JEE Main 2023 Feb 1 Shift 1 Mathematics Question Paper with Solutions

Time Allowed :3 Hours	Maximum Marks :300	Total Questions :90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

MATHEMATICS

Section-A

1. If

$$\lim_{n\to\infty}\left(\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+\cdots+\frac{1}{2n}\right)$$

is equal to:

(1) 0

(2) $\log_e 2$

(3) $\log_e\left(\frac{3}{2}\right)$

(4) $\log_e\left(\frac{2}{3}\right)$

Correct Answer: (2) $\log_e 2$

Solution:

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r+r}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{1+r}$$

Using integration approximation:

$$\int_1^n \frac{dx}{1+x}=[\ln(1+x)]_1^n=\ln(1+n)-\ln(2)\approx\log 2$$

Quick Tip

For infinite series, integral approximation is a useful tool to estimate the limit.

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2. The negation of the expression

$$q\vee(\neg q\wedge p)$$

is equivalent to:

(1) $(\neg p)\wedge(\neg q)$

(2) $p\wedge(\neg q)$

(3) $(\neg p)\vee(\neg q)$

(4) $(\neg p)\vee q$

Correct Answer: (1) $(\neg p) \wedge (\neg q)$

Solution:

$$\neg(q \vee (\neg q \wedge p))$$

$$= \neg q \wedge \neg(\neg q \wedge p)$$

$$= \neg q \wedge (\neg\neg q \vee \neg p) = \neg q \wedge (q \vee \neg p)$$

$$= (\neg q \wedge q) \vee (\neg q \wedge \neg p) = 0 \vee (\neg q \wedge \neg p)$$

$$= (\neg p) \wedge (\neg q)$$

Quick Tip

For logical expressions, use De Morgan's laws and distributive properties to simplify.

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3. In a binomial distribution $B(n, p)$, the sum and product of the mean and variance are 5 and 6, respectively. Then $6(n + p - q)$ is equal to:

(1) 51

(2) 52

(3) 53

(4) 50

Correct Answer: (2) 52

Solution:

Step 1: Using the formulas for mean and variance in a binomial distribution:

$$\text{Mean} = np, \quad \text{Variance} = npq$$

Given:

$$np + npq = 5, \quad npq = 6$$

$$np(1 + q) = 5, \quad 6(np^2q) = 6$$

$$(1 + q)^2 = 25 \Rightarrow q^2 + 12q + 6 = 25q$$

Solving for q, we get:

$$q = \frac{2}{3}$$
$$p = \frac{1}{3}$$
$$n = 9$$

Now, computing $6(n + p - q)$:

$$6(9 + \frac{1}{3} - \frac{2}{3}) = 52$$

Quick Tip

For binomial distributions, use the given sum and product conditions to find individual values for n, p, q.

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4. The sum to 10 terms of the series

$$\frac{1}{1 + 1^2 + 1^4} + \frac{2}{1 + 2^2 + 2^4} + \frac{3}{1 + 3^2 + 3^4} + \dots$$

is:

(1) $\frac{59}{111}$

(2) $\frac{55}{111}$

(3) $\frac{56}{111}$

(4) $\frac{58}{111}$

Correct Answer: (2) $\frac{55}{111}$

Solution:

Using partial fractions:

$$T_r = \frac{x^2 + r + 1 - (x^2 - r + 1)}{2(r^2 + r + 1)}$$
$$T_1 = \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3}\right]$$
$$T_2 = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{7}\right]$$
$$T_{10} = \frac{1}{2}\left[\frac{1}{29} - \frac{1}{111}\right]$$

Summing over 10 terms:

$$\sum_{r=1}^{10} T_r = \frac{55}{111}$$

Quick Tip

For summation of series, express the terms in partial fractions to simplify calculations.

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5. The value of

$$\frac{1}{1!50!} + \frac{1}{3!48!} + \frac{1}{5!46!} + \cdots + \frac{1}{49!2!} + \frac{1}{51!1!}$$

is:

(1) $\frac{2^{50}}{50!}$

(2) $\frac{2^{50}}{51!}$

(3) $\frac{2^{51}}{51!}$

(4) $\frac{2^{51}}{50!}$

Correct Answer: (2) $\frac{2^{50}}{51!}$

Solution:

$$\sum_{r=1}^{26} \frac{1}{(2r-1)!(51-(2r-1))!} = \sum_{r=1}^{26} \binom{51}{2r-1} \frac{1}{51!}$$

$$= \frac{1}{51!} \sum_{r=1}^{26} \binom{51}{2r-1} = \frac{1}{51!}(2^{50})$$

Quick Tip

For binomial sums involving factorials, expressing terms using combinations simplifies evaluation.

