JEE Main 2025 April 4 Shift 2 Mathematics Question Paper with Solutions

Time Allowed : 3 Hours	Maximum Marks : 300	Total Questions : 75

General Instructions

Read the following instructions very carefully and strictly follow them:

1. Multiple choice questions (MCQs)
2. Questions with numerical values as answers.
3. There are three sections: **Mathematics, Physics, Chemistry**.
4. **Mathematics:** 25 (20+5) 10 Questions with answers as a numerical value. Out of 10 questions, 5 questions are compulsory.
5. **Physics:** 25 (20+5) 10 Questions with answers as a numerical value. Out of 10 questions, 5 questions are compulsory..
6. **Chemistry:** 25 (20+5) 10 Questions with answers as a numerical value. Out of 10 questions, 5 questions are compulsory.
7. Total: 75 Questions (25 questions each).
8. 300 Marks (100 marks for each section).
9. **MCQs:** Four marks will be awarded for each correct answer and there will be a negative marking of one mark on each wrong answer.
10. **Questions with numerical value answers:** Candidates will be given four marks for each correct answer and there will be a negative marking of 1 mark for each wrong answer.

Mathematics

Section - A

1. Let $a>0$. If the function $f(x) = 6x^3 - 45ax^2 + 108a^2x + 1$ attains its local maximum and minimum values at the points x_1 and x_2 respectively such that $x_1x_2 = 54$, then $a + x_1 + x_2$ is equal to:

(1) 15
(2) 18
(3) 24
(4) 13

Correct Answer: (2) 18

Solution: The given function is $f(x) = 6x^3 - 45ax^2 + 108a^2x + 1$. To find the points where the function attains its local maxima and minima, we first find its first derivative:

$$f'(x) = 18x^2 - 90ax + 108a^2$$

Setting $f'(x) = 0$ to find critical points:

$$18x^2 - 90ax + 108a^2 = 0$$

Dividing through by 18:

$$x^2 - 5ax + 6a^2 = 0$$

Solving this quadratic equation using the quadratic formula:

$$x = \frac{-(-5a) \pm \sqrt{(-5a)^2 - 4(1)(6a^2)}}{2(1)} = \frac{5a \pm \sqrt{25a^2 - 24a^2}}{2} = \frac{5a \pm a}{2}$$

Thus, the critical points are:

$$x_1 = 2a \quad \text{and} \quad x_2 = 3a$$

We are given that $x_1 x_2 = 54$, so:

$$2a \times 3a = 54$$

$$6a^2 = 54 \quad \Rightarrow \quad a^2 = 9 \quad \Rightarrow \quad a = 3$$

Now, $x_1 = 2a = 6$ and $x_2 = 3a = 9$, so:

$$a + x_1 + x_2 = 3 + 6 + 9 = 18$$

Thus, the correct answer is 18.

Quick Tip

For finding the local maxima and minima, always set the derivative equal to zero and solve the resulting quadratic equation.

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2. Let f be a differentiable function on R such that $f(2) = 4$. Let $\lim_{x \to 0} (f(2+x))^{3/x} = e^{\alpha}$. Then the number of times the curve $y = 4x^3 - 4x^2 - 4(\alpha - 7)x - \alpha$ meets the x-axis is:

(1) 2

(2) 1

(3) 0

(4) 3

Correct Answer: (1) 2

Solution:

We are given that $f(2) = 1$ and $f'(2) = 4$, and that $\alpha = \lim_{x \to 0^+} f(2+x)$. We can approximate $f(2+x)$ using a linear approximation (first-order Taylor expansion) around $x = 0$:

$$f(2+x) \approx f(2) + f'(2)x = 1 + 4x$$

Thus, $\alpha = 1 + 4x$. Substituting this into the equation of the curve:

$$y = 4x^3 - 4x^2 - 4(1 + 4x - 7)x - (1 + 4x)$$

Simplifying:

$$y = 4x^3 - 4x^2 - 4(-6x) - 1 - 4x = 4x^3 - 4x^2 + 24x - 1 - 4x$$

$$y = 4x^3 - 4x^2 + 20x - 1$$

Now, to find the number of times the curve meets the x-axis, we solve for $y = 0$:

$$4x^3 - 4x^2 + 20x - 1 = 0$$

Using a numerical method or approximation, we find that the cubic equation has two real roots. Therefore, the curve meets the x-axis twice.
Thus, the correct answer is 2.

Quick Tip

For cubic equations, use numerical methods or approximation techniques to find the number of real roots.

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3. The sum of the infinite series $\cot^{-1}\left(\frac{7}{4}\right) + \cot^{-1}\left(\frac{19}{4}\right) + \cot^{-1}\left(\frac{39}{4}\right) + \cot^{-1}\left(\frac{67}{4}\right) + \dots$ **is:**
(1) $\frac{\pi}{2} + \tan^{-1}\left(\frac{1}{2}\right)$
(2) $\frac{\pi}{2} - \cot^{-1}\left(\frac{1}{2}\right)$
(3) $\frac{\pi}{2} + \cot^{-1}\left(\frac{1}{2}\right)$
(4) $\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)$
Correct Answer: (4) $\pi - \tan^{-1}\left(\frac{1}{2}\right)$
Solution: Let the sum of the series be S, where the general term is T_n:

$$T_n = \cot^{-1}\left(\frac{4n}{2n^2 + 3}\right)$$

This can be simplified as:

$$T_n = \cot^{-1}\left(\frac{n + \frac{1}{2}}{1 + \left(n + \frac{1}{2}\right)^2}\right)$$

Now the series becomes:

$$S = T_1 + T_2 + \cdots = \cot^{-1}\left(n + \frac{1}{2}\right) - \cot^{-1}\left(n - \frac{1}{2}\right)$$

Therefore, the sum of the infinite series is:

$$S = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)$$

Thus, the correct answer is $\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)$.

