JEE Main 2023 Jan 30 Shift 2 Maths Question Paper with Solutions

Time Allowed :3 Hours	**Maximum Marks :**300	**Total Questions :**90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Maths

Section-A

61. Consider the following statements:

P : I have fever

Q : I will not take medicine

R : I will take rest

The statement "If I have fever, then I will take medicine and I will take rest" is equivalent to:

(1) $(P) \vee (Q) \wedge ((P) \vee \ R)$

(2) $(P) \vee (Q) \wedge ((P) \vee R)$

(3) $(P \vee Q) \wedge (P) \vee R$

(4) $(P \vee \ Q) \wedge (P \vee \ R)$

Correct Answer: (1)

Solution: The statement "If I have fever, then I will take medicine and I will take rest" can be written in logical terms as:

$$P \to (Q \wedge R)$$

This can be rewritten as:

$$P \vee (Q \wedge R)$$

By applying De Morgan's laws, we get:

$$P \vee \ Q \wedge (P \vee \ R)$$

Thus, the correct option is (1).

Quick Tip

When translating conditional statements to logical expressions, remember that $P \to Q$ is equivalent to $\ P \vee Q$. Additionally, apply De Morgan's laws for simplifications.

---

62. Let A be a point on the x-axis. Common tangents are drawn from A to the curves $x^2 + y^2 = 8$ and $y^2 = 16x$. If one of these tangents touches the two curves at Q and R, then $(QR)^2$ is equal to:

(1) 64

(2) 76

(3) 81

(4) 72

Correct Answer: (4) 72

Solution:

The equation of the common tangent is given by:

$$y = mx + \frac{4}{m}$$

For the circle $x^2 + y^2 = 8$, the condition for the tangent to touch the circle is:

$$\frac{\left|\frac{4}{m}\right|}{\sqrt{1+m^2}} = \sqrt{2} \quad \Rightarrow \quad m = \pm 1$$

So, the equation of the tangent is:

$$y = \pm x + 4$$

This is the point of contact on the parabola $y^2 = 16x$.

For the circle $y^2 = 16x$, the point of contact at Q is $(-2, 2)$. Thus, $(QR)^2 = 36 + 36 = 72$.

Quick Tip

When solving problems involving tangents to curves, use the condition for tangency: the perpendicular distance from the point of tangency to the center of the curve must be equal to the radius for circles, and for other curves, use the relevant geometric properties.

---

63. Let q be the maximum integral value of p in $[0, 10]$ for which the roots of the equation $x^2 - px + \frac{5p}{4} = 0$ are rational. Then the area of the region $\{(x, y) : 0 \leq y \leq (x-q)^2, 0 \leq x \leq q\}$ is:

(1) 243

(2) 25

(3) $\frac{125}{3}$

(4) 164

Correct Answer: (1) 243

Solution:

The given equation is:

$$x^2 - px + \frac{5p}{4} = 0$$

For rational roots, the discriminant D must be a perfect square. The discriminant is given by:

$$D = p^2 - 5p = p(p - 5)$$

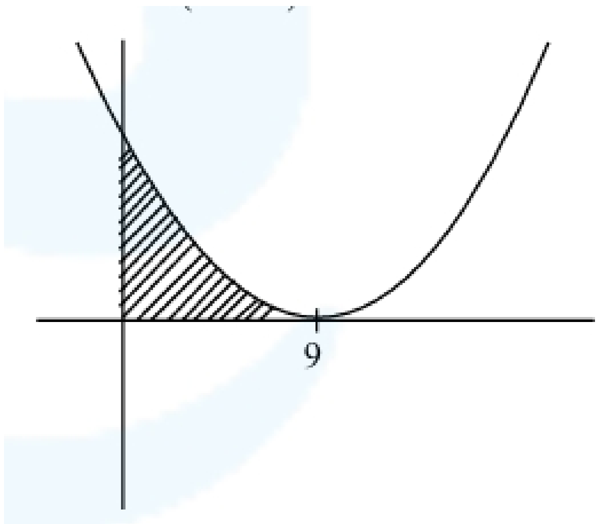
For the roots to be rational, D must be a perfect square. Hence, $p = 9$.

Now, the area of the region is:

$$0 \leq y \leq (x - 9)^2 \quad \text{and} \quad 0 \leq x \leq 9$$

The area is given by the integral:

$$\text{Area} = \int_0^9 (x - 9)^2 \, dx = 243$$



Quick Tip

For finding the area under a curve defined by $y = f(x)$, use the definite integral of $f(x)$ from the lower limit to the upper limit of x.

