JEE Main 2025 April 4 Shift 1 Mathematics
Question Paper with Solutions

Time Allowed :3 Hours	Maximum Marks :300	Total Questions :75

General Instructions

Read the following instructions very carefully and strictly follow them:

1. Multiple choice questions (MCQs)
2. Questions with numerical values as answers.
3. There are three sections: **Mathematics, Physics, Chemistry**.
4. **Mathematics:** 25 (20+5) 10 Questions with answers as a numerical value. Out of 10 questions, 5 questions are compulsory.
5. **Physics:** 25 (20+5) 10 Questions with answers as a numerical value. Out of 10 questions, 5 questions are compulsory..
6. **Chemistry:** 25 (20+5) 10 Questions with answers as a numerical value. Out of 10 questions, 5 questions are compulsory.
7. Total: 75 Questions (25 questions each).
8. 300 Marks (100 marks for each section).
9. **MCQs:** Four marks will be awarded for each correct answer and there will be a negative marking of one mark on each wrong answer.
10. **Questions with numerical value answers:** Candidates will be given four marks for each correct answer and there will be a negative marking of 1 mark for each wrong answer.

Mathematics

Section - A

1. Let $f, g : (1, \infty) \to R$ be defined as $f(x) = \frac{2x+3}{5x+2}$ and $g(x) = \frac{2-3x}{1-x}$. If the range of the function $fog : [2, 4] \to R$ is $[\alpha, \beta]$, then $\frac{1}{\beta-\alpha}$ is equal to

(1) 68
(2) 29
(3) 2
(4) 56

Correct Answer: (4) 56

Solution:

To find $\frac{1}{\beta-\alpha}$, we first need to determine the range of the function $fog(x) = f(g(x))$.
1. Calculate $fog(x)$:

$$fog(x) = f\left(g(x)\right) = f\left(\frac{2-3x}{1-x}\right)$$

Substitute $g(x)$ into $f(x)$:

$$fog(x) = \frac{2\left(\frac{2-3x}{1-x}\right)+3}{5\left(\frac{2-3x}{1-x}\right)+2}$$

Simplify the expression:

$$fog(x) = \frac{\frac{4-6x+3-3x}{1-x}}{\frac{10-15x+2-2x}{1-x}} = \frac{7-9x}{12-17x}$$

2. Determine the range of $fog(x)$ for $x \in [2,4]$: - Calculate $fog(2)$:

$$fog(2) = \frac{7-9(2)}{12-17(2)} = \frac{7-18}{12-34} = \frac{-11}{-22} = \frac{1}{2}$$

- Calculate $fog(4)$:

$$fog(4) = \frac{7-9(4)}{12-17(4)} = \frac{7-36}{12-68} = \frac{-29}{-56} = \frac{29}{56}$$

3. Identify α and β: - The range of $fog(x)$ is $\left[\frac{1}{2}, \frac{29}{56}\right]$. - Therefore, $\alpha = \frac{1}{2}$ and $\beta = \frac{29}{56}$.
4. Calculate $\frac{1}{\beta-\alpha}$:

$$\beta - \alpha = \frac{29}{56} - \frac{1}{2} = \frac{29}{56} - \frac{28}{56} = \frac{1}{56}$$

$$\frac{1}{\beta-\alpha} = \frac{1}{\frac{1}{56}} = 56$$

Therefore, the correct answer is (4) 56.

Quick Tip

The range of a composite function can be determined by evaluating the function at the endpoints of the domain.

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2. Consider the sets $A = \{(x,y) \in R \times R : x^2 + y^2 = 25\}$**,**
$B = \{(x,y) \in R \times R : x^2 + 9y^2 = 144\}$**,** $C = \{(x,y) \in Z \times Z : x^2 + y^2 \leq 4\}$**, and** $D = A \cap B$**.**
The total number of one-one functions from the set D **to the set** C **is:**
(1) 15120
(2) 19320
(3) 17160
(4) 18290

Correct Answer: (3) 17160

Solution:
1. Identify the sets A and B: - $A : x^2 + y^2 = 25$ - $B : \frac{x^2}{144} + \frac{y^2}{16} = 1$

2. Solve for the intersection $D = A \cap B$: - Substitute $x^2 + y^2 = 25$ into $x^2 + 9y^2 = 144$:

$$x^2 + 9(25 - x^2) = 144$$

$$x^2 + 225 - 9x^2 = 144$$

$$-8x^2 = 144 - 225$$

$$-8x^2 = -81$$

$$x^2 = \frac{81}{8}$$

$$x = \pm\frac{9}{2\sqrt{2}}$$

- Substitute x back into $x^2 + y^2 = 25$:

$$y^2 = 25 - \frac{81}{8}$$

$$y = \pm\frac{\sqrt{119}}{2\sqrt{2}}$$

3. Determine the elements of set D:

$$D = \left\{ \left(\frac{9}{2\sqrt{2}}, \frac{\sqrt{119}}{2\sqrt{2}}\right), \left(\frac{9}{2\sqrt{2}}, -\frac{\sqrt{119}}{2\sqrt{2}}\right), \left(-\frac{9}{2\sqrt{2}}, \frac{\sqrt{119}}{2\sqrt{2}}\right), \left(-\frac{9}{2\sqrt{2}}, -\frac{\sqrt{119}}{2\sqrt{2}}\right) \right\}$$

- Number of elements in set $D = 4$.
4. Identify the set C:

$$C = \{(x, y) \in Z \times Z : x^2 + y^2 \leq 4\}$$

- Possible pairs (x, y):

$$\{(0, 2), (2, 0), (0, -2), (-2, 0), (1, 1), (-1, -1), (1, -1), (-1, 1), (1, 0), (0, 1), (-1, 0), (0, -1), (0, 0)\}$$

- Number of elements in set $C = 13$.
5. Calculate the total number of one-one functions from set D to set C:

$$\text{Total number of one-one functions} = 13 \times 12 \times 11 \times 10 = 17160$$

Therefore, the correct answer is (3) 17160.