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6. If the orthocentre of a triangle, whose vertices are $(1, 2)$, $(2, 3)$, and $(3, 1)$ **is** (α, β), **then the quadratic equation whose roots are** $\alpha + 4\beta$ **and** $4\alpha + \beta$ **is:**

(1) $x^2 - 19x + 90 = 0$

(2) $x^2 - 18x + 80 = 0$

(3) $x^2 - 22x + 120 = 0$

(4) $x^2 - 20x + 99 = 0$

Correct Answer: (4) $x^2 - 20x + 99 = 0$

Solution:

Step 1: Finding slopes of perpendicular altitudes:

$$m = \frac{-1}{2} = -\frac{1}{2}$$

Here $m_{BH} \times m_{AC} = -1$

$$\beta - 3 = 2\alpha - 4$$

$$\beta = 2\alpha - 1$$

Step 2: Using midpoint formula and relations, we find:

$$\alpha = \frac{5}{3}, \quad \beta = \frac{7}{3}$$

Step 3: Now, computing roots:

$$\alpha + 4\beta = 11, \quad 4\alpha + \beta = 9$$

$$x^2 - (11 + 9)x + 11 \times 9 = 0$$

$$x^2 - 20x + 99 = 0$$

Quick Tip

For orthocenter problems, use perpendicular slopes and midpoint relations effectively.

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1. For a triangle ABC, the value of

$$\cos 2A + \cos 2B + \cos 2C$$

is least. If its inradius is 3 and incentre is M, then which of the following is NOT correct?

(1) Perimeter of $\triangle ABC$ is $18\sqrt{3}$

(2) $\sin 2A + \sin 2B + \sin 2C = \sin A + \sin B + \sin C$

(3) $\overline{MA} \cdot \overline{MB} = -18$

(4) Area of $\triangle ABC$ is $\frac{27\sqrt{3}}{2}$

Correct Answer: (4) $\frac{27\sqrt{3}}{2}$

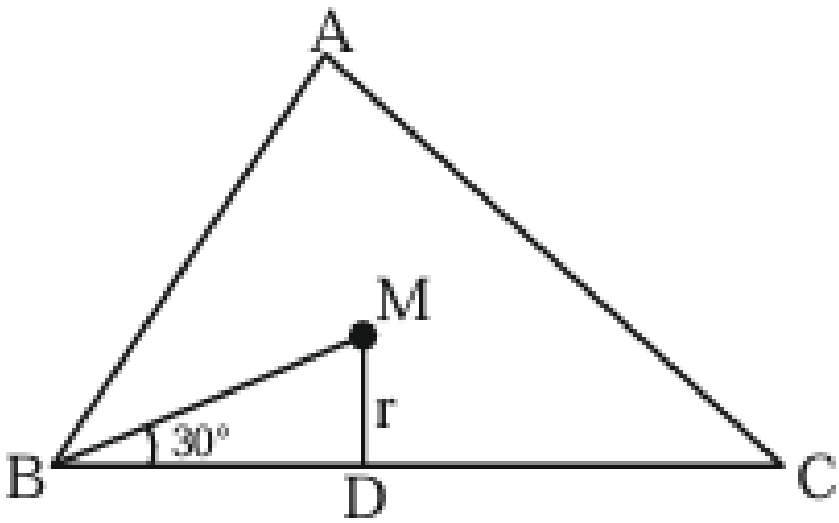
Solution:

If $\cos 2A + \cos 2B + \cos 2C$ is minimum, then

$$A = 60^\circ, \quad B = C = 60^\circ$$

$$\Rightarrow \triangle ABC \text{ is equilateral}$$

$$\text{In-radius} = r = 3$$



$$\text{Area} = \frac{\sqrt{3}}{4}a^2 = 27\sqrt{3}$$

$$\text{Thus, area is incorrectly given as } \frac{27\sqrt{3}}{2}$$

Quick Tip

For equilateral triangles, the area formula is $\frac{\sqrt{3}}{4}a^2$.