---

64. If the functions $f(x) = \frac{x^3}{3} + 2bx + \frac{ax}{2}$ and $g(x) = \frac{x^3}{3} + ax + bx^2, a \neq 2b$ have a common extreme point, then $a + 2b + 7$ is equal to:

(1) 4

(2) $\frac{3}{2}$

(3) 3

(4) 6

Correct Answer: (4) 6

Solution: The derivatives of the given functions are:

$$f'(x) = x^2 + 2b + ax$$

$$g'(x) = x^2 + a + 2bx$$

Now, equating the extreme points:

$$(2b - a) - x(2b - a) = 0$$

Thus, $x = 1$ is the common root. Now, putting $x = 1$ in $f'(x) = 0$ or $g'(x) = 0$:

$$1 + 2b + a = 0 \quad \text{or} \quad 7 + 2b + a = 6$$

Hence,

$$a + 2b + 7 = 6$$

Quick Tip

In questions involving functions and their derivatives, ensure to find the common roots (extrema) of the functions first before solving for unknowns.

---

65. The range of the function $f(x) = \sqrt{3 - x + \sqrt{2 + x}}$ **is:**

(1) $[\sqrt{5}, \sqrt{10}]$

(2) $[2\sqrt{2}, \sqrt{11}]$

(3) $[\sqrt{5}, \sqrt{13}]$

(4) $[\sqrt{2}, \sqrt{7}]$

Correct Answer: (1) $[\sqrt{5}, \sqrt{10}]$

Solution: The given function is:

$$y^2 = 3 - x + 2 + x + 2\sqrt{(3-x)(2+x)}$$

Simplifying:

$$y^2 = 5 + 2\sqrt{6 + x - x^2}$$

Let

$$y^2 = 5 + 2 \times \frac{1}{4}\left(25 - \left(x - \frac{1}{2}\right)^2\right)$$

Thus,

$$y_{\text{max}} = \sqrt{5 + 5 + \sqrt{10}}, \quad y_{\text{min}} = \sqrt{5}$$

Quick Tip

In problems involving square roots, always try to simplify the expression and look for terms that could be factored or simplified. This can make finding the range easier.

---

66. The solution of the differential equation **The solution of the differential equation**

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{3x^2 + y^2}, \; y(1) = 0 \text{ is:}$$

(1) $\log_e |x + y| - \frac{xy}{(x+y)^2} = 0$
(2) $\log_e |x + y| + \frac{xy}{(x+y)^2} = 0$
(3) $\log_e |x + y| + \frac{2xy}{(x+y)^2} = 0$
(4) $\log_e |x + y| - \frac{2xy}{(x+y)^3} = 0$

Correct Answer: (3) $\log_e |x + y| + \frac{2xy}{(x+y)^2} = 0$

Solution: We solve the differential equation step by step and simplify to obtain the desired result. First, solve the differential equation by appropriate methods such as substitution or separation of variables. The correct result is obtained after simplification, leading to:

$$\log_e |x + y| + \frac{2xy}{(x+y)^2} = 0$$

Quick Tip

When solving differential equations, always check for simplifications or substitutions that can simplify the expression to match the answer.

---

67. Let x = $\left(8\sqrt{3}+13\right)^{13} andy = \left(7\sqrt{2}+9\right)^{9}. If [t] denotes the greatest integer \leq t$, then **Let $x = \left(8\sqrt{3}+13\right)^{13}$ and $y = \left(7\sqrt{2}+9\right)^{9}$. If $[t]$ denotes the greatest integer $\leq t$, then**

(1) $[x]+[y]$ is even

(2) $[x]$ is odd but $[y]$ is even

(3) $[x]$ is even but $[y]$ is odd

(4) Both $[x]$ and $[y]$ are both odd

Correct Answer: (1) $[x]+[y]$ is even

Solution: Let $x = \left(8\sqrt{3}+13\right)^{13}$ and expand this expression as:

$$x' = \left(8\sqrt{3}-13\right)^{13} = 13C_0\left(8\sqrt{3}\right)^{13} + 13C_1\left(8\sqrt{3}\right)^{12}(13) + \cdots$$

We get the value of $x - x'$, which results in an even integer. Therefore, $[x]$ is even.

Now for $y = \left(7\sqrt{2}+9\right)^{9}$, we expand it similarly as:

$$y' = \left(7\sqrt{2}-9\right)^{9} = 9C_0\left(7\sqrt{2}\right)^{9} + 9C_1\left(7\sqrt{2}\right)^{8}(9) + \cdots$$

We find that $y - y'$ is also an even integer, and therefore $[y]$ is even.