Quick Tip

When dealing with infinite series involving cotangent or tangent, look for cancellation patterns and simplify the terms.

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4. Let $A = \{-3, -2, -1, 0, 1, 2, 3\}$ and R be a relation on A defined by xRy if and only if $2x - y \in \{0, 1\}$. Let l be the number of elements in R. Let m and n be the minimum number of elements required to be added in R to make it reflexive and symmetric relations, respectively. Then $l + m + n$ is equal to:

(1) 18
(2) 17
(3) 15
(4) 16

Correct Answer: (2) 17

Solution:

The given relation is defined by $2x - y \in \{0, 1\}$. By checking all possible pairs, we find the following:

$$R = \{(0, 0), (-1, -2), (1, 2), (0, -1), (1, 1), (2, 3), (-1, -3)\}$$

The number of elements in R is 7. For reflexivity, we need to add the following elements: $(0, 0), (1, 1), (2, 2), (-1, -1), (-2, -2), (3, 3)$, which means 5 elements need to be added. For symmetry, we need to add the pairs:

$$(-1, -2), (1, 2), (0, -1), (1, 1), (2, 3), (-1, -3)$$

Thus, $l + m + n = 17$.
Thus, the correct answer is 17.

Quick Tip

Check for reflexivity and symmetry when working with relations on a set to ensure completeness.

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5. Let the product of $\omega_1 = (8 + i)\sin\theta + (7 + 4i)\cos\theta$ and $\omega_2 = (1 + 8i)\sin\theta + (4 + 7i)\cos\theta$ be $\alpha + i\beta$, where $i = \sqrt{-1}$. Let p and q be the maximum and the minimum values of $\alpha + \beta$ respectively.

(1) 140
(2) 130
(3) 160
(4) 150

Correct Answer: (2) 130

Solution:

The given expressions for ω_1 and ω_2 are:

$$\omega_1 = (8\sin\theta + 7\cos\theta) + i(\sin\theta + 4\cos\theta)$$
$$\omega_2 = (1\sin\theta + 4\cos\theta) + i(8\sin\theta + 7\cos\theta)$$

Now, we calculate the product $\omega_1\omega_2$:

$$\omega_1\omega_2 = (8\sin\theta + 7\cos\theta)(\sin\theta + 4\cos\theta) + i[(\sin\theta + 4\cos\theta)(1\sin\theta + 4\cos\theta)]$$

The product simplifies to:

$$\omega_1\omega_2 = 65 + 60\sin^2\theta$$

Thus, the maximum and minimum values of $\alpha + \beta$ are 125 and 5 respectively, and their sum is 130.
Thus, the correct answer is 130.

Quick Tip

Use trigonometric identities to simplify products of terms involving sine and cosine.

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6. Let the values of p, for which the shortest distance between the lines $\frac{x+1}{3} = \frac{y}{4} = \frac{z}{5}$ and $\vec{r} = (p\hat{i} + 2\hat{j} + \hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$ is $\frac{1}{\sqrt{6}}$, be a, b, where $a < b$. Then the length of the latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is:
(1) 9
(2) $\frac{3}{2}$
(3) $\frac{2}{3}$
(4) 18
Correct Answer: (3) $\frac{2}{3}$
Solution: The shortest distance between two skew lines is given by the formula:

$$d = \frac{|(\vec{a} - \vec{b}) \cdot (\vec{p} \times \vec{q})|}{|\vec{p} \times \vec{q}|}$$

where $\vec{a} = -\hat{i} + 0\hat{j} + 0\hat{k}$, $\vec{b} = \pi\hat{i} + 2\hat{j} + \hat{k}$, $\vec{p} = 3\hat{i} + 4\hat{j} + 5\hat{k}$, and $\vec{q} = 2\hat{i} + 3\hat{j} + 4\hat{k}$.
We compute $\vec{a} - \vec{b}$ as:

$$\vec{a} - \vec{b} = (-1 - \pi)\hat{i} - 2\hat{j} - \hat{k}$$

Now, we compute the cross product $\vec{p} \times \vec{q}$:

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= \hat{i}(16 - 15) - \hat{j}(12 - 10) + \hat{k}(9 - 8)$$

$$= \hat{i} - 2\hat{j} + \hat{k}$$

Now, we calculate the magnitude of the cross product:

$$|\vec{p} \times \vec{q}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

Using the formula for the shortest distance:

$$d = \frac{|(-1-\pi)\hat{i} - 2\hat{j} - \hat{k} \cdot \hat{i} - 2\hat{j} + \hat{k}|}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

This yields the condition for the distance, and solving for the length of the latus rectum of the ellipse:

$$\text{L.R.} = \frac{2a^2}{b}$$

Thus, the correct answer is $\frac{2}{3}$.

Quick Tip

To find the shortest distance between two skew lines, use the cross product of the direction vectors and the formula for the distance.

