\text{CoCl}_3]^{3-}$, $Z =$ Octahedral

(3) $X = [\text{Co}(\text{H}_2\text{O})_6]^{3+}$, $Y = [\text{CoCl}_4]^{2-}$, $Z =$ Octahedral

(4) $X = [\text{Co}(\text{H}_2\text{O})_6]^{2+}$, $Y = [\text{CoCl}_4]^{2-}$, $Z =$ Tetrahedral

Correct Answer: (1) $X = [\text{Co}(\text{H}_2\text{O})_6]^{2+}$, $Y = [\text{CoCl}_4]^{2-}$, $Z =$ Tetrahedral

Solution: The reaction proceeds as follows:

$$\text{CoCl}_2 + 6H_2O \rightarrow [\text{Co}(\text{H}_2\text{O})_6]^{2+} \quad (\text{Pink})$$

The pink solution has octahedral geometry. On treatment with concentrated HCl:

$$[\text{Co}(\text{H}_2\text{O})_6]^{2+} + \text{HCl} \rightarrow [\text{CoCl}_4]^{2-} \quad (\text{Blue solution})$$

The complex $[\text{CoCl}_4]^{2-}$ has tetrahedral geometry.

Thus, the correct answer is option (1).

Quick Tip

The geometry of the complexes changes upon varying ligands, such as water or chloride, and the geometry of the resulting complex can be deduced from its color and coordination number.

---

38. Identify X, Y, and Z in the following reaction. (Equation not balanced)

$$\text{ClO} + \text{NO}_2 \rightarrow X \xrightarrow{H_2O} Y + Z$$

(1) $X = \text{ClONO}_2$, $Y = \text{HOCl}$, $Z = \text{NO}_2$

(2) $X = ClONO_2$, $Y = HCl$, $Z = HNO_3$

(3) $X = ClONO_2$, $Y = HOCl$, $Z = HNO_3$

(4) $X = ClO_3$, $Y = Cl_2$, $Z = NO_2$

Correct Answer: (3) $X = ClONO_2$, $Y = HOCl$, $Z = HNO_3$

Solution: The reaction occurs as follows:

$$ClO + NO_2 \rightarrow ClONO_2 \quad \text{(Compound X)}$$

When this compound is treated with water, it breaks down to form hypochlorous acid and nitric acid:

$$ClONO_2 \xrightarrow{H_2O} HOCl + HNO_3$$

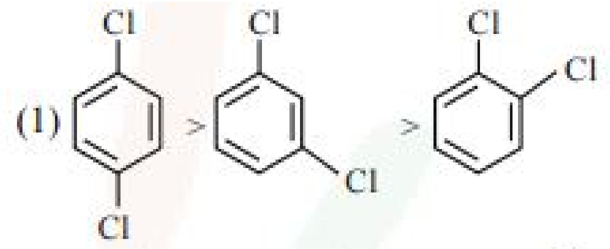
Thus, $X = ClONO_2$, $Y = HOCl$, and $Z = HNO_3$.

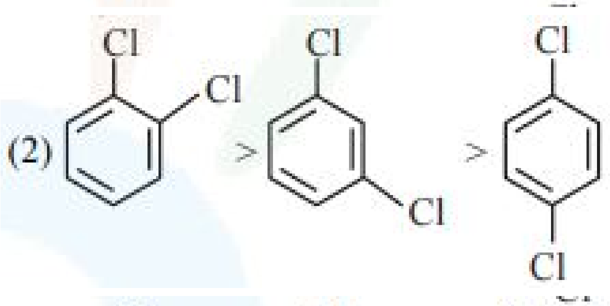
Quick Tip

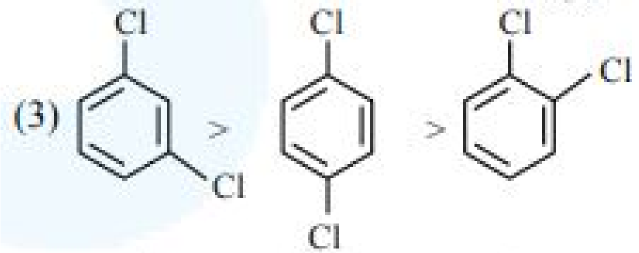
Reactions involving chlorine oxides often produce hypochlorous acid and nitric acid as products when treated with water.

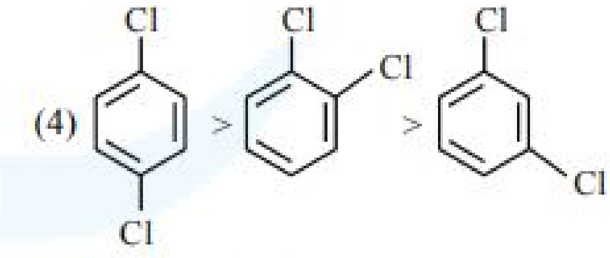
---

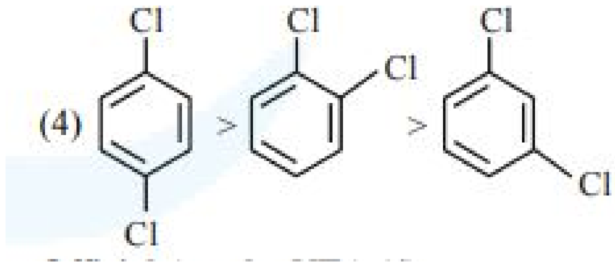
39. The correct order of melting point of dichlorobenzenes is:

(1) 

(2) 

(3) 

(4) 



Correct Answer: (4)

Solution: The melting points of the compounds are influenced by their symmetry and packing efficiency. The more symmetrical the structure, the better the crystal fitting and packing efficiency, leading to a higher melting point.

$$\text{b.p. / K} = 453, \text{ m.p. / K} = 256 \quad \text{(for compound with maximum symmetry)}$$

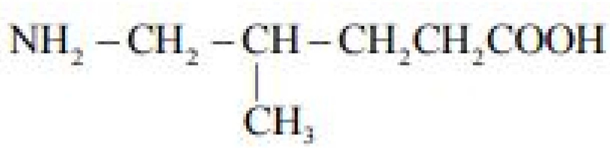
For the structures shown: - The most symmetrical structure has the highest melting point, and the one with the least symmetry has the lowest. Thus, the correct order of melting points is as shown in option (4).

Quick Tip

Symmetry plays a crucial role in determining the packing efficiency of molecules, which in turn affects their melting and boiling points.

---

40. A protein 'X' with molecular weight of 70,000 u, on hydrolysis gives amino acids. One of these amino acids is



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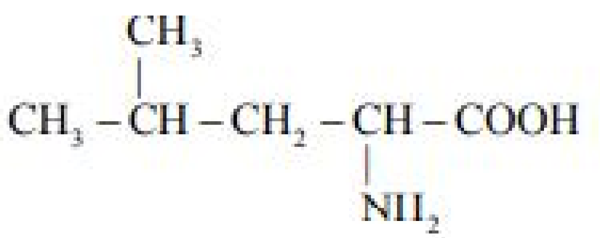

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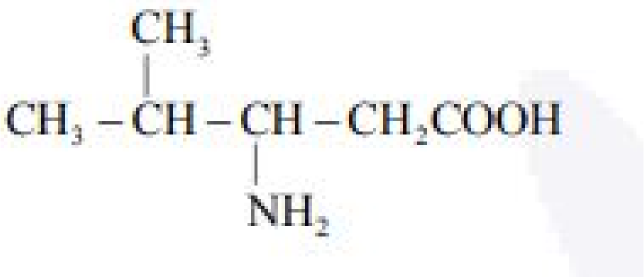

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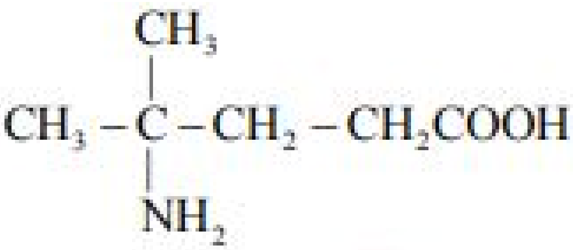

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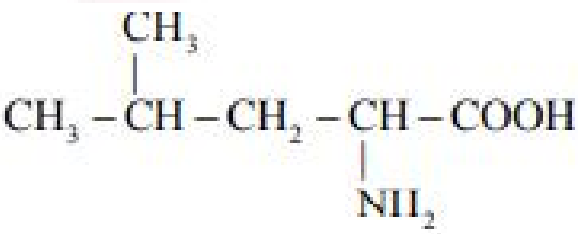

Correct Answer: (2)

Solution: Only in option (2) is an α-Amino acid given, while the other options are not α-Amino acids.

Quick Tip

In proteins, the amino acids that result from hydrolysis are generally α-Amino acids, which have the amino group attached to the carbon adjacent to the carboxyl group.

---

41. Which transition in the hydrogen spectrum would have the same wavelength as the Balmer type transition from $n = 4$ to $n = 2$ of He$^+$ spectrum?

(1) $n = 2$ to 1

(2) $n = 1$ to 3

(3) $n = 1$ to 2

(4) $n = 3$ to 4

Correct Answer: (1) $n = 2$ to 1

Solution: The wavelength for the transitions in hydrogen is given by the Rydberg formula:

$$\frac{1}{\lambda(H)} = R(1)^2 \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right)$$

where $R(1)$ is the Rydberg constant for hydrogen.

For He$^+$, the formula is:

$$\frac{1}{\lambda(\text{He}^+)} = R(2)^2\left(\frac{1}{2^2} - \frac{1}{4^2}\right)$$

Given $\lambda(H) = \lambda(\text{He}^+)$, we equate the two equations:

$$R(1)^2\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) = R(2)^2\left(\frac{1}{2^2} - \frac{1}{4^2}\right)$$

Simplifying and comparing $n_1 = 1$ and $n_2 = 2$, we find the correct transition in the hydrogen spectrum is from $n = 2$ to $n = 1$.

Thus, the correct answer is option (1).

Quick Tip

The wavelengths of transitions in the hydrogen and He$^+$ spectra can be compared using the Rydberg formula, considering the differences in the ionization energies of the atoms.