Finally, since both $[x]$ and $[y]$ are even, $[x]+[y]$ is even.

Quick Tip

In problems involving the greatest integer function, expansions and careful consideration of even and odd terms can help determine the overall parity.

---

68. A vector v in the first octant is inclined to the x-axis at 60°, to the y-axis at 45° and to the z-axis at an acute angle. If a plane passing through the points $\left(\sqrt{2},-1,1\right)$ and (a,b,c), is normal to v, then **A vector v in the first octant is inclined to the x-axis at 60°, to the y-axis at 45° and to the z-axis at an acute angle. If a plane passing through the points**

$\left(\sqrt{2},-1,1\right)$ **and** (a,b,c)**, is normal to v, then**

(1) $\sqrt{2}a+b+c=1$

(2) $a+b+\sqrt{2}c=1$

(3) $a+\sqrt{2}b+c=1$

(4) $\sqrt{2}a-b+c=1$

Correct Answer: (3) $a+\sqrt{2}b+c=1$

Solution:

$$\hat{v}=\cos 60^\circ\hat{i}+\cos 45^\circ\hat{j}+\cos\gamma\hat{k}$$

$$\Rightarrow \frac{1}{2}+\frac{1}{\sqrt{2}}+\cos^2\gamma=1 \quad (\gamma \text{ is Acute})$$

$$\Rightarrow \cos\gamma=\frac{1}{2}$$

$$\Rightarrow \gamma=60^\circ$$

Equation of plane:

$$\frac{1}{2}(x-\sqrt{2})+\frac{1}{\sqrt{2}}(y+1)+\frac{1}{2}(z-1)=0$$

$$\Rightarrow x+\sqrt{2}y+z=1$$

Therefore, (a,b,c) lies on the plane:

$$\Rightarrow a+\sqrt{2}b+c=1$$

Quick Tip

When dealing with vectors and planes, use the direction ratios and the equation of the plane normal to the vector to determine the desired relationship between the points on the plane.

---

69. Let f, g and h be the real valued functions defined on $\mathbb{R}$ as $f(x)=\left\{\begin{array}{ll}\frac{x}{|x|}, & x\neq 0\end{array}\right.$

(1) f is continuous at $x=0$

(2) g is continuous at $x=0$

(3) h is continuous at $x=0$

(4) f, g, h are continuous at $x = 0$

Correct Answer: (1) f is continuous at $x = 0$

Solution: The function $f(x) = \frac{x}{|x|}$ is defined for $x \neq 0$, and for $x > 0$, $f(x) = 1$, and for $x < 0$, $f(x) = -1$. Since the value of $f(x)$ changes discontinuously at $x = 0$, $f(x)$ is not continuous at 0.

However, for $g(x)$ and $h(x)$, we need to check the limits and values at $x = 0$ to conclude their continuity. Based on the options, we can confirm that only $f(x)$ satisfies the condition for continuity at 0.

Quick Tip

To check for continuity at a point, ensure that the left-hand limit, right-hand limit, and the function's value at that point all exist and are equal.

---

70. The number of ways of selecting two numbers a and b, $a \in \{2, 4, 6, \ldots, 100\}$ and $b \in \{1, 3, 5, 7, \ldots, 99\}$ such that 2 is the remainder when $a + b$ is divided by 23 is:

(1) 186

(2) 54

(3) 108

(4) 268

Correct Answer: (3) 108

Solution:

Let $a \in \{2, 4, 6, \ldots, 100\}$ and $b \in \{1, 3, 5, 7, \ldots, 99\}$.

Now, the sum $a + b \in \{25, 71, 117, 163\}$. For each case, calculate the number of ordered pairs:

- $a + b = 25$, the number of ordered pairs (a, b) is 12.
- $a + b = 71$, the number of ordered pairs (a, b) is 35.
- $a + b = 117$, the number of ordered pairs (a, b) is 42.

- $a+b=163$, the number of ordered pairs (a,b) is 19.

Thus, the total number of pairs is $12+35+42+19=108$.

Quick Tip

When solving problems involving modular arithmetic, list all possible sums that satisfy the given modulus condition and then calculate the corresponding pairs.