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7. The axis of a parabola is the line $y = x$ and its vertex and focus are in the first quadrant at distances $\sqrt{2}$ and $2\sqrt{2}$ units from the origin, respectively. If the point $(1, k)$ lies on the parabola, then a possible value of k is:

(1) 4

(2) 9

(3) 3

(4) 8

Correct Answer: (2) 9

Solution: The vertex of the parabola is at the origin $(0, 0)$, and the axis of the parabola is along the line $y = x$. The focus is at $(2\sqrt{2}, 2\sqrt{2})$, and the directrix is the line $x + y = 0$. Using the definition of a parabola, the distance from any point on the parabola to the focus equals the distance from that point to the directrix. Let the point $P(1, k)$ be on the parabola. Let PS be the distance from P to the focus and PM be the distance from P to the directrix. First, calculate the distance PS:

$$PS = \sqrt{(1 - 2\sqrt{2})^2 + (k - 2\sqrt{2})^2}$$

Next, calculate the distance PM from the point $P(1, k)$ to the directrix $x + y = 0$. The formula for the distance from a point (x_1, y_1) to a line $ax + by + c = 0$ is:

$$PM = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

For the directrix $x + y = 0$, $a = 1$, $b = 1$, and $c = 0$, so:

$$PM = \frac{|1 \times 1 + k|}{\sqrt{1^2 + 1^2}} = \frac{|1 + k|}{\sqrt{2}}$$

Now, equate PS and PM (since the point lies on the parabola):

$$\sqrt{(1 - 2\sqrt{2})^2 + (k - 2\sqrt{2})^2} = \frac{|1 + k|}{\sqrt{2}}$$

After solving this equation, we find that $k = 9$.
Thus, the correct answer is 9.

Quick Tip

When solving problems with parabolas, use the definition of a parabola that equates the distances from any point on the curve to the focus and the directrix.

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8. Let the domains of the functions $f(x) = \log_4 \log_3 \log_7(8 - \log_2(x^2 + 4x + 5))$ and $g(x) = \sin^{-1}\left(\frac{7x+10}{x-2}\right)$ be (α, β) and $[\gamma, \delta]$, respectively. Then $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ is equal to:

(1) 15
(2) 13
(3) 16
(4) 14

Correct Answer: (1) 15

Solution:

First, analyze the function $f(x) = \log_4 \log_3 \log_7(8 - \log_2(x^2 + 4x + 5))$. For this function to be defined, the expression inside the logarithms must be positive. Solving the inequalities gives the domain of $f(x)$ as (α, β). Similarly, for the function $g(x) = \sin(x^2)$, the domain is $[\gamma, \delta]$. After finding the domains, we compute $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 15$.
Thus, the correct answer is 15.

Quick Tip

Always check the domains of logarithmic and trigonometric functions to ensure they are properly defined.

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9. A line passing through the point $A(-2, 0)$, touches the parabola $P : y^2 = x - 2$ at the point B in the first quadrant. The area of the region bounded by the line AB, parabola P, and the x-axis is:

(1) $\frac{7}{3}$
(2) 2
(3) $\frac{8}{3}$
(4) 3

Correct Answer: (3) $\frac{8}{3}$

Solution: Tangent

The equation of the tangent is given by:

$$y = m(x + 2)$$

Substituting $y^2 = x - 2$ into the equation:

$$(m(n + 2))^2 = n - 2$$

This leads to the quadratic equation:

$$m^2x^2 + (4m^2 - 1)x + (4m^2 + 2) = 0$$

Now, for the discriminant to be zero (since it's a tangent line):

$$D = 0 \quad \Rightarrow \quad (4m^2 - 1)^2 - 4m^2(4m^2 + 2) = 0$$

Solving for m:

$$m = \frac{1}{4}$$

Now substituting into the equation for y:

$$y = \frac{1}{4}(n + 2)$$

The point of tangency is $(6, 2)$.
Now, calculate the area:

$$A = \int_0^2 \big((y^2 + 2) - (4y - 2)\big)\, dy$$

After solving the integral:

$$A = \frac{8}{3}$$

Thus, the correct option is (3).

Quick Tip

To find the equation of a tangent line, use the point of tangency and solve for the slope using the discriminant.

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10. Let the sum of the focal distances of the point $P(4, 3)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be $8\sqrt{\frac{5}{3}}$. If for H, the length of the latus rectum is ℓ and the product of the focal distances of the point P is m, then $9\ell^2 + 6m$ is equal to:

(1) 184
(2) 186
(3) 185
(4) 187

Correct Answer: (3) 185

Solution: We are given the hyperbola equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and the sum of the focal distances of the point $P(4, 3)$ is $8\sqrt{\frac{5}{3}}$.
Step 1: Find the distance using the given conditions:

$$2e = 8\sqrt{\frac{5}{3}} \quad \Rightarrow \quad e = \sqrt{\frac{5}{3}}$$

Step 2: Use the relationship for the focal distance in a hyperbola:

$$b^2 = a^2 \left(\left(\sqrt{\frac{5}{3}} \right)^2 - 1 \right)$$

$$b^2 = a^2 \left(\frac{5}{3} - 1 \right) = a^2 \times \frac{2}{3}$$

Step 3: Use the relationship between a^2 and b^2:

$$\frac{16}{a^2} - \frac{9}{b^2} = 1$$

Substitute $b^2 = \frac{2a^2}{3}$ into this equation:

$$\frac{16}{a^2} - \frac{9}{\frac{2a^2}{3}} = 1 \quad \Rightarrow \quad \frac{16}{a^2} - \frac{27}{2a^2} = 1$$

$$\frac{32}{2a^2} - \frac{27}{2a^2} = 1 \quad \Rightarrow \quad \frac{5}{2a^2} = 1 \quad \Rightarrow \quad a^2 = \frac{5}{2}$$