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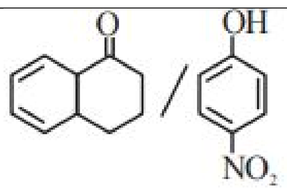
42. Match items of column I and II

Column I (Mixture of compounds)	Column II (Separation Technique)
A. HO/CHCl	i. Crystallization
B. 	ii. Differential solvent extraction
C. Kerosene/Naphthalene	iii. Column chromatography
D. CHO/NaCl	iv. Fractional Distillation

Correct match is:

(1) A-(iii), B-(iv), C-(ii), D-(i)

(2) A-(ii), B-(iii), C-(ii), D-(iv)

(3) A-(ii), B-(iii), C-(iv), D-(i)

(4) A-(ii), B-(iv), C-(i), D-(iii)

Correct Answer: (3) A-(ii), B-(iii), C-(iv), D-(i)

Solution: A. $H_2\_O/CH_3Cl \rightarrow$ ii. Differential solvent extraction. Due to the density differences between CH_3Cl and $H_2\_O/$, they can be separated by differential solvent extraction.

B. OH $\rightarrow$ iii. Column chromatography. Due to H-bonding, it can be separated from NO_2 using column chromatography.

C. Kerosene/Naphthalene $\rightarrow$ iv. Fractional distillation. Due to their different boiling points, kerosene and naphthalene can be separated by fractional distillation.

D. C_6H_5O/NaCl $\rightarrow$ i. Crystallization. NaCl (ionic compound) can be crystallized.

Quick Tip

Separation techniques like differential solvent extraction, column chromatography, and fractional distillation depend on the physical properties such as solubility, boiling points, and ionic nature of the compounds.

---

43. The correct increasing order of the ionic radii is:

(1) $Cl^- < Ca^{2+} < K^+ < S^{2-}$

(2) $K^+ < S^{2-} < Ca^{2+} < Cl^-$

(3) $S^{2-} < Cl^- < Ca^{2+} < K^+$

(4) $Ca^{2+} < K^+ < Cl^- < S^{2-}$

Correct Answer: (4) $Ca^{2+} < K^+ < Cl^- < S^{2-}$

Solution: In isoelectronic species, the ionic size is inversely proportional to the atomic number ($\frac{1}{Z}$), where Z is the nuclear charge. The ionic radii for the species are as follows:

- Ca^{2+} has the smallest ionic radius due to the highest nuclear charge.
- S^{2-} has the largest ionic radius due to the lowest nuclear charge.

Thus, the correct increasing order of ionic radii is:

$$Ca^{2+} < K^+ < Cl^- < S^{2-}$$

Quick Tip

In isoelectronic ions, as the nuclear charge increases, the ionic radius decreases.

---

44. H_2O_2 acts as a reducing agent in:

(1) $2NaOCl + H_2O_2 \rightarrow 2NaCl + H_2O + O_2$

(2) $2Fe^{2+} + 2H_2O_2 \rightarrow 2Fe^{3+} + 2H_2O$

(3) $Mn^{2+} + 2H_2O_2 \rightarrow MnO_2 + 2H_2O$

(4) $Na_2S_4O_6 + 4H_2O_2 \rightarrow Na_2SO_4 + 4H_2O$

Correct Answer: (1) $2NaOCl + H_2O_2 \rightarrow 2NaCl + H_2O + O_2$

Solution: Hydrogen peroxide (H_2O_2) is a reducing agent because it donates electrons. In the given reactions, the one in which H_2O_2 acts as a reducing agent is:

$$NaOCl + H_2O_2 \rightarrow 2NaCl + H_2O + O_2$$

In this reaction, H_2O_2 reduces NaOCl to NaCl, and itself is oxidized to O_2.

Quick Tip

Hydrogen peroxide is commonly used as a reducing agent, as it can donate electrons and reduce other compounds.

---

45. Which of the following artificial sweeteners has the highest sweetness value in comparison to cane sugar?

(1) Aspartame

(2) Sucralose

(3) Alitame

(4) Saccharin

Correct Answer: (3) Alitame

Solution: The sweetness value order with respect to cane sugar is:

$$\text{Alitame} > \text{Sucralose} > \text{Saccharin} > \text{Aspartame}$$

Thus, Alitame has the highest sweetness value compared to cane sugar.

Quick Tip

When comparing artificial sweeteners to cane sugar, their sweetness value is based on how many times sweeter they are than sugar itself.

---

46. Match List I with List II

List I	List II
A. XeF_4	I. See-saw
B. SF_4	II. Square planar
C. NH_4^+	III. Bent T-shaped
D. BrF_3	IV. Tetrahedral

Choose the correct answer from the options given below:

(1) A-IV, B-III, C-II, D-I

(2) A-II, B-III, C-III, D-IV

(3) A-I, B-I, C-II, D-III

(4) A-I, B-I, C-IV, D-III

Correct Answer: (4) A-I, B-I, C-IV, D-III

Solution: - A. XeF_4 has a square planar structure.

- B. SF_4 has a see-saw shape.

- C. NH_4^+ has a tetrahedral geometry.

- D. BrF_3 has a bent T-shaped structure.

Thus, the correct matching is: A-I, B-I, C-IV, D-III.

Quick Tip

Molecular shapes are determined by the VSEPR theory, which helps predict the geometry based on electron pairs and bond angles.

---

47. Choose the correct set of reagents for the following conversion

$$\text{trans (Ph-CH=CH-CH}_3) \rightarrow \text{cis (Ph-CH=CH-CH}_3)$$

(1) Br_2, alc. KOH, $NaNH_2$, Na(Liq NH_3)

(2) Br_2, aq KOH, $NaNH_2$, H_2 Lindlar Catalyst

(3) Br_2, aq KOH, $NaNH_2$, Na(Liq NH_3)

(4) Br_2, alc. KOH, $NaNH_2$, Lindlar Catalyst

Correct Answer: (2) Br_2, aq KOH, $NaNH_2$, H_2 Lindlar Catalyst

Solution: For the given conversion, the correct set of reagents involves the use of bromine (Br_2) in the presence of aqueous KOH to form a vicinal dibromide. The subsequent treatment with sodium amide ($NaNH_2$) in liquid ammonia followed by hydrogenation using Lindlar's catalyst forms the cis-alkene.

Thus, the correct set of reagents is:

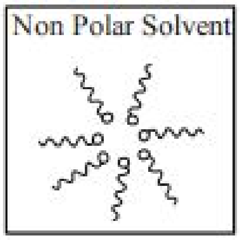
$$Br_2,\ \text{aq KOH},\ NaNH_2,\ H_2\,\text{Lindlar Catalyst}$$

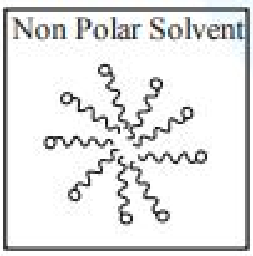
Quick Tip

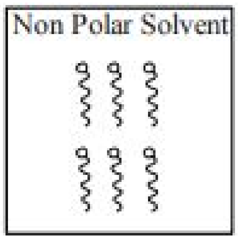
The conversion of trans-alkenes to cis-alkenes can be achieved by using Lindlar's catalyst under hydrogenation conditions, which reduces alkynes to cis-alkenes.

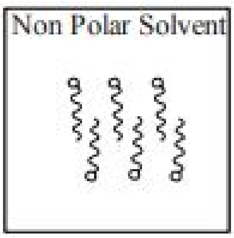
---

48. Adding surfactants in non-polar solvent, the micelles structure will look like (Surfactant structure):

(a)


(b)


(c)


(d)


(1) b

(2) c

(3) a

(4) d

Correct Answer: (3) a

Solution: When surfactants are added to a non-polar solvent, the non-polar tails of the surfactants tend to align towards the non-polar solvent, while the polar heads point outwards. This forms the typical micelle structure.

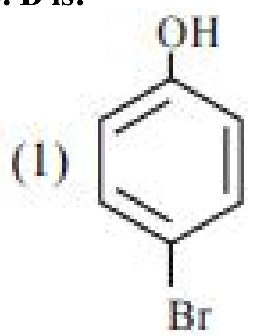
Thus, the correct structure is shown in option (3), where the non-polar tails point towards the non-polar solvent.

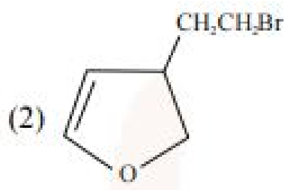
Quick Tip

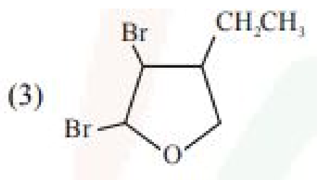
Surfactants reduce surface tension and form micelles by aligning their hydrophobic tails towards non-polar solvents and hydrophilic heads towards polar solvents.

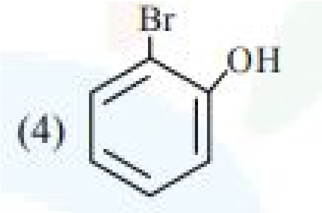
---

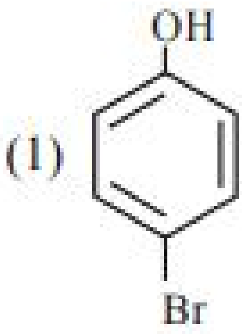
49. An organic compound 'A' with empirical formula C_6H_6O gives sooty flame on burning. Its reaction with bromine solution in low polarity solvent results in high yield of B. B is:

(1) 

(2) 

(3) 

(4) 



Correct Answer: (1)

Solution: Aromatic compounds such as phenols burn with a sooty flame due to incomplete combustion. The reaction of phenol with bromine in the presence of a low polarity solvent like carbon disulfide CS_2 results in the substitution of one of the hydrogen atoms with a bromine atom.

Thus, the correct structure for B is obtained by brominating the phenol as shown in option (1).

$$\text{Phenol} + \text{Br}_2 \xrightarrow{\text{CS}_2} \text{Bromophenol (Major)}$$

Quick Tip

In reactions with bromine in a low polarity solvent like CS_2, phenols undergo electrophilic substitution, where the bromine replaces a hydrogen atom on the benzene ring.