8. The combined equation of the two lines

$$ax + by + c = 0 \quad \text{and} \quad a'x + b'y + c' = 0$$

can be written as

$$(ax + by + c)(a'x + b'y + c') = 0$$

The equation of the angle bisectors of the lines represented by the equation

$$2x^2 + xy - 3y^2 = 0$$

is:

(1) $3x^2 + 5xy + 2y^2 = 0$

(2) $x^2 - y^2 + 10xy = 0$

(3) $3x^2 + xy - 2y^2 = 0$

(4) $x^2 - y^2 - 10xy = 0$

Correct Answer: (4) $x^2 - y^2 - 10xy = 0$

Solution:

Step 1: Equation of the pair of angle bisectors for the homogeneous equation

$$ax^2 + 2hxy + by^2 = 0$$

is given by:

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

Step 2: Here,

$$a = 2, \quad h = \frac{1}{2}, \quad b = -3$$

Thus, equation becomes:

$$\frac{x^2 - y^2}{2 - (-3)} = \frac{xy}{\frac{1}{2}}$$

$$x^2 - y^2 - 10xy = 0$$

Quick Tip

For angle bisectors of conic sections, use the standard formula and compare coefficients.

9. The shortest distance between the lines

$$\frac{x-5}{1}=\frac{y-2}{2}=\frac{z-4}{-3}, \quad \frac{x+3}{1}=\frac{y+5}{4}=\frac{z-1}{-5}$$

is:

(1) $7\sqrt{3}$

(2) $5\sqrt{3}$

(3) $6\sqrt{3}$

(4) $4\sqrt{3}$

Correct Answer: (3) $6\sqrt{3}$

Solution:

Step 1: Shortest distance formula between skew lines:

$$d=\frac{|(a_1b_2-a_2b_1)+(a_2b_3-a_3b_2)+(a_3b_1-a_1b_3)|}{\sqrt{(a_1-a_2)^2+(b_1-b_2)^2+(c_1-c_2)^2}}$$

Step 2: Substituting values,

$$d=\frac{8(-10+12)-7(-5+3)+3(4-2)}{\sqrt{4+4+4}}$$
$$=\frac{16+14+6}{\sqrt{12}}=\frac{36}{\sqrt{12}}=\frac{36}{2\sqrt{3}}$$
$$=\frac{18}{\sqrt{3}}=6\sqrt{3}$$

Quick Tip

For shortest distance problems, identify direction ratios and apply vector formulas systematically.

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10. Let S denote the set of all real values of λ such that the system of equations

$$\lambda x+y+z=1$$

$$x+\lambda y+z=1$$

$$x + y + \lambda z = 1$$

is inconsistent. Then

$$\sum_{\lambda \in S} (|\lambda|^2 + |\lambda|)$$

is equal to:

(1) 2

(2) 12

(3) 4

(4) 6

Correct Answer: (4) 6

Solution:

Step 1: Solving determinant:

$$\begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

Expanding:

$$(\lambda + 2)(\lambda - 1)^2 - 1(\lambda - 1) + (1 - \lambda) = 0$$

Step 2: Finding roots,

$$\lambda = -2, \quad \lambda = 1$$

$$\sum_{\lambda \in S} (|\lambda|^2 + |\lambda|) = 6$$

Quick Tip

For system consistency, solve determinant equations and analyze root conditions.

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11. Let

$$S = \left\{ x : x \in \mathbb{R} \text{ and } (\sqrt{3} + \sqrt{2})^{x-4} + (\sqrt{3} - \sqrt{2})^{x-4} = 10 \right\}$$

Then $|S|$ is equal to:

(1) 2

(2) 4

(3) 6

(4) 0

Correct Answer: (2) 4

Solution:

Let $(\sqrt{3}+\sqrt{2})^{x-4} = t$, then

$$t + \frac{1}{t} = 10$$

Solving for t, we get

$$t = 5 + 2\sqrt{6}, \quad t = 5 - 2\sqrt{6}$$

Thus,

$$(\sqrt{3}+\sqrt{2})^{x-4} = 5 + 2\sqrt{6}, \quad 5 - 2\sqrt{6}$$

Squaring both sides,

$$x^2 - 4 = 2, -2 \Rightarrow x^2 = 6, 2$$

$$x = \pm\sqrt{2}, \pm\sqrt{6}$$

Quick Tip

When solving exponential equations, try using substitutions to simplify the exponent expressions.