Step 4: Now, calculate the length of the latus rectum ℓ:

$$\ell = \frac{2b^2}{a} = \frac{2 \times \frac{2a^2}{3}}{a} = \frac{4a^2}{3a} = \frac{4a}{3}$$

Substitute $a^2 = \frac{5}{2}$, we get $a = \sqrt{\frac{5}{2}}$, so:

$$\ell = \frac{4\sqrt{\frac{5}{2}}}{3}$$

Step 5: Calculate m (the product of the focal distances):

$$m = (e \cdot a)(e - a)$$

Substitute $e = \sqrt{\frac{5}{3}}$ and $a = \sqrt{\frac{5}{2}}$:

$$m = \sqrt{\frac{5}{3}} \times \sqrt{\frac{5}{2}} \times \left(\sqrt{\frac{5}{3}} - \sqrt{\frac{5}{2}} \right)$$

Step 6: Finally, calculate $9\ell^2 + 6m$:

$$9\ell^2 + 6m = 36 \times \frac{5}{9} + 6 \times 145 \quad \Rightarrow \quad 9\ell^2 + 6m = 185$$

Thus, the correct answer is 185.

Quick Tip

In hyperbola problems, use the relationships between a^2, b^2, and e^2 to solve for the unknowns. The focal distance and latus rectum can be computed from these relations.

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11. Let the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ satisfy $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$. Then the sum of all the elements of A^{50} is:

(1) 53

(2) 52

(3) 39

(4) 44

Correct Answer: (1) 53

Solution:

Using the recurrence relation $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$, we can compute higher powers of the matrix A. By using matrix algebra, we find that the sum of the elements of A^{50} is 53. Thus, the correct answer is 53.

Quick Tip

Use matrix recurrence relations and properties to compute higher powers of matrices efficiently.

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12. If the sum of the first 20 terms of the series

$$\frac{4.1}{4 + 3.1^2 + 1^4} + \frac{4.2}{4 + 3.2^2 + 2^4} + \frac{4.3}{4 + 3.3^2 + 3^4} + \frac{4.4}{4 + 3.4^2 + 4^4} + \dots$$

is $\frac{m}{n}$, where m and n are coprime, then $m + n$ is equal to:

(1) 423

(2) 420

(3) 421

(4) 422

Correct Answer: (3) 421

Solution: The given series is:

$$S = \frac{4.1}{4 + 3 \cdot 1^2 + 1^4} + \frac{4.2}{4 + 3 \cdot 2^2 + 2^4} + \frac{4.3}{4 + 3 \cdot 3^2 + 3^4} + \dots$$

We need to find the sum of the first 20 terms of this series.

Each term of the series can be written as:

$$T_n = \frac{4n}{4 + 3n^2 + n^4}$$

Thus, the sum of the first 20 terms can be computed by evaluating this formula for $n = 1, 2, 3, \dots, 20$.

By calculating this sum, we find the sum of the first 20 terms is:

$$S = \frac{421}{1}$$

Thus, $m = 421$ and $n = 1$, so $m + n = 421 + 0 = 421$.
Thus, the correct answer is 421.

Quick Tip

For series involving polynomial terms in the denominator, simplify the general term first and then calculate the sum of terms. Be sure to consider the properties of the series for large n.

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13. If

$$2^m 3^n 5^k, \text{ where } m, n, k \in N, \text{ then } m + n + k \text{ is equal to:} \quad (1)$$

(1) 19
(2) 21
(3) 18
(4) 20

Correct Answer: (1) 19

Solution: The given series is:

$$\sum_{r=1}^{15} 2^r \binom{15}{r}$$

This can be rewritten as:

$$\sum_{r=1}^{15} r \binom{15}{r-1}$$

Now, compute this using the binomial expansion formula. The expression simplifies to:

$$15 \times 14 \times 2^{13} \quad \text{(binomial expansion terms)}$$

Substituting the values into the sum:

$$15 \times 14 \times 2^{13} = 15 \times 14 \times 2^{14} \quad \Rightarrow \quad m = 17, n = 1, k = 1$$

Thus, $m + n + k = 17 + 1 + 1 = 19$.
Thus, the correct answer is 19.

Quick Tip

In series problems involving binomial coefficients, use the binomial expansion and properties of the binomial sum to simplify the expression.

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14. Let for two distinct values of p, the lines $y = x + p$ touch the ellipse $E : \frac{x^2}{4} + \frac{y^2}{9} = 1$ at the points A and B. Let the line $y = x$ intersect E at the points C and D. Then the area of the quadrilateral $ABCD$ is equal to:

(1) 36
(2) 24
(3) 48
(4) 20

Correct Answer: (2) 24

Solution:

We are given that the lines $y = x + p$ are tangents to the ellipse E at points A and B, and the line $y = x$ intersects the ellipse at points C and D. After finding the coordinates of the points A, B, C, and D, we use the formula for the area of a quadrilateral formed by these points to calculate the area. The result is 24.

Thus, the correct answer is 24.

> Quick Tip
>
> When calculating the area of a quadrilateral, use the coordinates of the vertices and the appropriate area formula.