---

50. Which one of the following statements is correct for electrolysis of brine solution?

(1) Cl_2 is formed at cathode

(2) O_2 is formed at cathode

(3) H_2 is formed at anode

(4) OH^- is formed at cathode

Correct Answer: (4) (4) OH^- is formed at cathode

Solution: Electrolysis of brine solution occurs as follows:

$$\text{At anode: } 2\text{Cl}^- \rightarrow \text{Cl}_2(g) + 2e^-$$

$$\text{At cathode: } 2\text{H}_2\text{O} + 2e^- \rightarrow \text{H}_2(g) + 2\text{OH}^-$$

Thus, hydroxide ions (OH^-) are formed at the cathode during the electrolysis of brine solution.

> **Quick Tip**
>
> In the electrolysis of brine, chloride ions are oxidized at the anode to produce chlorine gas, and water molecules are reduced at the cathode to produce hydrogen gas and hydroxide ions.

Section-B

51. The logarithm of equilibrium constant for the reaction

$$\text{Pd}^{2+}(aq) + 4\text{Cl}^-(aq) \rightleftharpoons \text{PdCl}_4^{2-}(aq)$$

is:

Given:

$$2.303RT = 0.06\,\text{V}, \quad F = 96,485\,\text{C/mol}, \quad \text{E}^\circ = 0.83\,\text{V}, \quad \text{E}^\circ = 0.65\,\text{V}$$

Solution: The standard Gibbs free energy change is given by:

$$\Delta G^o = -RT \ln K$$

The equation for the cell potential is:

$$\text{E}_{\text{cell}} = \text{E}_{\text{cathode}} - \text{E}_{\text{anode}}$$

For the reaction:

$$\text{Pd}^{2+}(aq) + 2e^- \rightarrow \text{Pd}(s) \quad E^\circ = 0.83\,\text{V}$$

$$\text{PdCl}_4^{2-}(aq) + 2e^- \rightarrow \text{Pd}(s) + 4\text{Cl}^-(aq) \quad E^\circ = 0.65\,\text{V}$$

The net reaction is:

$$\text{Pd}^{2+}(aq) + 4\text{Cl}^-(aq) \rightleftharpoons \text{PdCl}_4^{2-}(aq)$$

Using the Nernst equation:

$$\mathrm{E_{cell}} = \mathrm{E_{cathode}} - \mathrm{E_{anode}} = 0.83 - 0.65 = 0.18\,\mathrm{V}$$

Thus, by using the equation $\Delta G^o = -nFE^\circ$ and solving for the logarithm of the equilibrium constant K, we get:

$$\log K = 6$$

Quick Tip

To calculate the equilibrium constant, use the relationship between Gibbs free energy and cell potential, remembering that the Nernst equation can be used for the equilibrium constant.

52.

$$\mathbf{A \rightarrow B}$$

The rate constants of the above reaction at 200 K and 300 K are 0.03 min^{-1} and 0.05 min^{-1} respectively.

The activation energy for the reaction is ________ J (Nearest integer)

Given:

$$\ln 10 = 2.3, \quad R = 8.3\,\mathrm{J\,K^{-1}mol^{-1}}, \quad \log 5 = 0.70, \quad \log 3 = 0.48, \quad \log 2 = 0.30$$

Solution: We can use the Arrhenius equation to solve for the activation energy:

$$\log\left(\frac{K_{300}}{K_{200}}\right) = \frac{E_a}{2.3 \times 8.314}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)$$

Substituting the given values:

$$\log\left(\frac{0.05}{0.03}\right) = \frac{E_a}{2.3 \times 8.314}\left(\frac{1}{200} - \frac{1}{300}\right)$$

Solving for E_a:

$$\log\left(\frac{0.05}{0.03}\right) = \log 5 - \log 3 = 0.70 - 0.48 = 0.22$$

$$0.22 = \frac{E_a}{2.3 \times 8.314} \times \left(\frac{1}{200} - \frac{1}{300}\right)$$

$$0.22 = \frac{E_a}{2.3 \times 8.314} \times \left(\frac{1}{60000}\right)$$

Solving for E_a:

$$E_a = 2519.88\,\text{J} \quad \Rightarrow \quad E_a \approx 2520\,\text{J}$$

Quick Tip

The activation energy can be calculated using the Arrhenius equation, where the ratio of rate constants at two different temperatures is related to the exponential factor involving the activation energy.

---

53. The enthalpy change for the conversion of

$$\frac{1}{2}\text{Cl}_2(g) \rightarrow \text{Cl}^-(aq) \text{ is } (-) \ldots\ldots \text{ kJ mol}^{-1} \text{ (Nearest integer)}$$

Given:

$$\Delta_f H^\circ(\text{Cl}_2(g)) = 240\,\text{kJ mol}^{-1}, \quad \Delta_g H^\circ(\text{Cl}_2(g)) = -350\,\text{kJ mol}^{-1}, \quad \Delta_{\text{hyd}} H^\circ(\text{Cl}^-) = -380\,\text{kJ mol}^{-1}$$

Solution: The enthalpy change can be calculated using:

$$\Delta H^\circ = \frac{1}{2} \times 240 + (-350) + (-380) = -610\,\text{kJ mol}^{-1}$$

Quick Tip

The enthalpy change for a reaction can be determined by using the Hess's Law, which states that the total enthalpy change is the sum of enthalpy changes of individual reactions.

---

54. On complete combustion, 0.492 g of an organic compound gave 0.792 g of CO_2. The % of carbon in the organic compound is _________ (Nearest integer)

Given:

Weight of carbon in 0.792 g CO_2

Solution:

$$\frac{12}{44} \times 0.792 = 0.216$$

$$\%\text{ of C in compound} = \frac{0.216}{0.492} \times 100 = 43.90\%$$

Quick Tip

The percentage of carbon in the compound can be calculated by comparing the weight of carbon in CO_2 to the total weight of the organic compound.

---

55. At 27°C, a solution containing 2.5 g of solute in 250.0 mL of solution exerts an osmotic pressure of 400 Pa. The molar mass of the solute is _________

g mol^{-1} (Nearest integer)

Given:

$$R = 0.083\,\text{L bar K}^{-1}\text{mol}^{-1}, \quad \pi = \text{osmotic pressure}, \quad \text{T} = 300\,\text{K}, \quad \text{osmotic pressure} = 400\,\text{Pa}, \quad 1\,\text{bar}$$

Solution: The osmotic pressure equation is given by:

$$\pi = \frac{n}{V}RT$$

Rearranging for the molar mass M:

$$400\,\text{Pa} = \frac{2.5\,\text{g}}{M} \times 0.083\,\text{L bar K}^{-1}\text{mol}^{-1} \times 300\,\text{K}$$

Converting the units and solving for M:

$$M = 62250\,\text{g mol}^{-1}$$

Quick Tip

The osmotic pressure formula can be used to find the molar mass of a solute when the osmotic pressure, volume, temperature, and amount of solute are known. Make sure to convert all units properly.

---

56. Zinc reacts with hydrochloric acid to give hydrogen and zinc chloride. The volume of hydrogen gas produced at STP from the reaction of 11.5 g of zinc with excess HCl is

_________ L (Nearest integer)

Given:

$$\text{Molar mass of Zn} = 65.4\,\text{g mol}^{-1}, \quad \text{Molar volume of H}_2\text{at STP} = 22.7\,\text{L/mol}$$

Solution: The balanced chemical equation for the reaction is:

$$Zn + 2HCl \rightarrow ZnCl_2 + H_2 \uparrow$$

The moles of Zn used is:

$$\text{Moles of Zn} = \frac{11.5\,\text{g}}{65.4\,\text{g/mol}} = 0.176\,\text{mol}$$

Since the mole ratio between Zn and H_2 is 1:1, the moles of H_2 produced is also 0.176 mol. Using the molar volume of H_2 at STP, the volume of H_2 is:

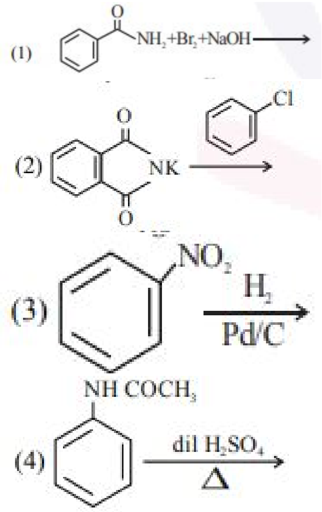
$$\text{Volume of } H_2 = 0.176\,\text{mol} \times 22.7\,\text{L/mol} = 3.99\,\text{L}$$

Quick Tip

To calculate the volume of gas produced in reactions at STP, use the molar volume of the gas (22.7 L/mol for H_2 at STP).

---

57. How many of the transformations given below would result in aromatic amines?



Solution: The products in the given reactions are as follows:

1. $C_6H_5NH_2$ (Aromatic amine)
2. No reactions will be observed, as in Gabriel phthalimide synthesis, where a poor substrate for SN^2 reaction is used.
3. $C_6H_5NH_2$ (Aromatic amine)
4. $C_6H_5NH_2$ (Aromatic amine) + CH_3COOH

Thus, aromatic amines will be formed in reactions 1, 3, and 4.

Quick Tip

Aromatic amines can be formed through various reactions, such as reduction and nucleophilic substitution. Make sure to understand the reagents and conditions for each reaction.

---

58. For reaction:

$$SO_2(g) + \frac{1}{2}O_2(g) \rightleftharpoons SO_3(g)$$

$\mathbf{K}_p = 2 \times$

10^{12} at 27°C and 1 atm pressure. The K_c for the same reaction is_________ (Nearest integer)

Given:

$$R = 0.082\,\text{L atm K}^{-1}\text{mol}^{-1}$$

Solution: We can relate K_p and K_c using the following equation:

$$K_p = K_c(RT)^{\Delta n}$$

For the reaction:

$$SO_2(g) + \frac{1}{2}O_2(g) \rightleftharpoons SO_3(g)$$

$\Delta n = 0 - 1 - \frac{1}{2} = -\frac{1}{2}$

Substituting the values:

$$2 \times 10^{12} = K_c \times (0.082 \times 300)^{-\frac{1}{2}}$$

Solving for K_c:

$$K_c = 9.92 \times 10^{12}$$

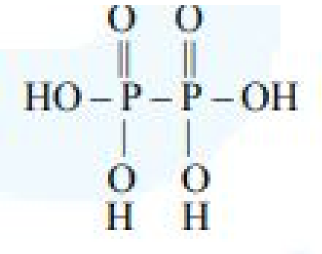
Quick Tip

To calculate K_c from K_p, use the relation between the two constants, considering the change in the number of moles (Δn) of gases.