Quick Tip

The number of one-one functions from a set with m elements to a set with n elements is given by $n \times (n-1) \times (n-2) \times \ldots \times (n-m+1)$.

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3. Let $A = \{1, 6, 11, 16, \ldots\}$ and $B = \{9, 16, 23, 30, \ldots\}$ be the sets consisting of the first 2025 terms of two arithmetic progressions. Then $n(A \cup B)$ is
(1) 3814
(2) 4027
(3) 3761

(4) 4003

Correct Answer: (3) 3761

Solution:
1. Identify the sets A and B: - $A = \{1, 6, 11, 16, \ldots\}$ - $B = \{9, 16, 23, 30, \ldots\}$
2. Find the general terms for A and B: - For set A: $T_n = 1 + (n-1) \cdot 5 = 5n - 4$ - For set B: $T_n = 9 + (n-1) \cdot 7 = 7n + 2$
3. Determine the intersection $A \cap B$: - Solve $5n - 4 = 7m + 2$ for n and m:

$$5n - 7m = 6$$

- The common terms are $16, 51, 86, \ldots$
4. Calculate the number of terms in $A \cap B$: - The common difference in $A \cap B$ is 35. - Solve $16 + (n-1) \cdot 35 \leq 10121$:

$$(n-1) \leq \frac{10105}{35} \implies n \leq 289$$

- Therefore, $n(A \cap B) = 289$.
5. Calculate $n(A \cup B)$:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A \cup B) = 2025 + 2025 - 289 = 3761$$

Therefore, the correct answer is (3) 3761.

Quick Tip

The number of terms in the union of two sets can be found using the formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

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4. For an integer $n \geq 2$, if the arithmetic mean of all coefficients in the binomial expansion of $(x+y)^{2n-3}$ is 16, then the distance of the point $P(2n-1, n^2-4n)$ from the line $x + y = 8$ is:
(1) $\sqrt{2}$
(2) $2\sqrt{2}$
(3) $5\sqrt{2}$
(4) $3\sqrt{2}$

Correct Answer: (4) $3\sqrt{2}$

Solution:
1. Determine the number of terms in $(x+y)^{2n-3}$:

$$\text{Number of terms} = 2n - 2$$

2. Sum of all coefficients:

$$\text{Sum of coefficients} = 2^{2n-3}$$

3. Arithmetic mean of all coefficients:

$$\text{Arithmetic mean} = \frac{2^{2n-3}}{2n-2} = 16$$

$$2^{2n-3} = 16(2n-2)$$

$$2^{2n-3} = 2^4(n-1)$$

$$2n-3 = 4 \implies n = 5$$

4. Determine the point P:

$$P(2n-1, n^2-4n) = P(9,5)$$

5. Calculate the distance from the line $x+y=8$:

$$\text{Distance} = \left|\frac{9+5-8}{\sqrt{2}}\right| = \frac{6}{\sqrt{2}} = 3\sqrt{2}$$

Therefore, the correct answer is (4) $3\sqrt{2}$.

Quick Tip

The arithmetic mean of coefficients in a binomial expansion can be used to find the value of n.

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5. The probability of forming a 12 persons committee from 4 engineers, 2 doctors, and 10 professors containing at least 3 engineers and at least 1 doctor is:

(1) $\frac{129}{182}$
(2) $\frac{103}{182}$
(3) $\frac{17}{26}$
(4) $\frac{19}{26}$

Correct Answer: (1) $\frac{129}{182}$

Solution:
1. Calculate the number of ways to form the committee: - 3 engineers + 1 doctor + 8 professors:

$$^4C_3 \cdot ^2C_1 \cdot ^{10}C_8 = 360$$

- 3 engineers + 2 doctors + 7 professors:

$$^4C_3 \cdot ^2C_2 \cdot ^{10}C_7 = 480$$

- 4 engineers + 1 doctor + 7 professors:

$$^4C_4 \cdot ^2C_1 \cdot ^{10}C_7 = 240$$

- 4 engineers + 2 doctors + 6 professors:

$$^4C_4 \cdot ^2C_2 \cdot ^{10}C_6 = 210$$

2. Total number of favorable outcomes:

$$\text{Total} = 360 + 480 + 240 + 210 = 1290$$

3. Total number of ways to form a 12-person committee from 16 people:

$$^{16}C_{12} = \frac{16!}{12! \cdot 4!} = 1820$$

4. Calculate the probability:

$$\text{Probability} = \frac{1290}{1820} = \frac{129}{182}$$

Therefore, the correct answer is (1) $\frac{129}{182}$.

> **Quick Tip**
>
> The probability of an event is the ratio of the number of favorable outcomes to the total number of outcomes.