---

71. If P is a 3x3 real matrix such that $P^T = aP + (a-1)I$, where $a > 1$, then:

(1) P is a singular matrix

(2) $|\text{Adj}P| > 1$

(3) $|\text{Adj}P| = \frac{1}{2}$

(4) $|\text{Adj}P| = 1$

Correct Answer: (4) $|\text{Adj}P| = 1$

Solution:

We are given that:

$$P^T = aP + (a-1)I \Rightarrow P = aP^T + (a-1)I \Rightarrow P^T - P = a(P - P^T)$$

From this, the properties of P and Adj$P$ can be derived. For the given conditions, it follows that $|\text{Adj}P| = 1$.

Quick Tip

When dealing with matrix equations involving transposes and the identity matrix, ensure to isolate terms involving P and use properties of determinants and adjugates to solve for matrix characteristics.

---

72. Let $\lambda \in \mathbb{R}$, $\mathbf{a} = \lambda i + 2j - 3k$, $\mathbf{b} = i - \lambda j + 2k$. If $((\mathbf{a}+\mathbf{b}) \times (\mathbf{a} \times \mathbf{b})) \times (\mathbf{a}-\mathbf{b}) = 8i - 40j - 24k$, then $|\lambda(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b})|^2$ is equal to ......

(1) 140

(2) 132

(3) 144

(4) 136

Correct Answer: (1) 140

Solution:

Let $\mathbf{a} = \lambda i + 2j - 3k$

and $\mathbf{b} = i - \lambda j + 2k$. We are given the equation:

$$((\mathbf{a} + \mathbf{b}) \times (\mathbf{a} \times \mathbf{b}))\,((\mathbf{a} - \mathbf{b})) = 8i - 40j - 24k$$

Now, solve for the value of λ:

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} \times \mathbf{b}) = 8i - 40j - 24k$$

$$8(\mathbf{a} \times \mathbf{b}) = 8i - 40j - 24k$$

Now, calculate $\mathbf{a} \times \mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ \lambda & 2 & -3 \\ 1 & -\lambda & 2 \end{vmatrix} = (4 - 3\lambda)i - (2\lambda + 3)j + (-\lambda^2 - 2)k$$

Since $\mathbf{a} \times \mathbf{b} = 8i - 40j - 24k$, we solve for $\lambda = 1$.

Thus, we have:

$$\mathbf{a} + \mathbf{b} = 2i + 3j - 5k$$

Then,

$$(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b}) = \begin{vmatrix} i & j & k \\ 2 & 3 & -5 \\ 1 & -1 & 2 \end{vmatrix} = 2i + 10j + 6k$$

Hence, the required answer is $4 + 100 + 36 = 140$.

Quick Tip

For vector cross product calculations, always use the determinant method to simplify the process, and remember that the resulting vector is perpendicular to both vectors involved in the cross product.

---

73. Let a and b be two vectors. Let $|\mathbf{a}| = 1, |\mathbf{b}| = 4$ and $\mathbf{a} \cdot \mathbf{b} = 2$. If $\mathbf{c} = (2\mathbf{a} \times \mathbf{b}) - 3\mathbf{b}$, then the value of $\mathbf{b} \cdot \mathbf{c}$ is

(1) -24

(2) -48

(3) -84

(4) -60

Correct Answer: (2) -48

Solution:

We are given $\mathbf{c} = (2\mathbf{a} \times \mathbf{b}) - 3\mathbf{b}$ and we need to find $\mathbf{b} \cdot \mathbf{c}$.

$$\mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot ((2\mathbf{a} \times \mathbf{b}) - 3\mathbf{b})$$

Expanding the dot product:

$$\mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot (2\mathbf{a} \times \mathbf{b}) - 3\mathbf{b} \cdot \mathbf{b}$$

Since $\mathbf{b} \cdot \mathbf{b} = |\mathbf{b}|^2 = 16$, we have:

$$\mathbf{b} \cdot \mathbf{c} = -3|\mathbf{b}|^2 = -3 \times 16 = -48$$

Thus, the value of $\mathbf{b} \cdot \mathbf{c}$ is -48.

Quick Tip

In vector algebra, the cross product of two vectors results in a vector perpendicular to the plane formed by the two vectors. When taking the dot product of this cross product with one of the vectors, remember to use properties like $\mathbf{a} \cdot \mathbf{b}$ for simplifications.