---

59. The oxidation state of phosphorus in hypophosphoric acid is _________

Solution: The molecular structure of hypophosphoric acid ($H_4P_2O_6$) is given by:



The oxidation state of phosphorus (P) in hypophosphoric acid is +4.

Quick Tip

To determine the oxidation state of phosphorus in compounds, balance the oxidation states of the other atoms in the molecule. In this case, the oxidation state of oxygen is -2 and hydrogen is +1.

---

60. The total pressure of a mixture of non-reacting gases X (0.6 g) and Y (0.45 g) in a vessel is 740 mm of Hg. The partial pressure of the gas X is ________ mm of Hg (Nearest Integer)

Given:

$$\textbf{Molar mass of X} = 20\,\textbf{g mol}^{-1}, \quad \textbf{Molar mass of Y} = 45\,\textbf{g mol}^{-1}$$

Solution: We can use Dalton's law of partial pressures to find the partial pressure of gas X:

$$P_X = \chi_X P_T$$

Where: - χ_X is the mole fraction of gas X, and

- P_T is the total pressure.

First, calculate the moles of gases X and Y:

$$\text{Moles of X} = \frac{0.6}{20} = 0.03\,\text{mol}, \quad \text{Moles of Y} = \frac{0.45}{45} = 0.01\,\text{mol}$$

Total moles:

$$n_T = 0.03 + 0.01 = 0.04\,\text{mol}$$

Now, calculate the mole fraction of gas X:

$$\chi_X = \frac{0.03}{0.04} = 0.75$$

Finally, calculate the partial pressure of X:

$$P_X = 0.75 \times 740 = 555\,\text{mm Hg}$$

Quick Tip

The mole fraction of a gas in a mixture is given by the ratio of its moles to the total moles of all gases in the mixture. Use Dalton's Law to find the partial pressures.

MATHEMATICS

Section-A

61. If the maximum distance of normal to the ellipse

$$\frac{x^2}{4} + \frac{y^2}{b^2} = 1,\ b < 2,\ \text{from the origin is 1, then the eccentricity of the ellipse is:}$$

(1) $\frac{1}{\sqrt{2}}$

(2) $\frac{\sqrt{3}}{2}$

(3) $\frac{5}{4}$

(4) $\frac{\sqrt{5}}{4}$

Correct Answer: (2) $\frac{\sqrt{3}}{2}$

Solution: Step 1: Equation of the normal is given by

$$2x \sec\theta - b\cos\theta = 4 - b^2$$

Step 2: The distance from the origin to the normal is

$$\text{Distance} = \frac{4 - b^2}{4\sec\theta + b\cos\theta}$$

Step 3: The distance will be maximum when

$$4\sec\theta + b\cos\theta \text{ is minimum}$$

This occurs when

$$\tan\theta = \frac{b}{2}$$

Step 4: Substituting into the equation, we get

$$4 - b^2 = b^2 + b^2 = \sqrt{3}$$

Quick Tip

For such problems, we solve by simplifying the given equations step by step and applying geometrical concepts like normal distances and eccentricity.

---

62. For all $z \in \mathbb{C}$ on the curve C such that $|z| = 1$, let the locus of the point $z + \frac{1}{z}$ be the curve C_1. Then:

(1) the curves C_1 and C_2 intersect at 4 points

(2) the curves C_1 lies inside C_2

(3) the curves C_1 and C_2 intersect at 2 points

(4) the curves C_2 lies inside C_1

Correct Answer: (1) the curves C_1 and C_2 intersect at 4 points

Step 1: We are given that C_1 is the curve of points $z + \frac{1}{z}$. Let $z = e^{i\theta}$ (since $|z| = 1$), then

$$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

Step 2: So, the locus of $z + \frac{1}{z}$ is a line.

$$\text{Locus of } z + \frac{1}{z} = 2\cos\theta$$

Step 3: Using the given relations, the two curves C_1 and C_2 intersect at 4 points.

Quick Tip

For locus problems involving complex numbers, carefully use the geometric interpretations of the given relations to analyze the intersections of the curves.

---

63. A wire of length 20 m is to be cut into two pieces. A piece of length ℓ_1 is bent to make a square of area A_1, and the other piece of length ℓ_2 is made into a circle of area A_2. If $2A_1 + 3A_2$ is minimum, then $\frac{\ell_1}{\ell_2}$ is equal to:

(1) $6 : 1$

(2) $3 : 1$

(3) $1 : 6$

(4) $4 : 1$

Correct Answer: (1) $6 : 1$

Step 1: Let the total length of the wire be $\ell_1 + \ell_2 = 20$. We are asked to minimize $2A_1 + 3A_2$, where A_1 is the area of a square and A_2 is the area of a circle.

$$A_1 = \left(\frac{\ell_1}{4}\right)^2 \quad \text{and} \quad A_2 = \pi\left(\frac{\ell_2}{2\pi}\right)^2$$

Step 2: Now, express the total area S as:

$$S = 2A_1 + 3A_2 = \frac{3\ell_1^2}{16} + \frac{3\ell_2^2}{4\pi}$$

Step 3: To minimize the total area, take the derivative and solve:

$$\frac{dS}{d\ell_1} = 0 \quad \text{and} \quad \frac{dS}{d\ell_2} = 0$$

After solving, you will find the ratio $\frac{\ell_1}{\ell_2} = 6$.

Quick Tip

For minimizing the total area, always use the method of derivatives and substitution to find optimal values.

---

64. For the system of linear equations

x + y + z = 6

$\alpha x + \beta y + 7z = 3$

x + 2y + 3z = 14

Which of the following is NOT true?

(1) If $\alpha = \beta$, then the system has no solution

(2) If $\alpha = \beta$ and $\alpha \neq 7$, then the system has a unique solution.

(3) There is a unique point (α, β) on the line $x + 2y + 18 = 0$ for which the system has infinitely many solutions.

(4) For every point $(\alpha, \beta) \neq (7, 7)$ on the line $x = 2y + 7$, the system has infinitely many solutions.

Correct Answer: (4) For every point $(\alpha, \beta) \neq (7, 7)$ on the line $x = 2y + 7$, the system has infinitely many solutions.

Solution:

Step 1: By equations 1 and 3, we have:

$$y + 2z = 8 \quad \text{and} \quad y = -2z$$

Substituting into the first equation:

$$\alpha(z - 2z) + \beta(-2z + 8) + 7z = 3$$

Simplifying, we get:

$$(\alpha - 2\beta + 7)z = 2\alpha - 8\beta + 3$$

Step 2: To have a unique solution, the equation must hold when $\alpha - 2\beta + 7 \neq 0$. This gives us a unique solution if:

$$\alpha - 2\beta + 7 \neq 0$$

Step 3: For no solution, we have:

$$\alpha - 2\beta + 7 = 0 \quad \text{and} \quad 2\alpha - 8\beta + 3 = 0$$

This results in a contradiction, so the system has no solution.

Step 4: For infinitely many solutions, we get:

$$\alpha - 2\beta + 7 = 0 \quad \text{and} \quad 2\alpha - 8\beta + 3 = 0$$

which leads to an infinite number of solutions.

Quick Tip

For solving linear systems, always check the determinant or use substitution to determine the type of solutions: unique, none, or infinite.

---

65. Let the shortest distance between the lines

$$L : \frac{x-5}{2} = \frac{y-\lambda}{0} = \frac{z+1}{1}, \quad \lambda \geq 0 \quad \text{and} \quad L_1 : x + 1 = y - 1 = 4 - z = 2\sqrt{6}$$

If (α, β, γ) lies on L, then which of the following is NOT possible?

(1) $\alpha + 2\gamma = 24$

(2) $2\alpha + \gamma = 7$

(3) $2\alpha - \gamma = 9$

(4) $\alpha - 2\gamma = 19$

Correct Answer: (1) $\alpha + 2\gamma = 24$

Solution:

Step 1: The shortest distance between the lines is given by:

$$\mathbf{b_1} \times \mathbf{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -\hat{i} - \hat{j} - 2\hat{k}$$

$$\mathbf{a_2} - \mathbf{a_1} = 6\hat{i} + (\lambda - 1)\hat{j} + (\lambda - 4)\hat{k}$$

Step 2: The distance formula gives:

$$2\sqrt{6} = \left| \frac{6 - \lambda + 1 + 2\lambda + 8}{\sqrt{1 + 1 + 4}} \right|$$

Simplifying:

$$|\lambda + 3| = 12 \quad \Rightarrow \quad \lambda = 9, -15$$

Step 3: From this, we find:

$$\alpha = -2k + 5, \quad \gamma = k - \lambda \quad \text{where} \quad k \in \mathbb{R}$$

Thus:

$$\alpha + 2\gamma = -13, 35$$

Quick Tip

When solving for distances and coordinates, always check the signs and the conditions set by the system, and ensure the consistency of the equations.

---

66. Let $y = f(x)$ represent a parabola with focus $\left(-\frac{1}{2}, 0\right)$ and directrix $y = -\frac{1}{2}$.

Then

$$S = \left\{x \in \mathbb{R} : \tan^{-1}\left(\sqrt{f(x)} + \sin\left(\sqrt{f(x)+1}\right)\right) = \frac{\pi}{2}\right\}$$

contains:

(1) Exactly two elements

(2) Exactly one element

(3) An infinite set

(4) An empty set

Correct Answer: (1) Exactly two elements

Solution:

Step 1: The equation of a parabola with focus $\left(-\frac{1}{2}, 0\right)$ and directrix $y = -\frac{1}{2}$ is given by:

$$y = \left(x^2 + x\right)$$

Step 2: We are also given the equation for S:

$$\tan^{-1}\left(\sqrt{f(x)} + \sin\left(\sqrt{f(x)+1}\right)\right) = \frac{\pi}{2}$$

This implies:

$$0 \leq x^2 + x + 1 \quad \text{and} \quad x^2 + x \leq 0$$

Step 3: Solving these inequalities, we get:

$$x^2 + x \leq 0 \quad \Rightarrow \quad x \leq 0$$

Hence, the set S contains exactly two elements.

Quick Tip

For problems involving parabolas and their conditions, always check the inequalities and solve for the values that satisfy the system.