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12. Let S be the set of all solutions of the equation

$$\cos^{-1}(2x) - 2\cos^{-1}(\sqrt{1-x^2}) = \pi, \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Then

$$\sum_{x \in S} 2\sin^{-1}(x^2 - 1)$$

is equal to:

(1) 0

(2) $\frac{-2\pi}{3}$

(3) $\pi - \sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

(4) $\pi - 2\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

Correct Answer: (1) 0

Solution:

Rewriting the equation,

$$\cos^{-1}(2x) = \pi + 2\cos^{-1}(\sqrt{1-x^2})$$

Since the LHS belongs to the interval $[0, \pi]$, the equation must be meaningful. Solving,

$$\cos^{-1}(2x) = \pi, \quad \cos^{-1}(\sqrt{1-x^2}) = 0$$

which gives

$$x = -\frac{1}{2}, \quad x = 0$$

Now sum over the set:

$$\sum_{x \in S} 2\sin^{-1}(x^2 - 1) = 0$$

Quick Tip

For inverse trigonometric functions, always check the domain constraints before solving.

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13. If the center and radius of the circle

$$\left|\frac{z-2}{z-3}\right| = 2$$

are respectively (α, β) **and** γ**, then**

$$3(\alpha + \beta + \gamma)$$

is equal to:

(1) 11

(2) 9

(3) 10

(4) 12

Correct Answer: (4) 12

Solution:

Using transformation:

$$\sqrt{(x-2)^2+y^2} = 2\sqrt{(x-3)^2+y^2}$$

Squaring both sides and simplifying,

$$x^2+y^2-20x+32=0$$

Comparing with standard form,

$$\alpha = \frac{10}{3}, \quad \beta = 0$$

$$\gamma = \sqrt{\frac{100}{9}+\frac{32}{3}} = \frac{2}{3}$$

$$3(\alpha+\beta+\gamma) = 3\left(\frac{10}{3}+0+\frac{2}{3}\right) = 12$$

Quick Tip

For modulus-based circle problems, use transformation equations and algebraic manipulation.

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14. If $y = y(x)$ is the solution curve of the differential equation

$$\frac{dy}{dx} + y\tan x = x\sec x, \quad 0 \le x \le \frac{\pi}{3}$$

and $y(0) = 1$, then $y\left(\frac{\pi}{6}\right)$ is equal to:

(1) $\frac{\pi}{12} + \frac{\sqrt{3}}{2}\log_e\left(\frac{2}{e\sqrt{3}}\right)$

(2) $\frac{\pi}{12} + \frac{\sqrt{3}}{2}\log_e\left(\frac{2\sqrt{3}}{e}\right)$

(3) $\frac{\pi}{12} - \frac{\sqrt{3}}{2}\log_e\left(\frac{2\sqrt{3}}{e}\right)$

(4) $\frac{\pi}{12} + \frac{\sqrt{3}}{2}\log_e\left(\frac{2}{e\sqrt{3}}\right)$

Correct Answer: (1)

Solution:

The integrating factor (I.F.) for this differential equation is:

$$I.F. = \sec x$$

Multiplying throughout by $\sec x$, we get:

$$\frac{d}{dx}(y \sec x) = x$$

Integrating both sides:

$$y \sec x = x \tan x - \ln(\sec x) + C$$

Using the given initial condition $y(0) = 1$, we determine $C = 1$.

$$y(\sec x) = x \tan x - \ln(\sec x) + 1$$

Substituting $x = \frac{\pi}{6}$:

$$y = \frac{\pi}{12} + \frac{\sqrt{3}}{2} \log_e \left(\frac{2}{e\sqrt{3}} \right)$$

Quick Tip

For first-order linear differential equations, always compute the integrating factor and solve step-by-step.

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15. Let R be a relation on $\mathbb{R}$, given by

$$R = \{(a, b) : 3a - 3b + \sqrt{7} \text{ is an irrational number}\}$$

Then R is:

(1) Reflexive but neither symmetric nor transitive

(2) Reflexive and transitive but not symmetric

(3) Reflexive and symmetric but not transitive

(4) An equivalence relation

Correct Answer: (1) Reflexive but neither symmetric nor transitive

Solution:

Check for reflexivity:

Since

$$3(a-a)+\sqrt{7}=\sqrt{7}$$

which is irrational, the relation is reflexive.

Check for symmetry:

Let $a=\frac{\sqrt{7}}{3}, b=0$. Then $(a,b)\in R$ but $(b,a)\notin R$.

Since

$$3(b-a)+\sqrt{7}=0$$

which is rational, the relation is not symmetric.