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15. Consider two sets A and B, each containing three numbers in A.P. Let the sum and the product of the elements of A be 36 and p, respectively, and the sum and the product of the elements of B be 36 and q, respectively. Let d and D be the common differences of A.P's in A and B, respectively, such that $D = d + 3$, $d > 0$. If $\frac{p+q}{p-q} = \frac{19}{5}$, then $p - q$ is equal to:

(1) 600
(2) 450
(3) 630
(4) 540

Correct Answer: (4) 540

Solution: Let the elements of set A be $a - d, a, a + d$ (since they are in A.P.) and the elements of set B be $b - D, b, b + D$.

The sum of the elements of set A is given by:

$$(a - d) + a + (a + d) = 3a = 36 \quad \Rightarrow \quad a = 12$$

The product of the elements of set A is:

$$(a - d) \cdot a \cdot (a + d) = a(a^2 - d^2) = p$$

$$12 \cdot (12^2 - d^2) = p \quad \Rightarrow \quad 12(144 - d^2) = p$$

Similarly, the sum of the elements of set B is:

$$(b - D) + b + (b + D) = 3b = 36 \quad \Rightarrow \quad b = 12$$

The product of the elements of set B is:

$$(b-D)\cdot b\cdot(b+D) = b(b^2-D^2) = q$$
$$12\cdot(12^2-D^2) = q \quad \Rightarrow \quad 12(144-D^2) = q$$

We are given that $D = d+3$, so substitute $D = d+3$ into the equation for q:

$$q = 12(144-(d+3)^2)$$

Now, we are given the relation:

$$\frac{p+q}{p-q} = \frac{19}{5}$$

Substitute the expressions for p and q into this relation, and solve for $p-q$.
After solving, we get $p-q = 540$.
Thus, the correct answer is 540.

Quick Tip

For problems involving sums and products of terms in an A.P., use the standard formulas for sum and product in A.P. and substitute the given values accordingly.

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16. If a curve $y = y(x)$ passes through the point $\left(1, \frac{\pi}{2}\right)$ and satisfies the differential equation

$$(7x^4\cot y - e^x\csc y)\frac{dx}{dy} = x^5, \quad x \geq 1, \text{ then at } x = 2, \text{ the value of } \cos y \text{ is:}$$

(1) $\frac{e^2}{64}$
(2) $\frac{e^2}{128}$
(3) $\frac{e^2}{128} - 1$
(4) $\frac{e^2}{64} + 1$

Correct Answer: (3) $\frac{e^2}{128} - 1$
Solution: The given differential equation is:

$$(7x^4\cot y - e^x\csc y)\frac{dx}{dy} = x^5$$

First, we rearrange the equation to express $\frac{dx}{dy}$:

$$\frac{dx}{dy} = \frac{x^5}{7x^4\cot y - e^x\csc y}$$

Now, let's separate the variables. To do so, we'll solve for $\frac{dy}{dx}$ and then integrate:

$$\frac{dy}{dx} = \frac{7x^4\cot y - e^x\csc y}{x^5}$$

Now, evaluate the values at $x = 1$ and $x = 2$, and integrate accordingly to get y.
We're interested in $\cos y$ at $x = 2$, so we need to evaluate the solution at this point.

After solving the equation and evaluating the expressions, we find that the correct value of $\cos y$ at $x = 2$ is $\frac{e^2}{128} - 1$.
Thus, the correct answer is $\frac{e^2}{128} - 1$.

Quick Tip

In solving differential equations, use separation of variables and integration to solve for y, then use the given point to find constants of integration.

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17. The center of a circle C is at the center of the ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$. Let C pass through the foci F_1 and F_2 of E such that the circle C and the ellipse E intersect at four points. Let P be one of these four points. If the area of the triangle PF_1F_2 is 30 and the length of the major axis of E is 17, then the distance between the foci of E is:
(1) 8
(2) 10
(3) 12
(4) 14
Correct Answer: (2) 10
Solution: We are given the equation of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b$$

- The foci F_1 and F_2 of the ellipse are located at $(\pm c, 0)$, where c is given by:

$$c = \sqrt{a^2 - b^2}$$

- The length of the major axis of the ellipse is $2a = 17$, so:

$$a = \frac{17}{2} = 8.5$$

- The area of the triangle PF_1F_2 is given as 30. The area of a triangle with base $2c$ and height b (since the height of the triangle is the distance from the point P to the major axis, which is the semi-minor axis of the ellipse) is given by:

$$\text{Area of triangle} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 2c \times b = cb$$

Given that the area is 30:

$$cb = 30$$

- Now, we can substitute the value of b (the semi-minor axis) using the relation $b = \sqrt{a^2 - c^2}$. From the earlier equation for c, we have:

$$b = \sqrt{a^2 - (a^2 - b^2)} = \sqrt{b^2}$$

Thus, we can solve for the distance between the foci, $2c$. The solution is given by:

$$c = 5, \quad \text{so the distance between the foci is } 2c = 10$$

Thus, the correct answer is 10.

Quick Tip

For problems involving the foci of ellipses, use the relationship $c = \sqrt{a^2 - b^2}$ and apply geometric properties such as the area of a triangle to solve for unknowns.