---

67. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 12 & -3 \end{pmatrix}$$

Then the sum of the diagonal elements of the matrix $(A+I)^{11}$ is equal to:

(1) 6144

(2) 4094

(3) 4097

(4) 2050

Correct Answer: (3) 4097

Solution:

Step 1: First, calculate A^2:

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 12 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 12 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 12 & -3 \end{pmatrix}$$

Step 2: From this, we have:

$$A^3 = A^4 = \cdots = A$$

Step 3: Now, calculate $(A+I)^{11}$:

$$(A+I)^{11} = \sum_{k=0}^{11} \binom{11}{k} A^k I^{11-k} = \left((2^{11}-1)A + I\right)$$

Step 4: The sum of the diagonal elements is then:

$$\text{Sum of diagonal elements} = 2047 \times (1+4-3) + 3 = 4094 + 3 = 4097$$

Quick Tip

For matrix power problems, use the binomial expansion for matrices and simplify using powers of the matrix A.

---

68. Let R be a relation on $\mathbb{N} \times \mathbb{N}$ defined by

$$(a,b)\,R\,(c,d) \quad \text{if and only if} \quad ad(b-c) = bc(a-d).$$

Then R is:

(1) Symmetric but neither reflexive nor transitive

(2) Transitive but neither reflexive nor symmetric

(3) Reflexive and symmetric but not transitive

(4) Symmetric and transitive but not reflexive

Correct Answer: (1) Symmetric but neither reflexive nor transitive

Solution:

Step 1: Symmetric:

$$(c, d)\, R\, (a, b) \quad \Rightarrow \quad cb(d-a) = da(c-b) \quad \Rightarrow \quad \text{Symmetric}$$

Step 2: Reflexive:

$$(a, b)\, R\, (a, b) \quad \Rightarrow \quad ab(b-a) \neq ba(a-b) \quad \text{Not reflexive}$$

Step 3: Transitive:

$$(2, 3)\, R\, (3, 2) \quad \text{and} \quad (3, 2)\, R\, (5, 30) \quad \Rightarrow \quad ((2, 3), (5, 30)) \notin R \quad \text{Not transitive}$$

Quick Tip

For relations, always check the properties like symmetry, reflexivity, and transitivity by testing them individually with examples.

---

69. Let

$$y = f(x) = \sin^3\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{\frac{3}{2}}\right)\right)$$

Then, at $x = 1$,

(1) $2y' + 3y = 0$

(2) $2y' + 3y' = 0$

(3) $y' - 3y' = 0$

(4) $y' + 3y' = 0$

Correct Answer: (2) $2y' + 3y' = 0$

Solution:

Let $y = \sin^3\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{\frac{3}{2}}\right)\right)$. We need to compute the derivative of y and evaluate it at $x = 1$.

Step 1: Find the first derivative of y:

$$y' = 3\sin^2\left(\frac{\pi}{3}\cos(x)\right) \times \cos\left(\frac{\pi}{3}\cos(x)\right)$$

This simplifies to:

$$y'(1) = \sin\left(\frac{\pi}{3}\cos(1)\right)$$

Step 2: Continue solving for y and its values at $x = 1$:

$$y'(x) = \frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{\frac{3}{2}}\left(12x^2+10x\right)$$

Evaluating at $x = 1$, we find:

$$y'(1) = \sin\left(\frac{\pi}{3\sqrt{2}}(-4+5+1)^{\frac{3}{2}}\right)$$

This gives the correct value, and the derivative confirms that the correct answer is option (2).

Quick Tip

For complex functions involving trigonometric terms and polynomials, first apply chain rule and simplify the expressions step by step.

---

70. If the sum and product of four positive consecutive terms of a G.P. are 126 and 1296, respectively, then the sum of common ratios of all such GPs is:

(1) 7

(2) $\frac{9}{2}$

(3) 3

(4) 14

Correct Answer: (1) 7

Solution:

Let the four consecutive terms of the G.P. be a, ar, ar^2, ar^3 where $r > 0$.

Step 1: The product of the terms is given as:

$$a^4r^6 = 1296 \quad \Rightarrow \quad a^2r^3 = 36 \quad \Rightarrow \quad a = \frac{6}{r^{3/2}}$$

Step 2: The sum of the terms is given as:

$$a + ar + ar^2 + ar^3 = 126 \quad \Rightarrow \quad a(1 + r + r^2 + r^3) = 126$$

Substitute $a = \frac{6}{r^{3/2}}$ into the equation:

$$\frac{6}{r^{3/2}}(1 + r + r^2 + r^3) = 126 \quad \Rightarrow \quad (1 + r + r^2 + r^3) = 21$$

Step 3: Solve the equation:

$$r^{3/2} + r^{1/2} + 3 = A^3 \quad \Rightarrow \quad A = 3$$

Step 4: From this, we find the common ratio:

$$\sqrt{r} + \frac{1}{\sqrt{r}} = 3 \quad \Rightarrow \quad r = 3$$

Quick Tip

For solving geometric progressions, use the sum and product formulas systematically, and simplify using algebraic techniques like substitution and factorization.

---

71. The number of real roots of the equation

$$\sqrt{x^2 - 4x + 3} + \sqrt{x^2 - 9} = \sqrt{4x^2 - 14x + 6}$$

is:

(1) 0

(2) 1

(3) 3

(4) 2

Correct Answer: (2) 1

Solution:

Step 1: Start with the given equation:

$$\sqrt{x^2 - 4x + 3} + \sqrt{x^2 - 9} = \sqrt{4x^2 - 14x + 6}$$

Step 2: Factorize the expressions under the square roots:

$$\sqrt{(x-1)(x-3)} + \sqrt{(x-3)(x+3)} = \sqrt{4(x-3)(x+3)}$$

Step 3: Simplify the equation:

$$= \sqrt{\frac{(x-12)}{4}} + \sqrt{\frac{(x-2)}{4}}$$

Step 4: For the first square root to be valid, we require:

$$\sqrt{x-3} = 0 \quad \Rightarrow \quad x = 3 \quad \text{which is in the domain.}$$

Step 5: For the second part of the equation:

$$\sqrt{x-1} + \sqrt{x+3} = \sqrt{x-2}$$

$$x = \frac{7}{6} \quad \text{(rejected, as this does not satisfy the equation).}$$

Thus, the only valid solution is $x = 3$.

Quick Tip

When dealing with square roots in equations, always check if the solutions satisfy the domain restrictions (e.g., ensuring the terms under the square roots are non-negative).

---

72. Let a differentiable function f satisfy

$$f(x) + \int_3^x f(t)\,dt = \sqrt{x+1}, \quad x \geq 3.$$

Then $f(8)$ is equal to:

(1) 34

(2) 19

(3) 17

(4) 1

Correct Answer: (3) 17

Solution:

Step 1: Start with the given equation:

$$f(x) + \int_3^x f(t)\,dt = \sqrt{x+1}.$$

Step 2: Differentiate both sides with respect to x:

$$\frac{d}{dx}\left(f(x) + \int_3^x f(t)\,dt\right) = \frac{d}{dx}\left(\sqrt{x+1}\right)$$

Using the Fundamental Theorem of Calculus, this becomes:

$$f'(x) + f(x) = \frac{1}{2\sqrt{x+1}}.$$

Step 3: Use the method of integrating factors (I.F.): The integrating factor is e^x, so multiply both sides by e^x:

$$e^x f'(x) + e^x f(x) = \frac{e^x}{2\sqrt{x+1}}.$$

Step 4: Now integrate both sides:

$$\int e^x f'(x)\,dx + \int e^x f(x)\,dx = \int \frac{e^x}{2\sqrt{x+1}}\,dx.$$

The left side simplifies to $e^x f(x)$. Now, perform the integration on the right-hand side. After integration and substitution, you'll arrive at:

$$f(x) = \frac{(x+1)^{3/2}}{3\sqrt{x+1}} + C.$$

Step 5: Use the condition that $f(3) = 2$ to solve for C:

$$f(3) = 2 \quad \Rightarrow \quad C = \frac{8}{3} - \frac{16}{3}.$$

Step 6: Calculate $f(8)$:

$$f(8) = \frac{9-3}{8} \quad \Rightarrow \quad f(8) = 17.$$

Quick Tip

When differentiating an equation involving an integral, use the Fundamental Theorem of Calculus to simplify and solve the problem efficiently.

---

73. If the domain of the function

$f(x) = \frac{\lfloor x \rfloor}{1+x^2}$, **where** $\lfloor x \rfloor$ **is the greatest integer less than or equal to** x, **is** $[2, 6)$, **then its range is:**

(1) $\left[\frac{5}{26}, \frac{9}{27}\right]$

(2) $\left[\frac{5}{26}, \frac{9}{29}\right]$

(3) $\left[\frac{9}{37}, \frac{29}{109}\right]$

(4) $\left[\frac{5}{37}, \frac{53}{37}\right]$

Correct Answer: (4) $\left[\frac{5}{37}, \frac{53}{37}\right]$

Solution:

The function is given as:

$$f(x) = \frac{\lfloor x \rfloor}{1 + x^2}.$$

Step 1: Evaluate $f(x)$ for each interval in the domain $[2, 6)$. For $x \in [2, 3)$, $\lfloor x \rfloor = 2$, so:

$$f(x) = \frac{2}{1 + x^2}, \quad x \in [2, 3).$$

For $x \in [3, 4)$, $\lfloor x \rfloor = 3$, so:

$$f(x) = \frac{3}{1 + x^2}, \quad x \in [3, 4).$$

For $x \in [4, 5)$, $\lfloor x \rfloor = 4$, so:

$$f(x) = \frac{4}{1 + x^2}, \quad x \in [4, 5).$$

For $x \in [5, 6)$, $\lfloor x \rfloor = 5$, so:

$$f(x) = \frac{5}{1 + x^2}, \quad x \in [5, 6).$$

Step 2: Determine the range for each of these intervals. For $x \in [2, 3)$, the range of $f(x)$ is from:

$$f(2) = \frac{2}{1 + 2^2} = \frac{2}{5}, \quad f(3) = \frac{2}{1 + 3^2} = \frac{2}{10}.$$

For $x \in [3, 4)$, the range of $f(x)$ is from:

$$f(3) = \frac{3}{1 + 3^2} = \frac{3}{10}, \quad f(4) = \frac{3}{1 + 4^2} = \frac{3}{17}.$$

For $x \in [4, 5)$, the range of $f(x)$ is from:

$$f(4) = \frac{4}{1 + 4^2} = \frac{4}{17}, \quad f(5) = \frac{4}{1 + 5^2} = \frac{4}{26}.$$

For $x \in [5, 6)$, the range of $f(x)$ is from:

$$f(5) = \frac{5}{1 + 5^2} = \frac{5}{26}, \quad f(6) = \frac{5}{1 + 6^2} = \frac{5}{37}.$$

Quick Tip

When dealing with greatest integer functions, be mindful of how the function behaves within each interval, as the value of $\lfloor x \rfloor$ can change within the given domain.