Check for transitivity:

Let $(a,b)=\left(1,\frac{2\sqrt{7}}{3}\right)$ and $(b,c)=\left(1,\frac{2\sqrt{7}}{3}\right)$. Then, $(a,b)\in R$ and $(b,c)\in R$, but $(a,c)\notin R$, so the relation is not transitive.

Quick Tip

To check for equivalence relations, verify reflexivity, symmetry, and transitivity systematically.

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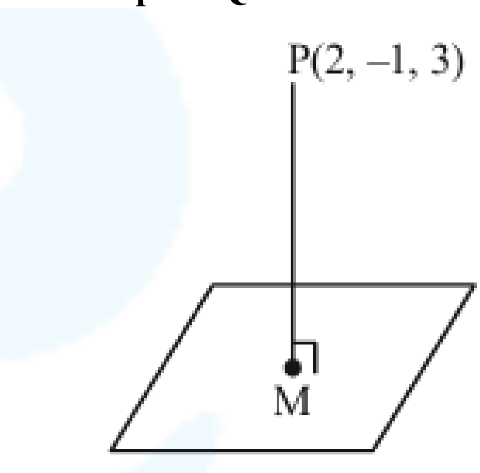
16. Let the image of the point $P(2,-1,3)$ in the plane

$$x+2y-z=0$$

be Q. Then the distance of the plane

$$3x+2y+z+29=0$$

from the point Q is:



(1) $\frac{22\sqrt{2}}{7}$

(2) $\frac{24\sqrt{2}}{7}$

(3) $2\sqrt{14}$

(4) $3\sqrt{14}$

Correct Answer: (4) $3\sqrt{14}$

Solution:

Equation of line PM:

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-3}{-1} = \lambda$$

Any point on this line is:

$$(\lambda + 2, 2\lambda - 1, -\lambda + 3)$$

Solving for λ, we get $\lambda = \frac{1}{2}$. Thus, the midpoint M is:

$$\left(\frac{5}{2}, 0, \frac{5}{2}\right)$$

For the image $Q(\alpha, \beta, \gamma)$, we compute:

$$\alpha + 2 = \frac{5}{2}, \quad \beta - 1 = 0, \quad \gamma + 3 = \frac{5}{2}$$

$$\alpha = \frac{1}{2}, \quad \beta = 1, \quad \gamma = -\frac{1}{2}$$

Using the distance formula:

$$d = \frac{|3(3) + 2(1) + (-1)(-1) + 29|}{\sqrt{3^2 + 2^2 + (-1)^2}}$$

$$= \frac{42}{\sqrt{14}} = 3\sqrt{14}$$

Quick Tip

For reflections in a plane, find the midpoint of the perpendicular segment and use distance formulas.

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17. Let

$$f(x) = \begin{bmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{bmatrix}$$

where $x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$**. If** α, β **are respectively the maximum and the minimum values of** f**, then**

(1) $\beta^2 - 2\sqrt{\alpha} = \frac{19}{4}$

(2) $\beta^2 + 2\sqrt{\alpha} = \frac{19}{4}$

(3) $\alpha^2 - \beta^2 = 4\sqrt{3}$

(4) $\alpha^2 + \beta^2 = \frac{9}{2}$

Correct Answer: (1) $\beta^2 - 2\sqrt{\alpha} = \frac{19}{4}$

Solution:

Applying column operations:

$$C_1 \to C_1 + C_2 + C_3$$

$$f(x) = \begin{bmatrix} 2+\sin 2x & \cos^2 x & \sin 2x \\ 2+\sin 2x & 1+\cos^2 x & \sin 2x \\ 2+\sin 2x & \cos^2 x & 1+\sin 2x \end{bmatrix}$$

Subtracting rows:

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$$

$$f(x) = (2+\sin 2x)\begin{bmatrix} 1 & \cos^2 x & \sin 2x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (2+\sin 2x)(1)$$

$$= 2 + \sin 2x$$

For given range $x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$:

$$\sin 2x \in \left[\frac{\sqrt{3}}{2}, 1\right]$$

Thus,

$$2 + \sin 2x \in \left[2 + \frac{\sqrt{3}}{2}, 3\right]$$

$$\alpha = 3, \quad \beta = 2 + \frac{\sqrt{3}}{2}$$
$$\beta^2 - 2\sqrt{\alpha} = \frac{19}{4}$$

Quick Tip

For determinant problems involving matrices, applying column/row operations simplifies the evaluation.