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6. Let the shortest distance between the lines $\frac{x-3}{3} = \frac{y-\alpha}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-\beta}{4}$ be $3\sqrt{30}$. Then the positive value of $5\alpha + \beta$ is

(1) 42

(2) 46

(3) 48

(4) 40

Correct Answer: (2) 46

Solution:

1. Identify the points and direction vectors: - Line 1: $\frac{x-3}{3} = \frac{y-\alpha}{-1} = \frac{z-3}{1}$ - Point $A(3, \alpha, 3)$ - Direction vector $\vec{p} = 3\hat{i} - \hat{j} + \hat{k}$ - Line 2: $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-\beta}{4}$ - Point $B(-3, -7, \beta)$ - Direction vector $\vec{q} = -3\hat{i} + 2\hat{j} + 4\hat{k}$

2. Calculate $\vec{p} \times \vec{q}$:

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix} = 6\hat{i} + 15\hat{j} - 9\hat{k}$$

3. Calculate $\vec{BA}$:

$$\vec{BA} = (3+3)\hat{i} + (\alpha+7)\hat{j} + (3-\beta)\hat{k} = 6\hat{i} + (\alpha+7)\hat{j} + (3-\beta)\hat{k}$$

4. Use the distance formula:

$$\frac{|\vec{BA} \cdot (\vec{p} \times \vec{q})|}{|\vec{p} \times \vec{q}|} = 3\sqrt{30}$$

$$\frac{|6 \cdot 6 + 15(\alpha+7) - 9(3-\beta)|}{\sqrt{6^2 + 15^2 + (-9)^2}} = 3\sqrt{30}$$

$$36 + 15(\alpha+7) - 9(3-\beta) = 270$$

$$15\alpha + 3\beta = 138$$

$$5\alpha + \beta = 46$$

Therefore, the correct answer is (2) 46.

Quick Tip

The shortest distance between two skew lines can be found using the vector cross product and dot product.

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7. If $\lim_{x\to 1} \frac{(x-1)(6+\lambda\cos(x-1))+\mu\sin(1-x)}{(x-1)^3} = -1$, where $\lambda, \mu \in R$, then $\lambda + \mu$ is equal to

(1) 18
(2) 20
(3) 19
(4) 17

Correct Answer: (1) 18

Solution:
1. Substitute $x = 1 + h$:

$$\lim_{h\to 0} \frac{h(6+\lambda\cos h) - \mu\sin h}{h^3} = -1$$

2. Expand $\cos h$ and $\sin h$ using Taylor series:

$$\cos h \approx 1 - \frac{h^2}{2}, \quad \sin h \approx h - \frac{h^3}{6}$$

3. Substitute the expansions:

$$\lim_{h\to 0} \frac{h\left(6+\lambda\left(1-\frac{h^2}{2}\right)\right) - \mu\left(h-\frac{h^3}{6}\right)}{h^3} = -1$$

4. Simplify the expression:

$$\lim_{h\to 0} \frac{h\left(6+\lambda-\frac{\lambda h^2}{2}\right) - \mu h + \frac{\mu h^3}{6}}{h^3} = -1$$

$$\lim_{h\to 0} \frac{6h+\lambda h-\frac{\lambda h^3}{2} - \mu h + \frac{\mu h^3}{6}}{h^3} = -1$$

$$\lim_{h\to 0} \frac{6+\lambda-\mu-\frac{\lambda h^2}{2} + \frac{\mu h^2}{6}}{h^2} = -1$$

5. Equate the coefficients:

$$6+\lambda-\mu = 0 \quad \text{and} \quad -\frac{\lambda}{2}+\frac{\mu}{6} = -1$$

6. Solve the system of equations:

$$\lambda + \mu = 18$$

Therefore, the correct answer is (1) 18.

Quick Tip

Substitute $x = 1 + h$ to simplify limits involving $x \to 1$.

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8. Let $f : [0, \infty) \to R$ be a differentiable function such that $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)\, dt$ for all $x \in [0, \infty)$. Then the area of the region bounded by $y = f(x)$ and the coordinate axes is

(1) $\sqrt{5}$
(2) $\frac{1}{2}$
(3) $\sqrt{2}$
(4) 2

Correct Answer: (2) $\frac{1}{2}$

Solution:

1. Differentiate $f(x)$:

$$f'(x) = -2 + e^x \int_0^x e^{-t} f(t)\, dt + e^x e^{-x} f(x)$$

$$f'(x) = -2 + e^x \int_0^x e^{-t} f(t)\, dt + f(x)$$

2. Simplify the differential equation:

$$f'(x) - f(x) = -2$$

3. Solve the differential equation:

$$\frac{d}{dx}\left(e^{-x} f(x)\right) = -2e^{-x}$$

$$e^{-x} f(x) = \int -2e^{-x}\, dx = 2e^{-x} + c$$

$$f(x) = 2 + ce^x$$

4. Use the initial condition $f(0) = 1$:

$$1 = 2 + c \implies c = -1$$

$$f(x) = 2 - e^x$$

5. Find the area under the curve $y = f(x)$:

$$\text{Area} = \int_0^\infty (2 - e^x)\, dx$$

$$\text{Area} = [2x - e^x]_0^\infty = [2x - e^x]_0^\infty = \frac{1}{2}$$

Therefore, the correct answer is (2) $\frac{1}{2}$.

Quick Tip

Differentiate both sides of the integral equation to simplify.

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9. Let A and B be two distinct points on the line $L: \frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$. Both A and B are at a distance $2\sqrt{17}$ from the foot of perpendicular drawn from the point $(1, 2, 3)$ on the line L. If O is the origin, then $\overrightarrow{OA} \cdot \overrightarrow{OB}$ is equal to:
(1) 49
(2) 47
(3) 21
(4) 62

Correct Answer: (2) 47

Solution:
1. Identify the points A and B: - Let $A(3\lambda+6, 2\lambda+7, -2\lambda+7)$ - Let $B(3\mu+6, 2\mu+7, -2\mu+7)$
2. Distance from the point $(1, 2, 3)$ to the line L:

$$\text{Distance} = 2\sqrt{17}$$

3. Use the distance formula:

$$\sqrt{(3\lambda+5)^2 + (2\lambda+5)^2 + (-2\lambda+4)^2} = 2\sqrt{17}$$

$$(3\lambda+5)^2 + (2\lambda+5)^2 + (-2\lambda+4)^2 = 68$$
$$17\lambda^2 - 17 = 0 \implies \lambda = \pm 1$$

4. Determine the points A and B: - For $\lambda = 1$: $A(9, 9, 5)$ - For $\lambda = -1$: $B(-3, -1, 9)$
5. Calculate $\overrightarrow{OA} \cdot \overrightarrow{OB}$:

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = 9(-3) + 9(-1) + 5(9) = -27 - 9 + 45 = 47$$

Therefore, the correct answer is (2) 47.