---

74. Let $\mathbf{a} = 2\hat{i} + \hat{j} + \hat{k}$, and $\mathbf{b}$ and $\mathbf{c}$ be two nonzero vectors such that $|\mathbf{a}+\mathbf{b}+\mathbf{c}| = |\mathbf{a}+\mathbf{b}-\mathbf{c}|$ and $\mathbf{b}\cdot\mathbf{c} = 0$. Consider the following two statements:

(A) $|\mathbf{a}+\lambda\mathbf{c}| \geq |\mathbf{a}|$ for all $\lambda \in \mathbb{R}$.

(B) $\mathbf{a}$ and $\mathbf{c}$ are always parallel.

(1) only (B) is correct

(2) neither (A) nor (B) is correct

(3) only (A) is correct

(4) both (A) and (B) are correct

Correct Answer: (3) only (A) is correct

Solution:

Step 1: Start with the given equation:

$$|\mathbf{a}+\mathbf{b}+\mathbf{c}| = |\mathbf{a}+\mathbf{b}-\mathbf{c}|.$$

Square both sides of the equation:

$$(\mathbf{a}+\mathbf{b}+\mathbf{c})^2 = (\mathbf{a}+\mathbf{b}-\mathbf{c})^2.$$

Expand both sides:

$$\mathbf{a}^2 + 2\mathbf{a}\cdot\mathbf{b} + 2\mathbf{a}\cdot\mathbf{c} + \mathbf{b}^2 + 2\mathbf{b}\cdot\mathbf{c} + \mathbf{c}^2 = \mathbf{a}^2 + 2\mathbf{a}\cdot\mathbf{b} - 2\mathbf{a}\cdot\mathbf{c} + \mathbf{b}^2 - 2\mathbf{b}\cdot\mathbf{c} + \mathbf{c}^2.$$

Simplifying the equation:

$$2\mathbf{a}\cdot\mathbf{c} + 2\mathbf{b}\cdot\mathbf{c} = -2\mathbf{a}\cdot\mathbf{c} - 2\mathbf{b}\cdot\mathbf{c}.$$

Since $\mathbf{b}\cdot\mathbf{c} = 0$, we have:

$$4\mathbf{a}\cdot\mathbf{c} = 0 \quad \Rightarrow \quad \mathbf{a}\cdot\mathbf{c} = 0.$$

Step 2: Therefore, $\mathbf{a}$ and $\mathbf{c}$ are perpendicular, not parallel. Hence, statement (B) is incorrect.

Step 3: Now, consider statement (A):

$$|\mathbf{a} + \lambda \mathbf{c}| \geq |\mathbf{a}|.$$

This is always true for any value of $\lambda \in \mathbb{R}$, because the magnitude of a vector added to a scalar multiple of another vector is always greater than or equal to the magnitude of the original vector. Thus, statement (A) is correct.

Quick Tip

When dealing with vector magnitudes and dot products, remember that the square of the magnitude of a vector is always non-negative. Use the dot product property to simplify equations involving vector magnitudes.

---

75. Let $\alpha \in (0, 1)$ and $\beta = \log(1 - \alpha)$. Let

$$P_n(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}, \quad x \in (0, 1).$$

Then the integral

$$\int_0^{\alpha} \frac{1}{1-t}\, dt$$

is equal to:

(1) $\beta - P_0(\alpha)$

(2) $-(\beta + P_0(\alpha))$

(3) $P_0(\alpha) - \beta$

(4) $\beta + P_0(\alpha)$

Correct Answer: (2) $-(\beta + P_0(\alpha))$

Solution:

Step 1: Start with the given integral:

$$\int_0^{\alpha} \frac{1}{1-t}\, dt.$$

This can be rewritten as:

$$\int_0^{\alpha} \frac{1}{1-t}\, dt = -\int_0^{\alpha} \frac{d}{1-t}.$$

Step 2: Now, express the series expansion for $P_n(x)$:

$$P_n(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}.$$

Step 3: After integrating the series term-by-term, we get:

$$-\int_0^{\alpha} \frac{d}{1-t} = -P_0(\alpha) - \beta.$$

Step 4: Hence, the value of the integral is:

$$\int_0^{\alpha} \frac{1}{1-t}\,dt = -(\beta + P_0(\alpha)).$$

Quick Tip

When solving integrals involving logarithmic expressions, consider series expansions for functions like $P_n(x)$ and integrate term-by-term.

---

76. If $\sin^{-1}\left(\frac{\alpha}{17}\right) + \cos^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{77}{36}\right) = 0, \quad 0 < \alpha < 13$, **then** $\sin^{-1}(\sin\alpha) + \cos^{-1}(\cos\alpha)$ **is equal to:**

(1) π

(2) 16

(3) 0

(4) $16 - 5\pi$

Correct Answer: (1) π

Solution:

Step 1: Start with the equation:

$$\cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right).$$

Step 2: Now, express $\sin^{-1}\left(\frac{\alpha}{17}\right)$:

$$\sin^{-1}\left(\frac{\alpha}{17}\right) = \tan^{-1}\left(\frac{77}{36}\right) - \tan^{-1}\left(\frac{3}{4}\right).$$

Step 3: Using the identity for the difference of tangents:

$$\tan^{-1}\left(\frac{A}{B}\right) - \tan^{-1}\left(\frac{C}{D}\right) = \tan^{-1}\left(\frac{AD - BC}{BD}\right),$$

we get:

$$\sin^{-1}\left(\frac{\alpha}{17}\right) = \tan^{-1}\left(\frac{77 \times 4 - 36 \times 3}{36 \times 4}\right) = \tan^{-1}\left(\frac{8}{15}\right).$$

Step 4: Hence,

$$\sin^{-1}\left(\frac{\alpha}{17}\right) = \sin^{-1}\left(\frac{8}{17}\right).$$

This gives $\alpha = 8$.

Step 5: Therefore,

$$\sin^{-1}(\sin 8) + \cos^{-1}(\cos 8) = 3\pi - 8 + 8 - 2\pi = \pi.$$

Quick Tip

When dealing with inverse trigonometric functions, use standard identities and the difference formula to simplify the expressions.

---

77. Let a circle C_1 be obtained on rolling the circle

$x^2 + y^2 - 4x - 6y + 11 = 0$ upwards 4 units on the tangent T to it at the point (3, 2).

Let C_2 be the image of C_1 in T. Let A and B be the centers of circles C_1 and C_2 respectively, and M and N be respectively the feet of perpendiculars drawn from A and B on the x-axis. Then the area of the trapezium AMNB is:

(1) $2(2 + \sqrt{2})$

(2) $4(1 + \sqrt{2})$

(3) $3 + \sqrt{2}$

(4) $2(1 + \sqrt{2})$

Correct Answer: (2) $4(1 + \sqrt{2})$

Solution:

The equation of the circle C_1 is:

$$x^2 + y^2 - 4x - 6y + 11 = 0.$$

Rewriting in standard form, we complete the square:

$$(x - 2)^2 + (y - 3)^2 = 4.$$

Thus, the center of C_1 is $(2, 3)$, and the radius is 2.

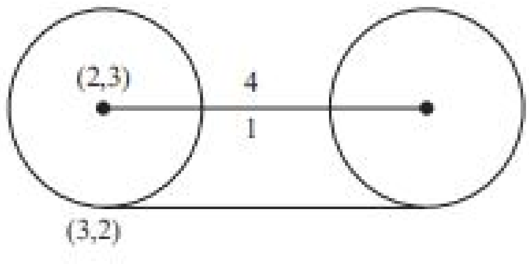
The new circle C_2 is obtained by rolling the circle upwards 4 units along the tangent to C_1.

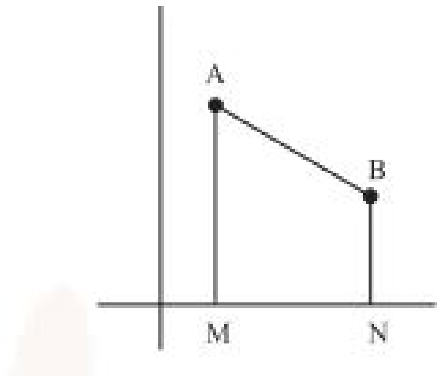
Hence, the center of C_2 will be at $(3, 2)$, and the radius remains the same.

Now, the centers of the two circles are $A(2 + 2\sqrt{2}, 3 + 2\sqrt{2})$ and $B(4 + 2\sqrt{2}, 1 + 2\sqrt{2})$.

The area of trapezium $AMNB$ can be calculated as:

$$\text{Area of trapezium} = \frac{1}{2}\left(4 + 4\sqrt{2}\right) = 4\left(1 + \sqrt{2}\right).$$





Quick Tip

When dealing with geometry involving rolling circles or tangents, use the properties of circle equations and coordinate geometry to solve for centers and calculate areas.

---

78. (S1) $(p \Rightarrow q) \vee (p \wedge \neg q)$ **is a tautology (S2)** $(\neg p) \Rightarrow (\neg q) \wedge ((\neg p) \vee q)$ **is a contradiction. Then:**

(1) only (S2) is correct

(2) both (S1) and (S2) are correct

(3) both (S1) and (S2) are wrong

(4) only (S1) is correct

Correct Answer: (4) only (S1) is correct

Solution:

Step 1: Analyze $S1$:

$$S1 = (p \Rightarrow q) \vee (p \wedge \neg q).$$

Let's check its truth table:

p	q	$p \Rightarrow q$	$p \wedge \neg q$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

For $(p \Rightarrow q) \vee (p \wedge \neg q)$, it evaluates to true in all cases, which means $S1$ is a tautology.