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18. Let

$$f(x) = 2x + \tan^{-1} x, \quad g(x) = \log_e(\sqrt{1 + x^2} + x)$$

where $x \in [0, 3]$. **Then:**

(1) There exists $x \in [0, 3]$ such that $f'(x) < g'(x)$

(2) $\max f(x) > \max g(x)$

(3) There exist $0 < x_1 < x_2 < 3$ such that $f(x) < g(x)$, $\forall x \in (x_1, x_2)$

(4) $\min f'(x) = 1 + \max g'(x)$

Correct Answer: (2)

Solution:

Computing derivatives:

$$g'(x) = \frac{1}{\sqrt{1 + x^2} + x}$$

Since $g(x)$ is always increasing in $[0, 3]$,

$$\min g'(x) = \frac{1}{\sqrt{10} + 3}, \quad \max g'(x) = \frac{1}{\sqrt{1} + 0} = 1$$

For $f(x)$,

$$f(x) = 2x + \tan^{-1} x$$

Since $f(x)$ is increasing,

$$\max f(x) = 6 + \tan^{-1} 3$$

For $g(x)$,

$$\max g(x) = \ln(3 + \sqrt{10})$$

$$6 + \tan^{-1} 3 > \ln(3 + \sqrt{10})$$

Quick Tip

Compare maximum values by checking function growth using derivatives.

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19. The mean and variance of 5 observations are 5 and 8, respectively. If 3 observations are 1, 3, 5, then the sum of cubes of the remaining two observations is:

(1) 1072

(2) 1792

(3) 1216

(4) 1456

Correct Answer: (1) 1072

Solution:

Using the mean formula:

$$\frac{1+3+5+a+b}{5} = 5$$

$$a + b = 16$$

Using variance formula:

$$\sigma^2 = \frac{\sum x_i^2}{5} - \left(\frac{\sum x_i}{5}\right)^2$$

$$8 = \frac{1^2 + 3^2 + 5^2 + a^2 + b^2}{5} - 25$$

$$a^2 + b^2 = 130$$

Solving equations:

$$a = 7, b = 9 \quad \text{or} \quad a = 9, b = 7$$

Computing sum of cubes:

$$a^3 + b^3 = 7^3 + 9^3 = 1072$$

Quick Tip

For mean and variance problems, use algebraic equations to find missing values and their statistical properties.

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20. The area enclosed by the closed curve C given by the differential equation

$$\frac{dy}{dx} + \frac{x+a}{y-2} = 0, \quad y(1) = 0$$

is 4π. **Let P and Q be the points of intersection of the curve C and the y-axis. If normals at P and Q on the curve C intersect the x-axis at points R and S respectively, then the length of the line segment RS is:**

(1) $2\sqrt{3}$

(2) $\frac{2\sqrt{3}}{3}$

(3) 2

(4) $\frac{4\sqrt{3}}{3}$

Correct Answer: (4) $\frac{4\sqrt{3}}{3}$

Solution:

Rewriting the given differential equation:

$$\frac{dy}{dx} = \frac{x+a}{2-y}$$

Separating variables and integrating,

$$(2-y)dy = (x+a)dx$$

$$2y - \frac{y^2}{2} = \frac{x^2}{2} + ax + C$$

Given $y(1) = 0$, solving for C,

$$a + C = -\frac{1}{2}$$

For the enclosed area,

$$X^2 + y^2 + 2ax - 4y - 1 - 2a = 0$$

Using standard circle form,

$$\pi r^2 = 4\pi$$

$$r^2 = 4$$

Solving for a,

$$4 = \sqrt{a^2 + 4 + 1 + 2a}$$

$$(a + 1)^2 = 0$$

Quick Tip

For area-related differential equations, solving for constants and using circle properties helps find enclosed regions.

Section-B

21. Let $a_1 = 8, a_2, a_3, \ldots a_n$ be an A.P. If the sum of its first four terms is 50 and the sum of its last four terms is 170, then the product of its middle two terms is:

Solution:

Using the sum formula for A.P.,

$$a_1 + a_2 + a_3 + a_4 = 50$$

$$8 + (8 + d) + (8 + 2d) + (8 + 3d) = 50$$

$$32 + 6d = 50 \Rightarrow d = 3$$

For the last four terms,

$$a_{n-3} + a_{n-2} + a_{n-1} + a_n = 170$$

$$32 + (4n - 10) \cdot 3 = 170$$

$$n = 14$$

Middle terms are:

$$a_7 = 26, \quad a_8 = 29$$

$$\Rightarrow a_7 \cdot a_8 = 754$$

Quick Tip

For A.P. problems, use the sum formula systematically and relate terms for efficient solving.