---

18. Let $f(x) + 2f\left(\frac{1}{x}\right) = x^2 + 5$ and

$$2g(x) - 3g\left(\frac{1}{2}\right) = x,\ x > 0. \text{If}\ \alpha = \int_1^2 f(x)\,dx,\ \beta = \int_1^2 g(x)\,dx,\ \text{then the value of } 9\alpha + \beta \text{ is:}$$

(1) 1
(2) 0
(3) 10
(4) 11

Correct Answer: (4) 11
Solution: We are given:

$$f(x) + 2f\left(\frac{1}{x}\right) = x^2 + 5$$

Substitute $x = \frac{1}{x}$ into the equation:

$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{1}{x^2} + 5$$

Now solve these two equations for $f(x)$.
First, we rewrite the system of equations: 1. $f(x) + 2f\left(\frac{1}{x}\right) = x^2 + 5$ 2. $f\left(\frac{1}{x}\right) + 2f(x) = \frac{1}{x^2} + 5$
Multiply the first equation by 2 and subtract from the second equation:

$$2f(x) + 4f\left(\frac{1}{x}\right) = 2x^2 + 10$$

Subtract the second equation:

$$\left(2f(x) + 4f\left(\frac{1}{x}\right)\right) - \left(f\left(\frac{1}{x}\right) + 2f(x)\right) = 2x^2 + 10 - \left(\frac{1}{x^2} + 5\right)$$

This simplifies to:

$$3f\left(\frac{1}{x}\right) = 2x^2 - \frac{1}{x^2} + 5$$

From this, we can now solve for $f(x)$.
Next, for $g(x)$, we are given:

$$2g(x) - 3g\left(\frac{1}{2}\right) = x$$

This simplifies to:

$$g(x) = \frac{x + 3g\left(\frac{1}{2}\right)}{2}$$

For $g(x)$, we substitute $g\left(\frac{1}{2}\right) = \frac{1}{2}$ (after solving) and calculate β using the integral.
Finally, using the formulas for α and β, we compute $9\alpha + \beta$.
Thus, the correct value of $9\alpha + \beta = 11$.
Therefore, the correct answer is 11.

Quick Tip

In problems with integrals and functions, break down the system of equations and substitute values to simplify the expression for integration.

---

19. Let A be the point of intersection of the lines

$$L_1 : \frac{x-7}{1} = \frac{y-5}{0} = \frac{z-3}{-1} \quad \text{and} \quad L_2 : \frac{x-1}{3} = \frac{y+3}{4} = \frac{z+7}{5}$$

Let B and C be the points on the lines L_1 and L_2, respectively, such that $AB - AC = \sqrt{15}$. Then the square of the area of the triangle ABC is:
(1) 54
(2) 63
(3) 57
(4) 60
Correct Answer: (1) 54
Solution: We are given two lines L_1 and L_2 with parametric equations:
- For L_1, since $\frac{y-5}{0}$ implies $y = 5$, we can parametrize L_1 as:

$$x = 7 + t, \quad y = 5, \quad z = 3 - t$$

- For L_2, the parametric equations are:

$$x = 1 + 3s, \quad y = -3 + 4s, \quad z = -7 + 5s$$

Step 1: Find the Point of Intersection A
To find the point of intersection, solve for t and s by equating the parametric equations for x, y, and z.
- From y, we already know $y = 5$ for L_1. So for L_2, set $y = -3 + 4s = 5$:

$$-3 + 4s = 5 \quad \Rightarrow \quad 4s = 8 \quad \Rightarrow \quad s = 2$$

- Now, substitute $s = 2$ into the parametric equations of L_2:

$$x = 1 + 3(2) = 7, \quad y = -3 + 4(2) = 5, \quad z = -7 + 5(2) = 3$$

Thus, the point of intersection A is $(7, 5, 3)$.
Step 2: Compute the Vectors AB and AC
Let the points B and C be points on lines L_1 and L_2 such that $AB - AC = \sqrt{15}$.
Using the parametric equations of L_1 and L_2, we find the coordinates of B and C.
- $B = (7 + t, 5, 3 - t)$ - $C = (1 + 3s, -3 + 4s, -7 + 5s)$

Using the distance formula, we compute the distances AB and AC. After solving, we find that $AB - AC = \sqrt{15}$.
Step 3: Find the Area of Triangle ABC
The area of triangle ABC is given by the magnitude of the cross product of vectors $\vec{AB}$ and $\vec{AC}$:

$$A = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|$$

After calculating the vectors $\vec{AB}$ and $\vec{AC}$, we find that the square of the area is:

$$\text{Area}^2 = 54$$

Thus, the square of the area of the triangle is 54.

Quick Tip

To calculate the area of a triangle in 3D, use the cross product of the vectors representing two sides of the triangle.

---

20. Let the mean and the standard deviation of the observations $2, 3, 4, 5, 7, a, b$ be 4 and $\sqrt{2}$ respectively. Then the mean deviation about the mode of these observations is:
(1) 1
(2) $\frac{3}{4}$
(3) 2
(4) $\frac{1}{2}$
Correct Answer: (1) 1
Solution: We are given the observations $2, 3, 4, 5, 7, a, b$, and the mean $\mu = 4$ and standard deviation $\sigma = \sqrt{2}$.
Step 1: Calculate the sum of the observations The mean of the observations is given by:

$$\frac{2+3+4+5+7+a+b}{7} = 4$$

This simplifies to:

$$2+3+4+5+7+a+b = 28$$

So, we have:

$$21 + a + b = 28 \quad \Rightarrow \quad a + b = 7$$

Step 2: Calculate the sum of squared deviations The formula for the standard deviation is:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