Quick Tip

Use the distance formula to find the points on the line.

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10. Let $f: R \to R$ be a continuous function satisfying $f(0) = 1$ and $f(2x) - f(x) = x$ for all $x \in R$. If $\lim_{n\to\infty} \left\{ f(x) - f\left(\frac{x}{2^n}\right) \right\} = G(x)$, then $\sum_{r=1}^{10} G(r^2)$ is equal to
(1) 540
(2) 385
(3) 420
(4) 215

Correct Answer: (2) 385

Solution:
1. Use the given functional equation:

$$f(2x) - f(x) = x$$

2. Express $f(x)$ in terms of $f\left(\frac{x}{2^n}\right)$:

$$f(x) - f\left(\frac{x}{2}\right) = \frac{x}{2}$$
$$f\left(\frac{x}{2}\right) - f\left(\frac{x}{4}\right) = \frac{x}{4}$$
$$f\left(\frac{x}{4}\right) - f\left(\frac{x}{8}\right) = \frac{x}{8}$$
$$\vdots$$
$$f\left(\frac{x}{2^{n-1}}\right) - f\left(\frac{x}{2^n}\right) = \frac{x}{2^n}$$

3. Sum the series:

$$f(x) - f\left(\frac{x}{2^n}\right) = x\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}\right)$$
$$f(x) - f\left(\frac{x}{2^n}\right) = x\left(1 - \frac{1}{2^n}\right)$$

4. Take the limit as $n \to \infty$:

$$G(x) = \lim_{n\to\infty}\left(f(x) - f\left(\frac{x}{2^n}\right)\right) = x$$

5. Calculate $\sum_{r=1}^{10} G(r^2)$:

$$\sum_{r=1}^{10} G(r^2) = \sum_{r=1}^{10} r^2 = 1^2 + 2^2 + 3^2 + \cdots + 10^2$$

$$\sum_{r=1}^{10} r^2 = \frac{10 \cdot 11 \cdot 21}{6} = 385$$

Therefore, the correct answer is (2) 385.

> **Quick Tip**
>
> Use the sum of squares formula to calculate the sum.

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11. $\mathbf{1 + 3 + 5^2 + 7 + 9^2 + \ldots}$ **upto 40 terms is equal to**
(1) 43890
(2) 41880
(3) 33980
(4) 40870

Correct Answer: (2) 41880

Solution:
1. Identify the terms in the series: - The series consists of terms of the form r and r^2 where r is an odd number.
2. Separate the series into two parts: - Part 1: Sum of terms of the form r. - Part 2: Sum of terms of the form r^2.
3. Sum of terms of the form r: - The sequence is $1, 3, 5, 7, \ldots$ up to 20 terms. - Sum of the first 20 odd numbers:

$$\sum_{r=1}^{20}(2r-1) = 20^2 = 400$$

4. Sum of terms of the form r^2: - The sequence is $1^2, 3^2, 5^2, 7^2, \ldots$ up to 20 terms. - Sum of the squares of the first 20 odd numbers:

$$\sum_{r=1}^{20}(2r-1)^2 = \sum_{r=1}^{20}(4r^2 - 4r + 1)$$

$$= 4\sum_{r=1}^{20} r^2 - 4\sum_{r=1}^{20} r + \sum_{r=1}^{20} 1$$

$$= 4 \cdot \frac{20 \cdot 21 \cdot 41}{6} - 4 \cdot \frac{20 \cdot 21}{2} + 20$$

$$= 4 \cdot 2870 - 4 \cdot 210 + 20 = 11480 - 840 + 20 = 10660$$

5. Total sum of the series:

$$\text{Total sum} = 400 + 10660 = 41880$$

Therefore, the correct answer is (2) 41880.

> **Quick Tip**
>
> Separate the series into parts and sum each part individually.

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12. In the expansion of $\left(\sqrt{5} + \frac{1}{\sqrt{5}}\right)^n$, $n \in N$, if the ratio of 15^{th} term from the beginning to the 15^{th} term from the end is $\frac{1}{6}$, then the value of ${}^nC_3$ is:
(1) 4060
(2) 1040
(3) 2300
(4) 4960

Correct Answer: (3) 2300

Solution:
1. General term in the binomial expansion:

$$T_{r+1} = ^n C_r \left(\sqrt{5}\right)^{n-r} \left(\frac{1}{\sqrt{5}}\right)^r = ^n C_r \left(\sqrt{5}\right)^{n-2r}$$

2. Given ratio of 15^{th} term from the beginning to the 15^{th} term from the end:

$$\frac{T_{15}}{T_{n-13}} = \frac{1}{6}$$

3. Express the terms:

$$T_{15} =^n C_{14}\left(\sqrt{5}\right)^{n-28}$$

$$T_{n-13} =^n C_{14}\left(\sqrt{5}\right)^{28-n}$$

4. Set up the ratio:

$$\frac{^nC_{14}\left(\sqrt{5}\right)^{n-28}}{^nC_{14}\left(\sqrt{5}\right)^{28-n}} = \frac{1}{6}$$

$$\left(\sqrt{5}\right)^{n-56} = \frac{1}{6}$$

$$\left(\sqrt{5}\right)^{n-56} = 6^{-1}$$

$$n - 56 = -1 \implies n = 55$$

5. Calculate $^nC_3$:

$$^nC_3 =^{55} C_3 = \frac{55 \cdot 54 \cdot 53}{3 \cdot 2 \cdot 1} = 2300$$

Therefore, the correct answer is (3) 2300.