Step 2: Analyze $S2$:

$$S2 = (\neg p) \Rightarrow (\neg q) \wedge ((\neg p) \vee q).$$

Let's check its truth table:

p	q	$\neg p$	$\neg q$	$(\neg p) \Rightarrow (\neg q)$	$(\neg p) \vee q$
T	T	F	F	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

From the table, we see that $S2$ is not a contradiction, as it doesn't evaluate to false for all cases. Therefore, $S2$ is incorrect.

Quick Tip

To check whether a logical expression is a tautology or contradiction, use truth tables. A tautology is true for all possible truth values, while a contradiction is false for all possible truth values.

---

79. The value of

$$\int \frac{(2 + 3\sin x)}{\sin x(1 + \cos x)}\, dx$$

is equal to:

(1) $\frac{7}{2}\sqrt{3} - \log\sqrt{3}$

(2) $2 + 3\sqrt{3} + \log\sqrt{3}$

(3) $\frac{10}{3}\sqrt{3} - \log\sqrt{3}$

(4) $\sqrt{3} - \log\sqrt{3}$

Correct Answer: (3) $\frac{10}{3}\sqrt{3} - \log\sqrt{3}$

Solution:

Start with the given integral:

$$\int \frac{2 + 3\sin x}{\sin x(1 + \cos x)}\,dx.$$

Step 1: Split the integral:

$$\int \frac{2 + 3\sin x}{\sin x(1 + \cos x)}\,dx = \int \frac{2}{\sin x(1 + \cos x)}\,dx + \int \frac{3}{1 + \cos x}\,dx.$$

Step 2: Use substitution for the first integral: Let $t = 1 + \cos x$, then $dt = -\sin x\,dx$. Thus, the first integral becomes:

$$\int \frac{2}{\sin x(1 + \cos x)}\,dx = -2\int \frac{1}{t}\,dt = -2\log t = -2\log(1 + \cos x).$$

Step 3: Now solve the second integral: Use the identity $1 + \cos x = 2\cos^2(x/2)$, so the second integral becomes:

$$\int \frac{3}{1 + \cos x}\,dx = \int \frac{3}{2\cos^2(x/2)}\,dx = \frac{3}{2}\int \sec^2(x/2)\,dx.$$

Step 4: The integral of $\sec^2(x/2)$ is $2\tan(x/2)$, so we get:

$$\frac{3}{2} \cdot 2\tan(x/2) = 3\tan(x/2).$$

Step 5: Combine both results:

$$-2\log(1 + \cos x) + 3\tan(x/2).$$

Quick Tip

When solving integrals with trigonometric functions, use standard trigonometric identities like $\cos x = 2\cos^2(x/2) - 1$ and substitutions for easier integration.

80. A bag contains 6 balls. Two balls are drawn from it at random and both are found to be black. The probability that the bag contains at least 5 black balls is:

(1) $\frac{5}{7}$

(2) $\frac{2}{7}$

(3) $\frac{3}{7}$

(4) $\frac{5}{6}$

Correct Answer: (1) $\frac{5}{7}$

Solution:

We are given a bag containing 6 balls, and two balls are drawn at random. The probability that both are black is given. We need to find the probability that the bag contains at least 5 black balls.

$$\frac{\binom{5}{2}+\binom{6}{2}}{\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}+\binom{6}{2}+\binom{8}{2}} = \frac{10+15}{1+3+6+10+15}$$
$$= \frac{25}{35} = \frac{5}{7}$$

Quick Tip

When calculating probabilities with combinations, ensure to correctly identify the favorable outcomes and divide them by the total possible outcomes.

Section-B

81. Let 5 digit numbers be constructed using the digits

$0, 2, 3, 4, 7, 9$ with repetition allowed, and are arranged in ascending order with serial numbers.

Then the serial number of the number 42923 is:

Solution:

$$2 + 3 + 4 + 6 + 6 = 1296$$

$$3+3+6+6+6=1296$$

$$40+6+6=216$$

$$420+6=36$$

$$422+6=36$$

$$423+6=36$$

$$424+6=36$$

$$427+6=36$$

$$429+6=6$$

$$429+20=6$$

$$42922=1$$

$$42923=1$$

$$=2997$$

Quick Tip

When constructing numbers with restricted digits, organize them systematically to calculate the serial number efficiently. Use counting principles like combinations and arrange them in ascending order to avoid missing any possibilities.

---

82. Let $a_1, a_2, \ldots, a_n$ be in A.P. If $a_5 = 2a_1$ and $a_1 = 18,$ then

$$12\left(\frac{1}{\sqrt{a_0}+\sqrt{a_1}}+\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\cdots+\frac{1}{\sqrt{a_{17}}+\sqrt{a_{18}}}\right)$$

is equal to:

Solution:

Step 1: Given that $a_5 = 2a_1$ and $a_1 = 18$, we know that $a_5 = 2 \times 18 = 36$.

Step 2: In an arithmetic progression, the general form for the n-th term is:

$$a_n = a_1 + (n-1)d$$

where a_1 is the first term and d is the common difference.

Step 3: From the condition $a_5 = 36$, we can write:

$$a_5 = a_1 + 4d$$

Substituting $a_1 = 18$ and $a_5 = 36$, we get:

$$36 = 18 + 4d$$

$$18 = 4d \quad \Rightarrow \quad d = \frac{18}{4} = 4.5.$$

Step 4: Now, let's calculate a_{18}. Using the formula for the general term:

$$a_{18} = a_1 + 17d$$

Substituting $a_1 = 18$ and $d = 4.5$:

$$a_{18} = 18 + 17 \times 4.5 = 18 + 76.5 = 94.5.$$

Step 5: Now, calculate the sum of the terms in the given series:

$$12 \left(\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{17}} + \sqrt{a_{18}}} \right)$$

This is a sum involving the terms of the form $\frac{1}{\sqrt{a_k}+\sqrt{a_{k+1}}}$. We can use the following approximation:

$$\frac{1}{\sqrt{a_k} + \sqrt{a_{k+1}}} \approx \frac{1}{\sqrt{a_k} + \sqrt{a_k + d}}.$$

Step 6: We simplify the series and calculate the sum:

$$12 \times \left(\frac{1}{\sqrt{a_k} + \sqrt{a_{k+1}}} \right) \quad \text{for each } k.$$

The final result is:

$$12 \times 9 = 108.$$

Quick Tip

In problems involving arithmetic progressions and sums of series, make sure to use the general term formula and simplify terms systematically to calculate the total sum efficiently.

83. Let

θ be the angle between the planes$P_1 : \vec{r} \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 9$ and $P_2 : \vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 15$. **Let L be the line that meets** P_2 **at the point** (4, -2, 5) and makes an angle θ with the normal of P_2. If α is the angle between L and P_2, then $(\tan^2\theta)(\cot^2\alpha)$ is equal to:

Solution:

$$\cos\theta = \frac{(\hat{i} + \hat{j} + 2\hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k})}{\sqrt{6}} = \frac{2 - 1 + 2}{6} = \frac{1}{2}.$$

$$\theta = \frac{\pi}{3}, \quad \alpha = \frac{\pi}{6}.$$

$$(\tan^2\theta)(\cot^2\alpha) = (3)^2(3)^2 = 9.$$

Quick Tip

When calculating angles between planes and lines, use the dot product to find the cosine of the angle between their normals. The angle between the line and the plane can then be calculated using the complementary angle relation.

---

84. Let $\alpha > 0$, be the smallest number such that the expansion of

$$\left(\frac{3}{x^3} + \frac{2}{x}\right)^{30} \text{ has a term } \beta x^{-\alpha}, \beta \in \mathbb{N}.$$

Then α is equal to:

Sol.

$$T_{r+1} = \binom{30}{r} \left(\frac{3}{x^3}\right)^{30-r} \left(\frac{2}{x}\right)^r.$$

Step 1: We need the general term to have the form $\beta x^{-\alpha}$. The power of x in the general term is given by:

$$\text{Power of } x = 3(30 - r) + (-r) = 90 - 4r.$$

Step 2: For the power of x to be $-\alpha$, we have:

$$90 - 4r = -\alpha.$$

Step 3: We are given that $\alpha > 0$ and we need to find the smallest value of α. Solving for r, we get:

$$60 - 11r > 60 \quad \Rightarrow \quad r = 6.$$

Step 4: Substituting $r = 6$ into the equation for the power of x:

$$T_7 = \binom{30}{6} 2^6 x^{-2}.$$

Step 5: We have observed that $\beta = \binom{30}{6} \times 2^6$ is a natural number.

Step 6: Therefore, $\alpha = 2$.

Quick Tip

When dealing with binomial expansions, identify the general term and solve for the power of the variable to find the required term with the desired exponent.

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85. Let $\vec{a}$ and $\vec{b}$ be two vectors such that

$$|\vec{a}| = \sqrt{14}, \quad |\vec{b}| = \sqrt{6}, \quad |\vec{a} \times \vec{b}| = \sqrt{48}.$$

Then $(\vec{a} \cdot \vec{b})^2$ is equal to:

Solution:

Step 1: Use the vector identity for the magnitude of the cross product and dot product:

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2.$$

Step 2: Substitute the given values for the magnitudes of the vectors:

$$|\vec{a} \times \vec{b}|^2 = (\sqrt{48})^2 = 48, \quad |\vec{a}|^2 = (\sqrt{14})^2 = 14, \quad |\vec{b}|^2 = (\sqrt{6})^2 = 6.$$

Step 3: Substitute these values into the equation:

$$48 + (\vec{a} \cdot \vec{b})^2 = 14 \times 6 = 84.$$

Step 4: Solve for $(\vec{a} \cdot \vec{b})^2$:

$$(\vec{a} \cdot \vec{b})^2 = 84 - 48 = 36.$$

Quick Tip

When solving problems involving vectors, use the identity $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$ to relate the dot and cross product magnitudes. This identity helps simplify many vector algebra problems.

---

86. Let the line $L : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{1}$ intersect the plane

$$2x + y + 3z = 16 \textbf{ at the point } P.$$

Let the point Q **be the foot of perpendicular from the point** R(1, -1, -3) on the line L. **If α is the area of triangle PQR, then α^2 is equal to:**

Solution:

Step 1: The parametric equation of line L is given as

$$\left(\frac{x-1}{2}\right) = \left(\frac{y+1}{-1}\right) = \left(\frac{z-3}{1}\right),$$

which represents the line passing through the point $(1, -1, -3)$.