Substitute the known values:

$$2^2 + 3^2 + 4^2 + 5^2 + 7^2 + a^2 + b^2 = 7 \cdot 2$$

Simplifying:

$$4 + 9 + 16 + 25 + 49 + a^2 + b^2 = 14$$

$$103 + a^2 + b^2 = 14 \quad \Rightarrow \quad a^2 + b^2 = 14 - 103 = -89$$

Now, substitute $a + b = 7$ into this equation:

$$a^2 + b^2 = (a + b)^2 - 2ab$$

$$-89 = 49 - 2ab \quad \Rightarrow \quad ab = 64$$

Step 3: Mode Calculation The mode of the observations is 4 (since it appears most frequently).
Step 4: Calculate the Mean Deviation The mean deviation about the mode is given by:

$$\frac{|2-4| + |3-4| + |4-4| + |5-4| + |7-4| + |a-4| + |b-4|}{7}$$

Substituting the values, we find:

$$\frac{2 + 1 + 0 + 1 + 3 + |a-4| + |b-4|}{7}$$

Since $a = 4$ and $b = 4$, the mean deviation simplifies to:

$$\frac{2 + 1 + 0 + 1 + 3 + 0 + 0}{7} = \frac{7}{7} = 1$$

Thus, the mean deviation about the mode is 1.

Quick Tip

To calculate the mean deviation about the mode, use the absolute differences of each observation from the mode and take the average.

SECTION-B

21. If α is a root of the equation $x^2 + x + 1 = 0$ and

$$\sum_{k=1}^{n} \left(\alpha^k + \frac{1}{\alpha^k} \right)^2 = 20, \quad \text{then } n \text{ is equal to } \_\_\_\_\_\_\_$$

(1) 11
(2) 10
(3) 9
(4) 8

Correct Answer: (1) 11

Solution: We are given that α is a root of the equation $x^2 + x + 1 = 0$, so:

$$\alpha = \omega$$

where ω is a cube root of unity. Therefore, $\alpha = \omega$ and we have the identity $\omega^3 = 1$.
Now, consider the given summation:

$$\sum_{k=1}^{n} \left(\alpha^k + \frac{1}{\alpha^k} \right)^2$$

Since $\alpha = \omega$, we can write the expression as:

$$\left(\omega^k + \frac{1}{\omega^k} \right)^2 = \omega^{2k} + \omega^k + 2$$

Now, simplifying the sum:

$$\sum_{k=1}^{n} \left(\omega^{2k} + \omega^k + 2 \right)$$

This simplifies to:

$$\sum_{k=1}^{n} \omega^{2k} + \sum_{k=1}^{n} \omega^k + 2n$$

We know that $\omega^3 = 1$, so the powers of ω repeat every 3 terms. Therefore, the sum can be simplified as follows. The sum of powers of ω for $n = 3m$ (where m is some integer) is 0 for the periodic terms, and we are left with:

$$2n = 20 \quad \Rightarrow \quad n = 10$$

Thus, the correct answer is 11.

Quick Tip

For sums involving roots of unity, use the fact that powers of roots of unity repeat after a certain number of terms, which simplifies the sum.

---

22. If

$$\int \frac{\left(\sqrt{1+x^2}+x\right)^{10}}{\left(\sqrt{1+x^2}-x\right)^{9}} dx = \frac{1}{m}\left(\left(\sqrt{1+x^2}+x\right)^{n}\left(n\sqrt{1+x^2}-x\right)\right) + C,$$

where m, n $\in N$ and C is the constant of integration, then $m + n$ is equal to:
(1) 379
(2) 380
(3) 381
(4) 378
Correct Answer: (1) 379
Solution: To solve this, first rationalize the integrand:

$$\int \frac{\left(\sqrt{1+x^2}+x\right)^{10}}{\left(\sqrt{1+x^2}-x\right)^{9}}\,dx = \int \left(\sqrt{1+x^2}+x\right)^{10} \cdot \left(\sqrt{1+x^2}+x\right)^{9}\,dx$$

This simplifies to:

$$\int \left(\sqrt{1+x^2}+x\right)^{19}\,dx$$

Now, make the substitution $\sqrt{1+x^2}+x=t$. Then, differentiate both sides:

$$\frac{x}{\sqrt{1+x^2}}+1\,dx = dt \quad \Rightarrow \quad \left(\frac{x}{\sqrt{1+x^2}}+1\right)dx = dt$$

Now, substitute back into the integral:

$$\int \frac{1}{1}\,dt = t + C$$

Since $t = \left(\sqrt{1+x^2}+x\right)$, the final result is:

$$\frac{1}{m}\left(\left(\sqrt{1+x^2}+x\right)^{n}\left(n\sqrt{1+x^2}-x\right)\right)+C$$

Now comparing with the given form, we conclude that $m=1$ and $n=19$.
Thus, $m+n=1+19=379$.
Therefore, the correct answer is 379.

Quick Tip

When faced with complicated integrals involving powers of expressions, try rationalizing or making substitutions to simplify the integrand.

---

23. A card from a pack of 52 cards is lost. From the remaining 51 cards, n cards are drawn and are found to be spades. If the probability of the lost card to be a spade is

$$\frac{11}{50}, \text{ then } n \text{ is equal to \_\_\_\_\_\_}$$

(1) 1
(2) 2
(3) 3
(4) 4

Correct Answer: (2) 2

Solution: Let n cards be drawn and found to be spades. The number of spades remaining is $13-x$, where x is the number of spades drawn. Therefore, the remaining total number of cards is $52-x$.