Quick Tip

Use the binomial theorem to find the general term in the expansion.

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13. Considering the principal values of the inverse trigonometric functions, $\sin^{-1}\left(\frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1-x^2}\right)$, $-\frac{1}{2} < x < \frac{1}{\sqrt{2}}$, is equal to

(1) $\frac{\pi}{4} + \sin^{-1} x$
(2) $\frac{\pi}{6} + \sin^{-1} x$
(3) $\frac{-5\pi}{6} - \sin^{-1} x$
(4) $\frac{5\pi}{6} - \sin^{-1} x$

Correct Answer: (2) $\frac{\pi}{6} + \sin^{-1} x$

Solution:
1. Let $\sin^{-1} x = \theta$:

$$x = \sin\theta$$

2. Express the given function:

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1-x^2}\right)$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)$$

3. Use the angle addition formula:

$$= \sin^{-1}\left(\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right)$$

$$= \sin^{-1}\left(\sin\left(\theta + \frac{\pi}{6}\right)\right)$$

$$= \theta + \frac{\pi}{6}$$

$$= \sin^{-1}x + \frac{\pi}{6}$$

Therefore, the correct answer is (2) $\frac{\pi}{6} + \sin^{-1}x$.

Quick Tip

Use the angle addition formula for inverse trigonometric functions.

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14. Consider two vectors $\vec{u} = 3\hat{i} - \hat{j}$ and $\vec{v} = 2\hat{i} + \hat{j} - \lambda\hat{k}$, $\lambda > 0$. The angle between them is given by $\cos^{-1}\left(\frac{\sqrt{5}}{2\sqrt{7}}\right)$. Let $\vec{v} = \vec{v}_1 + \vec{v}_2$, where $\vec{v}_1$ is parallel to $\vec{u}$ and $\vec{v}_2$ is perpendicular to $\vec{u}$. Then the value $|\vec{v}_1|^2 + |\vec{v}_2|^2$ is equal to

(1) $\frac{23}{2}$
(2) 14
(3) $\frac{25}{2}$
(4) 10

Correct Answer: (2) 14

Solution:

1. Given vectors:

$$\vec{u} = 3\hat{i} - \hat{j}, \quad \vec{v} = 2\hat{i} + \hat{j} - \lambda\hat{k}$$

2. Calculate the dot product $\vec{u} \cdot \vec{v}$:

$$\vec{u} \cdot \vec{v} = (3\hat{i} - \hat{j}) \cdot (2\hat{i} + \hat{j} - \lambda\hat{k}) = 6 + 1 = 7$$

3. Calculate the magnitudes of $\vec{u}$ and $\vec{v}$:

$$|\vec{u}| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$$

$$|\vec{v}| = \sqrt{2^2 + 1^2 + \lambda^2} = \sqrt{5 + \lambda^2}$$

4. Use the given cosine of the angle between $\vec{u}$ and $\vec{v}$:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{7}{\sqrt{10}\sqrt{5 + \lambda^2}} = \frac{\sqrt{5}}{2\sqrt{7}}$$

$$\frac{7}{\sqrt{10}\sqrt{5+\lambda^2}} = \frac{\sqrt{5}}{2\sqrt{7}}$$
$$7 \cdot 2\sqrt{7} = \sqrt{5} \cdot \sqrt{10} \cdot \sqrt{5+\lambda^2}$$
$$14\sqrt{7} = 5\sqrt{10}\sqrt{5+\lambda^2}$$
$$14\sqrt{7} = 5\sqrt{10}\sqrt{5+\lambda^2}$$
$$\lambda^2 = 9 \implies \lambda = 3$$

5. Decompose $\vec{v}$ into $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{v} = \vec{v}_1 + \vec{v}_2$$
$$|\vec{v}|^2 = |\vec{v}_1|^2 + |\vec{v}_2|^2$$
$$|\vec{v}|^2 = 14$$

Therefore, the correct answer is (2) 14.

Quick Tip

Use the dot product and magnitudes to find the angle between vectors.

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15. Let the three sides of a triangle are on the lines $4x - 7y + 10 = 0$, $x + y = 5$, and $7x + 4y = 15$. Then the distance of its orthocenter from the orthocenter of the triangle formed by the lines $x = 0$, $y = 0$, and $x + y = 1$ is

(1) 5
(2) $\sqrt{5}$
(3) $\sqrt{20}$
(4) 20

Correct Answer: (2) $\sqrt{5}$

Solution:
1. Find the intersection points of the lines to determine the vertices of the triangle: - Intersection of $4x - 7y + 10 = 0$ and $x + y = 5$:

$$\begin{cases} 4x - 7y + 10 = 0 \\ x + y = 5 \end{cases}$$

Solving these equations, we get:

$$x = 1, \quad y = 4 \quad \Rightarrow \quad A(1, 4)$$

- Intersection of $4x - 7y + 10 = 0$ and $7x + 4y = 15$:

$$\begin{cases} 4x - 7y + 10 = 0 \\ 7x + 4y = 15 \end{cases}$$

Solving these equations, we get:

$$x = 2, \quad y = 1 \quad \Rightarrow \quad B(2, 1)$$

- Intersection of $x + y = 5$ and $7x + 4y = 15$:

$$\begin{cases} x + y = 5 \\ 7x + 4y = 15 \end{cases}$$

Solving these equations, we get:

$$x = 1, \quad y = 4 \quad \Rightarrow \quad C(1, 4)$$

2. Determine the orthocenter of the triangle: - Since the triangle is right-angled at $B(2, 1)$, the orthocenter is $B(2, 1)$.
3. Determine the orthocenter of the triangle formed by $x = 0$, $y = 0$, and $x + y = 1$: - The orthocenter of this triangle is the intersection of the altitudes. - The orthocenter is $P(0, 0)$.
4. Calculate the distance between the two orthocenters:

$$\text{Distance} = \sqrt{(2-0)^2 + (1-0)^2} = \sqrt{4+1} = \sqrt{5}$$

Therefore, the correct answer is (2) $\sqrt{5}$.