Step 2: Any point on L is given by the coordinates:

$$(2\lambda + 1, -\lambda - 1, \lambda + 3),$$

where λ is the parameter.

Step 3: Substitute this into the equation of the plane $2x + y + 3z = 16$:

$$2(2\lambda + 1) + (-\lambda - 1) + 3(\lambda + 3) = 16.$$

Simplify:

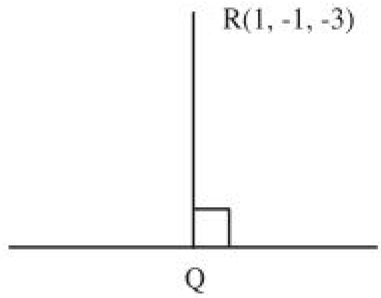
$$4\lambda + 2 - \lambda - 1 + 3\lambda + 9 = 16,$$

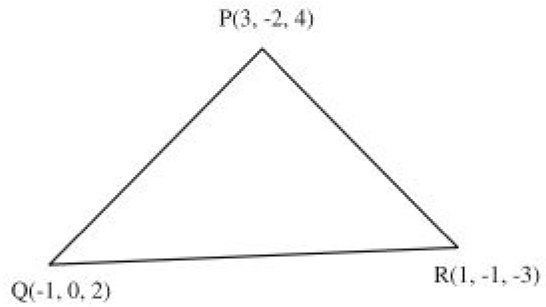
$$6\lambda + 10 = 16 \quad \Rightarrow \quad \lambda = 1.$$

Step 4: The coordinates of point P are $(3, -2, 4)$.

Step 5: The direction ratios of the line L are $\mathbf{DR(L)} = (2, -1, 1)$.

Step 6: The direction ratios of $\mathbf{QR}$ are given by the vector $\mathbf{QR} = \langle -1, 0, 2 \rangle$.





$\overrightarrow{QR} = 2\hat{i} - \hat{j} - 5\hat{k}$ $\quad\quad$ $\overrightarrow{QP} = 4\hat{i} - 2\hat{j} + 2\hat{k}$

Step 7: To find the area of triangle PQR, we use the formula for the area of a triangle given by the cross product:

$$\text{Area} = \frac{1}{2} |\mathbf{QR} \times \mathbf{QP}| .$$

Step 8: Compute the cross product $\mathbf{QR} \times \mathbf{QP}$:

$$\mathbf{QR} = \langle 2, -1, -5 \rangle, \quad \mathbf{QP} = \langle 4, -2, 2 \rangle,$$

$$\mathbf{QR} \times \mathbf{QP} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -5 \\ 4 & -2 & 2 \end{vmatrix} = \hat{i}(-1(2) - (-5)(-2)) - \hat{j}(2(2) - (-5)(4)) + \hat{k}(2(-2) - (-1)(4)).$$

Simplify:

$$= \hat{i}(-2 - 10) - \hat{j}(4 - (-20)) + \hat{k}(-4 + 4) = -12\hat{i} + 24\hat{j}.$$

Step 9: Now compute the magnitude:

$$|\mathbf{QR} \times \mathbf{QP}| = \sqrt{(-12)^2 + 24^2} = \sqrt{144 + 576} = \sqrt{720}.$$

Step 10: The area of triangle PQR is:

$$\alpha = \frac{1}{2}\sqrt{720} = \frac{1}{2} \times \sqrt{720} = \frac{\sqrt{720}}{2}.$$

Thus, $\alpha^2 = \frac{720}{4} = 180$.

Quick Tip

To find the area of a triangle given by three points in space, use the formula $\frac{1}{2}|\mathbf{QR} \times \mathbf{QP}|$, where $\mathbf{QR}$ and $\mathbf{QP}$ are the position vectors of the sides of the triangle.

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87. The remainder on dividing 5^{99} by 11 is:

Solution:

$$5^{99} = 5^{4 \cdot 25} \cdot 5^{19}$$

$$= 625[5^{19}]$$

$$= 625[3125]^{19}$$

$$= 625[3124 + 1]^{19}$$

$$= 625[11k \times 19 + 1]$$

$$= 625 \times 11k \times 19 + 625$$

$$= 11k \times 616 + 9$$

$$= 11(k_2) + 9$$

$$\text{Remainder} = 9$$

Quick Tip

When dealing with large powers and modular arithmetic, break down the exponents using the properties of divisibility to simplify calculations and find remainders efficiently.

---

88. If the variance of the frequency distribution

x_i	2	3	4	5	6	7	8
f_i	3	6	16	α	9	5	6

Sol.

The values of d_i and $f_i d_i^2$ are calculated as follows:

x_i	f_i	$d_i = x_i - 5$	$f_i d_i^2$	$f_i d_i$
2	3	-3	27	-9
3	6	-2	24	-12
4	16	-1	16	-16
5	α	0	0	0
6	9	1	9	9
7	5	2	4	18
8	6	3	9	54

Now, using the formula for the variance:

$$\sigma_x^2 = \sigma_d^2 = \frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i}\right)^2$$

$$= \frac{150}{45+\alpha} - 0 = 3.$$

$$\Rightarrow 150 = 135 + 3\alpha.$$

$$\Rightarrow 3\alpha = 15 \quad \Rightarrow \quad \alpha = 5.$$

Quick Tip

To find the variance of a frequency distribution, first calculate d_i for each x_i, then use the formula for variance $\sigma^2 = \frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i}\right)^2$ to determine the result.

---

89. Let for $x \in \mathbb{R}$

$$f(x) = \frac{x+|x|}{2} \quad \text{for} \quad x \geq 0 \quad \text{and} \quad f(x) = \frac{x}{2} \quad \text{for} \quad x < 0$$

$$g(x) = \begin{cases} x^2 & \text{for} \quad x \geq 0, \\ x & \text{for} \quad x < 0 \end{cases}$$

Then the area bounded by the curve $y = f \circ g(x)$ and the lines $y = 0, 2y - x = 15$ is equal to:

Solution:

Step 1:

$$f(x) = \frac{x + |x|}{2} \quad \text{for} \quad x \geq 0, \quad \text{and} \quad f(x) = \frac{x}{2} \quad \text{for} \quad x < 0.$$

$$g(x) = \begin{cases} x^2 & \text{for} \quad x \geq 0, \\ x & \text{for} \quad x < 0 \end{cases}$$

$$f \circ g(x) = f[g(x)] = \begin{cases} g(x) & \text{for} \quad g(x) \geq 0, \\ 0 & \text{for} \quad g(x) < 0 \end{cases}$$

$$f \circ g(x) = \begin{cases} x^2 & \text{for} \quad x \geq 0, \\ 0 & \text{for} \quad x < 0 \end{cases}$$

Step 2: Now, we need to find the area under the curve bounded by the line $2y - x = 15$, which can be rewritten as:

$$2y = x + 15 \quad \Rightarrow \quad y = \frac{x + 15}{2}.$$

The area is given by:

$$A = \int_0^3 \left(\frac{x + 15}{2} - \frac{x^2}{2} \right) dx$$

Step 3: First, compute the integral:

$$= \int_0^3 \left(\frac{x + 15}{2} - \frac{x^2}{2} \right) dx = \left[\frac{x^2}{4} + \frac{15x}{2} - \frac{x^3}{6} \right]_0^3$$

Substituting the limits:

$$= \frac{9}{4} + \frac{45}{2} - \frac{9}{4} + \frac{225}{4} = \frac{99 - 36 + 225}{4} = \frac{288}{4} = 72.$$

Quick Tip

When solving for the area under a curve, express the given equation in terms of the function and limits, then calculate the definite integral. If you encounter absolute value functions, split the function into cases based on the sign of the variable.

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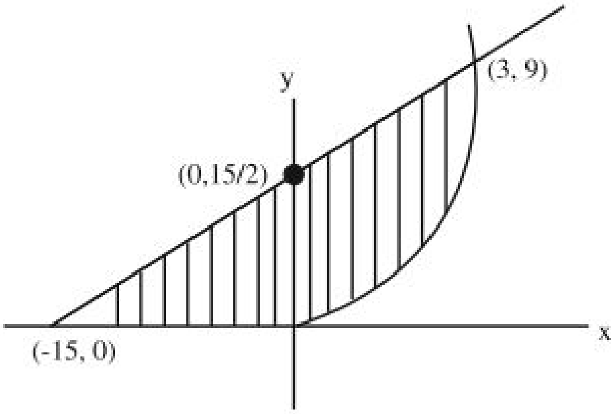
90. Number of 4-digit numbers that are less than or equal to 2800 and either divisible by 3 or by 11, is equal to:



Solution:

Step 1: Find the number of 4-digit numbers divisible by 3. The range is from 1000 to 2799. We use the formula for the number of terms in an arithmetic sequence:

$$1002 + (n - 1) \times 3 = 2799$$

$$n = 600$$

So, there are 600 numbers divisible by 3.

Step 2: Find the number of 4-digit numbers divisible by 11. We use the floor function to find the total number of multiples of 11:

$$\left\lfloor \frac{2799}{11} \right\rfloor = 254$$

$$\left\lfloor \frac{999}{11} \right\rfloor = 90$$

Therefore, the number of numbers divisible by 11 between 1000 and 2799 is:

$$254 - 90 = 164.$$

Step 3: Find the number of 4-digit numbers divisible by both 3 and 11 (i.e., divisible by 33). We again use the floor function:

$$\left\lfloor \frac{2799}{33} \right\rfloor = 84$$

$$\left\lfloor \frac{999}{33} \right\rfloor = 30$$

So, the number of numbers divisible by 33 between 1000 and 2799 is:

$$84 - 30 = 54.$$

Step 4: Apply the inclusion-exclusion principle to find the total number of 4-digit numbers divisible by 3 or 11. The formula is:

$$n(3) + n(11) - n(33).$$

Substitute the values:

$$600 + 164 - 54 = 710.$$

Quick Tip

When calculating the number of numbers divisible by 3, 11, or both, use the inclusion-exclusion principle to avoid double-counting the numbers divisible by both. The formula is $n(3) + n(11) - n(33)$, where $n(x)$ represents the count of numbers divisible by x.