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22. A(2, 6, 2), B(-4, 0, λ), C(2, 3, -1) and D(4, 5, 0), where $|\lambda| \leq 5$ are the vertices of a quadrilateral ABCD. If its area is 18 square units, then

$5 - 6\lambda$ is equal to:

Solution:

Using the area formula for a quadrilateral in 3D:

$$\text{Area} = \frac{1}{2}\left|\overrightarrow{BD} \times \overrightarrow{AC}\right|$$

Computing cross product:

$$\overrightarrow{AC} \times \overrightarrow{BD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -3 & -3 \\ 8 & 5 & -\lambda \end{vmatrix}$$

Expanding determinant:

$$= (3\lambda + 15)\hat{i} - (-24)\hat{j} + (-24)\hat{k}$$

$$= (3\lambda + 15)\hat{i} + 24\hat{j} - 24\hat{k}$$

$$\left|(3\lambda + 15)^2 + 24^2 + (-24)^2\right| = 36$$

$$\lambda^2 + 10\lambda + 9 = 0$$

Solving for λ:

$$\lambda = -1, -9$$

Since $|\lambda| \leq 5$, we take $\lambda = -1$,

$$5 - 6(-1) = 11$$

Quick Tip

For area problems in 3D, use determinant expansion carefully for accurate cross-product computations.

23. The number of 3-digit numbers that are divisible by either 2 or 3 but not divisible by 7 is:

Solution:

Total numbers divisible by:

$$\text{By 2} = 450, \quad \text{By 3} = 300, \quad \text{By 7} = 128$$

$$\text{By 2 and 7} = 64, \quad \text{By 3 and 7} = 43, \quad \text{By 2, 3, and 7} = 21$$

Applying the inclusion-exclusion principle,

$$450 + 300 - 150 - 64 - 43 + 21 = 514$$

Quick Tip

For counting problems involving multiple divisibility rules, use the inclusion-exclusion principle.

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24. The remainder when $19^{200} + 23^{200}$ is divided by 49, is:

Solution:

Using modular arithmetic,

$$(21 + 2)^{200} + (21 - 2)^{200}$$

Applying binomial expansion,

$$2\sum_{k=0}^{100} \binom{200}{2k} 21^{198-2k} 2^{2k}$$

$$\Rightarrow 2[49I_1 + 2^{200}]$$

Since $2^{200} = 49L + 470$,

$$\text{Remainder} = 49L + 470 \mod 49 = 29$$

Quick Tip

For large power modular problems, express numbers in terms of mod bases and apply binomial theorem.

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25. If

$$\int_0^1 (x^{21} + x^4 + x^7)(2x^{14} + 3x^7 + 6)^{1/7} dx = \frac{1}{7}(11)^{m/n}$$

where $l, m, n \in \mathbb{N}$**, and** m, n **are coprime, then**

$$l + m + n \text{ is equal to:}$$

Solution:

Applying integral transformation:

$$\int_0^1 x^l (2x^{14} + 3x^7 + 6)^{1/7} dx$$

Setting $t = 42(x^{20} + x^{13} + x^6)dx$,

$$\frac{1}{42} \int_0^{11} t^7 dt$$

$$= \frac{1}{48}(11^{8/7})$$

$$l = 48, \quad m = 8, \quad n = 7$$

$$l + m + n = 63$$

Quick Tip

For power integrals, use substitution and exponent transformation techniques.

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26. If

$$f(x) = x^2 + g'(1)x + g''(2)$$

and

$$g(x) = f(1)x^2 + xf'(x) + f''(x),$$

then the value of $f(4) - g(4)$ **is equal to:**

Solution:

Computing derivatives:

$$f'(x) = 2x + g'(1), \quad f''(x) = 2$$

Solving for $g(x)$,

$$g(x) = f(1)x^2 + x[2x + g'(1)] + 2$$

$$g'(x) = 2f(1)x + 4x + g'(1)$$

$$g''(x) = 2f(1) + 4$$

Setting boundary conditions:

$$f(1) = -2, \quad g'(1) = -3$$

$$f(x) = x^2 - 3x$$

$$g(x) = -3x + 2$$

$$f(4) - g(4) = 14$$

Quick Tip

For function transformations, differentiate and equate known conditions for solving unknowns.