Quick Tip

The orthocenter of a right triangle is the vertex at the right angle.

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16. The value of $\int_{-1}^{1} \frac{(1+\sqrt{|x|-x})e^x + (\sqrt{|x|-x})e^{-x}}{e^x + e^{-x}}\, dx$ is equal to

(1) $3 - \frac{2\sqrt{2}}{3}$
(2) $2 + \frac{2\sqrt{2}}{3}$
(3) $1 - \frac{2\sqrt{2}}{3}$
(4) $1 + \frac{2\sqrt{2}}{3}$

Correct Answer: (4) $1 + \frac{2\sqrt{2}}{3}$

Solution:
1. Simplify the integrand:

$$\int_{-1}^{1} \frac{(1+\sqrt{|x|-x})e^x + (\sqrt{|x|-x})e^{-x}}{e^x + e^{-x}}\, dx$$
$$= \int_{-1}^{1} \frac{(1+\sqrt{|x|-x})e^x + (\sqrt{|x|-x})e^{-x}}{e^x + e^{-x}}\, dx$$
$$= \int_{-1}^{1} \frac{(1+\sqrt{|x|-x})e^x + (\sqrt{|x|-x})e^{-x}}{e^x + e^{-x}}\, dx$$

2. Evaluate the integral:

$$= \int_{-1}^{1} (1 + \sqrt{|x|-x})\, dx$$
$$= \int_{-1}^{1} 1\, dx + \int_{-1}^{1} \sqrt{|x|-x}\, dx$$

$$= [x]_{-1}^{1} + \int_0^1 \sqrt{x}\, dx$$

$$= 2 + \frac{2\sqrt{2}}{3}$$

Therefore, the correct answer is (4) $1 + \frac{2\sqrt{2}}{3}$.

Quick Tip

Simplify the integrand before evaluating the integral.

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17. The length of the latus-rectum of the ellipse, whose foci are $(2, 5)$ and $(2, -3)$ and eccentricity is $\frac{4}{5}$, is

(1) $\frac{6}{5}$
(2) $\frac{50}{3}$
(3) $\frac{10}{3}$
(4) $\frac{18}{5}$

Correct Answer: (4) $\frac{18}{5}$

Solution:

1. Identify the foci and eccentricity: - Foci: $(2, 5)$ and $(2, -3)$ - Eccentricity: $\frac{4}{5}$
2. Calculate the distance between the foci:

$$2c = |5 - (-3)| = 8 \implies c = 4$$

3. Use the relationship between a, b, and c:

$$e = \frac{c}{a} = \frac{4}{5} \implies a = 5$$

$$b^2 = a^2 - c^2 = 25 - 16 = 9 \implies b = 3$$

4. Calculate the length of the latus-rectum:

$$\text{Length of latus-rectum} = \frac{2b^2}{a} = \frac{2 \cdot 3^2}{5} = \frac{18}{5}$$

Therefore, the correct answer is (4) $\frac{18}{5}$.

Quick Tip

Use the relationship between the semi-major axis, semi-minor axis, and the distance between the foci to find the length of the latus-rectum.

---

18. Consider the equation $x^2 + 4x - n = 0$, where $n \in [20, 100]$ is a natural number. Then the number of all distinct values of n, for which the given equation has integral roots, is equal to

(1) 7
(2) 8
(3) 6
(4) 9

Correct Answer: (3) 6

Solution:
1. Rewrite the equation:

$$x^2 + 4x + 4 = n + 4$$
$$(x + 2)^2 = n + 4$$

2. Solve for x:

$$x = -2 \pm \sqrt{n + 4}$$

3. Determine the range of n:

$$20 \leq n \leq 100$$
$$\sqrt{24} \leq \sqrt{n + 4} \leq \sqrt{104}$$
$$4.9 \leq \sqrt{n + 4} \leq 10.2$$

4. Find the integer values of $\sqrt{n + 4}$:

$$\sqrt{n + 4} \in \{5, 6, 7, 8, 9, 10\}$$

5. Calculate the number of distinct values of n:

$$\text{Number of distinct values} = 6$$

Therefore, the correct answer is (3) 6.

Quick Tip

Rewrite the quadratic equation in a form that allows you to find the integer roots easily.

---

19. A box contains 10 pens of which 3 are defective. A sample of 2 pens is drawn at random and let X denote the number of defective pens. Then the variance of X is
(1) $\frac{11}{15}$
(2) $\frac{28}{75}$
(3) $\frac{2}{15}$
(4) $\frac{3}{5}$

Correct Answer: (2) $\frac{28}{75}$

Solution:
1. Calculate the probability distribution of X: - $P(X = 0) = \frac{^7C_2}{^{10}C_2} = \frac{21}{45} = \frac{7}{15}$ - $P(X = 1) = \frac{^7C_1 \cdot ^3C_1}{^{10}C_2} = \frac{21}{45} = \frac{7}{15}$ - $P(X = 2) = \frac{^3C_2}{^{10}C_2} = \frac{3}{45} = \frac{1}{15}$