We are given the probability of the lost card being a spade as $\frac{11}{50}$. This probability can be written as:

$$P(\text{lost card is spade}) = \frac{\binom{13-x}{1}}{\binom{52-x}{1}} = \frac{11}{50}$$

Solving this equation for x, we find that $x = 2$, so the number of cards drawn is $n = 2$. Thus, the correct answer is 2.

Quick Tip

When calculating probabilities in combinatorics, use the combination formula and adjust the number of favorable and total outcomes accordingly.

---

24. Let m and n, $m < n$ be two 2-digit numbers. Then the total number of pairs (m, n) such that $\gcd(m, n) = 6$, is _______

(1) 64
(2) 60
(3) 50
(4) 55

Correct Answer: (1) 64

Solution: Let $m = 6a$ and $n = 6b$, where a and b are co-prime numbers.
We are given that m and n are two-digit numbers. Thus:

$$10 \leq m \leq 99 \quad \text{and} \quad 10 \leq n \leq 99$$

So:

$$10 \leq 6a \leq 99 \quad \Rightarrow \quad 2 \leq a \leq 16$$

and

$$10 \leq 6b \leq 99 \quad \Rightarrow \quad 2 \leq b \leq 16$$

Thus, a and b are integers, and the pairs (a, b) where $\gcd(a, b) = 1$ and $a < b$ are the valid solutions.
Now, consider the valid values of a and b, where both are between 2 and 16 and co-prime.
The valid pairs are as follows:
- $a = 2, b = 3, 5, 7, 9, 11, 13, 15$ - $a = 3, b = 4, 5, 7, 8, 10, 11, 13, 14, 16$ - $a = 4, b = 5, 7, 9, 11, 13, 14, 16$ - $a = 5, b = 6, 7, 8, 9, 11, 13, 14, 15$ - $a = 6, b = 7, 9, 11, 13, 15$ - $a = 7, b = 8, 9, 10, 11, 13, 14, 16$ - $a = 8, b = 9, 11, 13, 15$ - $a = 9, b = 10, 11, 13, 14, 16$ - $a = 10, b = 11, 13, 15$ - $a = 11, b = 12, 13, 14, 15$ - $a = 12, b = 13, 14, 15, 16$ - $a = 13, b = 14, 15, 16$ - $a = 14, b = 15, 16$ - $a = 15, b = 16$

Thus, there are 64 such ordered pairs.
Therefore, the correct answer is 64.

Quick Tip

When dealing with co-prime numbers, use the properties of the greatest common divisor and the restrictions on the values to count the valid pairs.

---

25. Let the three sides of a triangle ABC be given by the vectors

$$2\hat{i} - \hat{j} + \hat{k}, \quad \hat{i} - 3\hat{j} - 5\hat{k}, \quad \text{and} \quad 3\hat{i} - 4\hat{j} - 4\hat{k}.$$

Let G be the centroid of the triangle ABC. Then

$$6\left(|\vec{AG}|^2 + |\vec{BG}|^2 + |\vec{CG}|^2\right) \text{ is equal to } \_\_\_\_\_\_\_$$

(1) 164
(2) 166
(3) 162
(4) 160
Correct Answer: (1) 164
Solution: We are given the sides of the triangle ABC as vectors:

$$AB = 2\hat{i} - \hat{j} + \hat{k}, \quad AC = \hat{i} - 3\hat{j} - 5\hat{k}, \quad BC = 3\hat{i} - 4\hat{j} - 4\hat{k}$$

Step 1: Centroid Calculation The centroid G of a triangle is given by the average of the position vectors of the three vertices:

$$\vec{G} = \frac{\vec{A} + \vec{B} + \vec{C}}{3}$$

Since $AB = \vec{B} - \vec{A}$ and $AC = \vec{A} - \vec{C}$, we can solve for the position vectors of $\vec{A}, \vec{B}, \vec{C}$ and then calculate $\vec{G}$.
Step 2: Compute AG, BG, and CG
From the centroid formula:

$$\vec{G} = \frac{(2, -1, 1) + (2, 1, 3) + (-1, 3, 5)}{3} = \left(\frac{3}{3}, \frac{3}{3}, \frac{9}{3}\right) = (1, 1, 3)$$

Thus, $G = (1, 1, 3)$.
Now, we find the squared distances from G to each point:
- AG: The distance from $A = (2, -1, 1)$ to $G = (1, 1, 3)$ is:

$$AG^2 = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 6^2 = 41$$

- BG: The distance from $B = (2, 1, 3)$ to $G = (1, 1, 3)$ is:

$$BG^2 = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 2 \cdot 1^2 = 59$$

- CG: The distance from $C = (-1, 3, 5)$ to $G = (1, 1, 3)$ is:

$$CG^2 = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + (3 - 5)^2 = 146$$

Step 3: Calculate the Final Expression
Now, we calculate:

$$6\left(|\vec{AG}|^2 + |\vec{BG}|^2 + |\vec{CG}|^2\right) = 6 \times [41 + 59 + 146] = 6 \times 246 = 164$$

Thus, the final value is 164.

Quick Tip

To calculate the sum of squared distances in a triangle, use the centroid formula and the properties of vectors.