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27. Let

$$\vec{v} = \alpha\hat{i} + 2\hat{j} - 3\hat{k}, \quad \vec{w} = 2\alpha\hat{i} + \hat{j} - \hat{k}, \quad \text{and } \hat{u} \text{ be a vector such that } |\hat{u}| = \alpha > 0.$$

If the minimum value of the scalar triple product

$$[\vec{u} \quad \vec{v} \quad \vec{w}]$$

is $-\alpha\sqrt{3401}$, and

$$|\hat{u} \cdot \hat{i}|^2 = \frac{m}{n}$$

where m and n are coprime natural numbers, then

$$m + n \text{ is equal to:}$$

Solution:

Computing the scalar triple product:

$$[\vec{u} \quad \vec{v} \quad \vec{w}] = \hat{u} \cdot (\vec{v} \times \vec{w})$$

Given that the minimum value is $-\alpha\sqrt{3401}$,

$$\cos\theta = -1$$

$$|\hat{u}| = \alpha$$

Computing cross product:

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & 2 & -3 \\ 2\alpha & 1 & -1 \end{vmatrix}$$

Expanding the determinant:

$$\vec{v} \times \vec{w} = \hat{i}(-2+3) - \hat{j}(\alpha + 6\alpha) + \hat{k}(\alpha - 4\alpha)$$

$$= \hat{i} - 5\alpha\hat{j} - 3\alpha\hat{k}$$

Computing magnitude:

$$|\vec{v} \times \vec{w}| = \sqrt{1 + 25\alpha^2 + 9\alpha^2} = \sqrt{3401}$$

Equating for α^2,

$$34\alpha^2 = 3400$$

$$\alpha^2 = 100 \Rightarrow \alpha = 10$$

Thus,

$$\hat{u} = \lambda(\hat{i} - 5\alpha\hat{j} - 3\alpha\hat{k})$$

Computing:

$$|\hat{u} \cdot \hat{i}|^2 = \frac{100}{3401} = \frac{m}{n}$$

Since $m = 100, n = 3401$,

$$m + n = 3501$$

Quick Tip

For vector triple products, always compute the cross product first and then take the dot product with the given vector.

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28. The number of words, with or without meaning, that can be formed using all the letters of the word ASSASSINATION so that the vowels occur together, is:

Solution:

Grouping all vowels together as a single unit:

$$\text{Vowels: A, A, A, I, I, O}$$

$$\text{Consonants: S, S, S, S, N, N, T}$$

Total number of ways vowels occur together:

$$= \frac{8!}{4!2!} \times \frac{6!}{3!2!} = 50400$$

Quick Tip

For permutation problems with identical objects, use the formula $\frac{n!}{p!q!r!}$ where p, q, r are repeated elements.

---

29. Let A be the area bounded by the curve

$$y = x|x - 3|$$

the x-axis, and the ordinates $x = -1$ and $x = 2$. Then $12A$ is equal to:

Solution:

Computing the area:

$$A = \int_{-1}^{0} (x^2 - 3x)dx + \int_{0}^{2} (3x - x^2)dx$$

$$A = \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_{-1}^{0} + \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_{0}^{2}$$

$$A = \frac{11}{6} + \frac{10}{3} - \frac{31}{6}$$
$$\Rightarrow 12A = 62$$

Quick Tip

For area calculation using integrals, split the function where it changes form and integrate separately.

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30. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$f'(x) + f(x) = \int_0^2 f(t)dt.$$

If $f(0) = e^{-2}$, then

$2f(0) - f(2)$ is equal to:

Solution:

Given the differential equation:

$$\frac{dy}{dx} + y = k$$

Multiplying by the integrating factor e^x:

$$ye^x = k \cdot e^x + c$$

Using initial condition $f(0) = e^{-2}$:

$$c = e^{-2} - k$$

Thus, the general solution is:

$$y = k + (e^{-2} - k)e^{-x}$$

Integrating from 0 to 2:

$$k = \int_0^2 \left(k + (e^{-2} - k)e^{-x}\right) dx$$

Solving,

$$k = e^{-2} - 1$$

$$y = (e^{-2} - 1) + e^{-x}$$

Evaluating at $x = 2$:

$$f(2) = 2e^{-2} - 1, \quad f(0) = e^{-2}$$

$$2f(0) - f(2) = 1$$

Quick Tip

For first-order linear differential equations, use the integrating factor method and apply initial conditions to find constants.