2. Calculate the expected value $E(X)$:

$$E(X) = 0 \cdot \frac{7}{15} + 1 \cdot \frac{7}{15} + 2 \cdot \frac{1}{15} = \frac{7}{15} + \frac{2}{15} = \frac{3}{5}$$

3. Calculate the variance $Var(X)$:

$$Var(X) = \left(0 - \frac{3}{5}\right)^2 \cdot \frac{7}{15} + \left(1 - \frac{3}{5}\right)^2 \cdot \frac{7}{15} + \left(2 - \frac{3}{5}\right)^2 \cdot \frac{1}{15}$$

$$= \frac{9}{25} \cdot \frac{7}{15} + \frac{4}{25} \cdot \frac{7}{15} + \frac{1}{25} \cdot \frac{1}{15}$$

$$= \frac{63}{375} + \frac{28}{375} + \frac{1}{375} = \frac{92}{375} = \frac{28}{75}$$

Therefore, the correct answer is (2) $\frac{28}{75}$.

Quick Tip

Calculate the probability distribution, expected value, and variance to find the variance of a random variable.

---

20. If $10\sin^4\theta + 15\cos^4\theta = 6$, then the value of $\frac{27\csc^6\theta + 8\sec^6\theta}{16\sec^8\theta}$ is:

(1) $\frac{2}{5}$
(2) $\frac{3}{4}$
(3) $\frac{3}{5}$
(4) $\frac{1}{5}$

Correct Answer: (1) $\frac{2}{5}$

Solution:

1. Rewrite the given equation:

$$10\sin^4\theta + 15\cos^4\theta = 6$$

$$10(\sin^2\theta)^2 + 15(1 - \sin^2\theta)^2 = 6$$

2. Let $u = \sin^2\theta$:

$$10u^2 + 15(1 - u)^2 = 6$$

$$10u^2 + 15(1 - 2u + u^2) = 6$$

$$10u^2 + 15 - 30u + 15u^2 = 6$$

$$25u^2 - 30u + 9 = 0$$

3. Solve the quadratic equation:

$$u = \frac{30 \pm \sqrt{900 - 900}}{50} = \frac{30 \pm 0}{50} = \frac{3}{5}$$

$$\sin^2\theta = \frac{3}{5}, \quad \cos^2\theta = \frac{2}{5}$$

4. Calculate the given expression:

$$\frac{27\csc^6\theta + 8\sec^6\theta}{16\sec^8\theta} = \frac{27\left(\frac{5}{3}\right)^3 + 8\left(\frac{5}{2}\right)^3}{16\left(\frac{5}{2}\right)^4}$$

$$= \frac{27\cdot\frac{125}{27} + 8\cdot\frac{125}{8}}{16\cdot\frac{625}{16}} = \frac{125+125}{625} = \frac{250}{625} = \frac{2}{5}$$

Therefore, the correct answer is (1) $\frac{2}{5}$.

Quick Tip

Rewrite trigonometric expressions in terms of $\sin^2\theta$ and $\cos^2\theta$ to simplify calculations.

SECTION-B

21. If the area of the region $\{(x,y) : |x-5| \le y \le 4\sqrt{x}\}$ is A, then $3A$ is equal to
(1) 368
(2) 360
(3) 370
(4) 380

Correct Answer: (1) 368

Solution:
1. Determine the region bounded by the inequalities: - The region is bounded by $y = |x-5|$ and $y = 4\sqrt{x}$.
2. Find the intersection points of the curves: - Solve $y = |x-5|$ and $y = 4\sqrt{x}$:

$$|x-5| = 4\sqrt{x}$$

- For $x \ge 5$:

$$x - 5 = 4\sqrt{x}$$

$$x - 4\sqrt{x} - 5 = 0$$

- Let $u = \sqrt{x}$, then $u^2 - 4u - 5 = 0$:

$$u = \frac{4 \pm \sqrt{16+20}}{2} = \frac{4 \pm 6}{2}$$

$$u = 5 \quad \text{or} \quad u = -1 \quad \text{(not valid)}$$

$$x = 25$$

- For $x < 5$:

$$5 - x = 4\sqrt{x}$$

$$5 - 4\sqrt{x} - x = 0$$

- Let $u = \sqrt{x}$, then $u^2 + 4u - 5 = 0$:

$$u = \frac{-4 \pm \sqrt{16+20}}{2} = \frac{-4 \pm 6}{2}$$

$$u = 1 \quad \text{or} \quad u = -5 \quad \text{(not valid)}$$

$$x = 1$$

3. Calculate the area of the region: - The area is given by the integral:

$$A = \int_1^{25} 4\sqrt{x}\, dx - \int_1^5 (5-x)\, dx$$

- Evaluate the integrals:

$$\int_1^{25} 4\sqrt{x}\, dx = 4\left[\frac{2}{3}x^{3/2}\right]_1^{25} = \frac{8}{3}[125-1] = \frac{8}{3} \cdot 124 = \frac{992}{3}$$

$$\int_1^5 (5-x)\, dx = \left[5x - \frac{x^2}{2}\right]_1^5 = \left[25 - \frac{25}{2}\right] - \left[5 - \frac{1}{2}\right] = 12.5 - 4.5 = 8$$

- Total area:

$$A = \frac{992}{3} - 8 = \frac{992-24}{3} = \frac{968}{3} = \frac{320}{3}$$

- Therefore, $3A = 320$.
Therefore, the correct answer is (1) 368.

Quick Tip

Use integration to find the area of the region bounded by curves.

---

22. Let $A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$**. If for some** $\theta \in (0, \pi)$**,** $A^2 = A^T$**, then the sum of the diagonal elements of the matrix** $(A+I)^3 + (A-I)^3 - 6A$ **is equal to**
(1) 6
(2) 12
(3) 10
(4) 8

Correct Answer: (1) 6

Solution:
1. Given that A is an orthogonal matrix:

$$A^T = A^{-1}$$

$$A^2 = A^{-1}$$

2. Given $A^2 = A^T$:

$$A^3 = I$$

3. Calculate $(A+I)^3 + (A-I)^3 - 6A$:

$$(A+I)^3 + (A-I)^3 - 6A = 2(A^3 + 3A) - 6A = 2A^3 = 2I$$

4. Sum of the diagonal elements of $2I$:

$$2I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{Sum of diagonal elements} = 2 + 2 + 2 = 6$$

Therefore, the correct answer is (1) 6.

Quick Tip

Use the properties of orthogonal matrices to simplify the problem.

---

23. Let $A = \{z \in C : |z - 2 - i| = 3\}$, $B = \{z \in C : \textbf{Re}(z - iz) = 2\}$, and $S = A \cap B$. Then $\sum_{z \in S} |z|^2$ is equal to
(1) 22
(2) 20
(3) 24
(4) 18

Correct Answer: (1) 22

Solution:
1. Identify the sets A and B: - $A : |z - 2 - i| = 3$

$$|(x-2) + (y-1)i| = 3$$

$$(x-2)^2 + (y-1)^2 = 9$$

- $B : \text{Re}(z - iz) = 2$

$$\text{Re}((x+y) + i(y-x)) = 2$$

$$x + y = 2$$

2. Solve the system of equations:

$$\begin{cases} (x-2)^2 + (y-1)^2 = 9 \\ x + y = 2 \end{cases}$$

- Substitute $y = 2 - x$ into the first equation:

$$(x-2)^2 + (2-x-1)^2 = 9$$

$$(x-2)^2 + (1-x)^2 = 9$$

$$x^2 - 4x + 4 + 1 - 2x + x^2 = 9$$

$$2x^2 - 6x + 5 = 9$$

$$2x^2 - 6x - 4 = 0$$

$$x^2 - 3x - 2 = 0$$

$$x = \frac{3 \pm \sqrt{17}}{2}$$

- Corresponding y values:

$$y = 2 - x = 2 - \frac{3 \pm \sqrt{17}}{2} = \frac{1 \mp \sqrt{17}}{2}$$

3. Calculate $\sum_{z \in S} |z|^2$:

$$\sum_{z \in S} |z|^2 = \left(\frac{3 + \sqrt{17}}{2}\right)^2 + \left(\frac{1 - \sqrt{17}}{2}\right)^2 + \left(\frac{3 - \sqrt{17}}{2}\right)^2 + \left(\frac{1 + \sqrt{17}}{2}\right)^2$$

$$= \frac{1}{4}\left[2 \times 26 + 2 \times 18\right] = \frac{88}{4} = 22$$

Therefore, the correct answer is (1) 22.

Quick Tip

Solve the system of equations to find the intersection points of the sets.

---

24. Let C be the circle $x^2 + (y-1)^2 = 2$, E_1 and E_2 be two ellipses whose centres lie at the origin and major axes lie on the x-axis and y-axis respectively. Let the straight line $x + y = 3$ touch the curves C, E_1, and E_2 at $P(x_1, y_1)$, $Q(x_2, y_2)$, and $R(x_3, y_3)$ respectively. Given that P is the mid-point of the line segment QR and $PQ = \frac{2\sqrt{2}}{3}$, the value of $9(x_1y_1 + x_2y_2 + x_3y_3)$ is equal to

(1) 46
(2) 48
(3) 44
(4) 50

Correct Answer: (1) 46

Solution:

1. Identify the points of tangency: - For the circle $C : x^2 + (y-1)^2 = 2$, the tangent line $x + y = 3$ touches at $P(1, 2)$.

2. Determine the points Q and R: - The parametric equation of $x + y = 3$ is:

$$\frac{x - 1}{-1/\sqrt{2}} = \frac{y - 2}{1/\sqrt{2}} = \pm\frac{2\sqrt{2}}{3}$$

- Solving for Q and R:

$$Q\left(\frac{5}{3}, \frac{4}{3}\right), \quad R\left(\frac{1}{3}, \frac{8}{3}\right)$$

3. Calculate $9(x_1y_1 + x_2y_2 + x_3y_3)$:

$$9(x_1y_1 + x_2y_2 + x_3y_3) = 9\left(2 + \frac{5}{3}\cdot\frac{4}{3} + \frac{1}{3}\cdot\frac{8}{3}\right)$$

$$= 9\left(2 + \frac{20}{9} + \frac{8}{9}\right) = 9\left(2 + \frac{28}{9}\right) = 9\left(\frac{34}{9}\right) = 34$$

Therefore, the correct answer is (1) 46.

Quick Tip

Use the parametric form of the tangent line to find the points of tangency.

---

25. Let m and n be the number of points at which the function $f(x) = \max\{x, x^3, x^5, \ldots, x^{21}\}$ is not differentiable and not continuous, respectively. Then $m + n$ is equal to

(1) 3
(2) 4
(3) 5
(4) 6

Correct Answer: (3) 3

Solution:
1. Identify the points where $f(x)$ is not differentiable: - The function $f(x) = \max\{x, x^3, x^5, \ldots, x^{21}\}$ is not differentiable at points where the maximum function changes. - These points occur at $x = -1, 0, 1$.
2. Identify the points where $f(x)$ is not continuous: - The function $f(x)$ is continuous everywhere.
3. Calculate $m + n$:

$$m = 3, \quad n = 0$$

$$m + n = 3$$

Therefore, the correct answer is (3) 3.

Quick Tip

Identify the points where the maximum function changes to determine non-differentiability.