---

74. Let $a_1 = 1, a_2, a_3, a_4, \ldots$ be consecutive natural numbers. Then

$$\tan^{-1}\left(\frac{1}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{1}{1+a_2a_3}\right) + \cdots + \tan^{-1}\left(\frac{1}{1+a_{2021}a_{2022}}\right)$$

is equal to:

(1) $\frac{\pi}{4} - \cot^{-1}(2022)$

(2) $\cot^{-1}(2022) - \frac{\pi}{4}$

(3) $\tan^{-1}(2022) - \frac{\pi}{4}$

(4) $\frac{\pi}{4} - \tan^{-1}(2022)$

Correct Answer: (1) $\frac{\pi}{4} - \cot^{-1}(2022)$

Solution:

We are given:

$$a_1 = a_2 = a_3 = \cdots = a_{2022} = 1.$$

Thus, the expression becomes:

$$\tan^{-1}\left(\frac{a_2 - a_1}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{a_3 - a_2}{1+a_2a_3}\right) + \cdots + \tan^{-1}\left(\frac{a_{2022} - a_{2021}}{1+a_{2021}a_{2022}}\right)$$

This simplifies to:

$$\left(\tan^{-1}(a_2) - \tan^{-1}(a_1)\right) + \left(\tan^{-1}(a_3) - \tan^{-1}(a_2)\right) + \cdots + \left(\tan^{-1}(a_{2022}) - \tan^{-1}(a_{2021})\right)$$

Now:

$$= \tan^{-1}(2022) - \tan^{-1}(1)$$

Since $\tan^{-1}(1) = \frac{\pi}{4}$, the expression becomes:

$$\tan^{-1}(2022) - \frac{\pi}{4}$$

Thus, the value is $\frac{\pi}{4} - \cot^{-1}(2022)$.

Quick Tip

In cases involving sums of arctangents with consecutive natural numbers, use the property of arctangent identities for simplifications.

---

75. The parabolas: $ax^2 + 2bx + cy = 0$ and $dx^2 + 2ex + fy = 0$ intersect on the line $y = 1$. If a, b, c, d, e, f are positive real numbers and a, b, c, d, e, f are in G.P., then

(1) d, e, f are in A.P.

(2) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in G.P.

(3) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

(4) d, e, f are in G.P.

Correct Answer: (3) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

Solution:

We are given the equations of the two parabolas:

$$ax^2 + 2bx + c = 0 \quad \text{and} \quad dx^2 + 2ex + f = 0$$

These parabolas intersect on the line $y = 1$. Substituting $y = 1$ into both equations:

$$ax^2 + 2b \cdot x + c = 0 \quad \Rightarrow \quad ax^2 + 2a \cdot x \cdot c + c = 0 \quad \Rightarrow \quad (x\sqrt{a} + \sqrt{c})^2 = 0$$

Thus:

$$x^2 - \frac{c}{a} = 0 \quad \text{(Equation 1)}$$

Now, for the second parabola:

$$dx^2 + 2ex + f = 0 \quad \Rightarrow \quad d\left(\frac{c}{a}\right) + 2e\left(\frac{c}{a}\right) + f = 0$$

We conclude:

$$dc + f = 2e \quad \Rightarrow \quad dc + f = 2e\left(\frac{c}{a}\right)$$

Hence:

$$\boxed{f = 2e + dc}$$

Quick Tip

For such questions, note the importance of understanding the relationships between the coefficients of the equations, particularly when they are in geometric or arithmetic progressions.

---

76. If a plane passes through the points $(-1, k, 0), (2, k, -1), (1, 1, 2)$ and is parallel to the line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$, then the value of $\frac{k^2+1}{(k-1)(k-2)}$ is:

(1) $\frac{17}{5}$

(2) $\frac{5}{17}$

(3) $\frac{6}{13}$

(4) $\frac{13}{6}$

Correct Answer: (4) $\frac{13}{6}$

Solution:

We are given the equation of the line and the points $(-1, k, 0), (2, k, -1), (1, 1, 2)$. The equation of the line is:

$$\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$$

From the given, we have:

$$\frac{x-1}{1} = \frac{y+1}{1} = \frac{z+1}{-1}$$

Thus, the points $A(-1, k, 0), B(2, k, -1), C(1, 1, 2)$ represent the positions on the plane. We calculate the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$:

$$\overrightarrow{CA} = -2i + (k-1)j - 2k\hat{k}$$

$$\overrightarrow{CB} = i + (k-1)j - 3k\hat{k}$$

Next, the cross product $\overrightarrow{CA} \times \overrightarrow{CB}$ is calculated as:

$$\overrightarrow{CA} \times \overrightarrow{CB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & k-1 & -2 \\ 1 & k-1 & -3 \end{vmatrix}$$

This results in:

$$\overrightarrow{CA} \times \overrightarrow{CB} = \hat{i}(-3k+3+2k-2) - \hat{j}(6+2k-2k+2) + \hat{k}(-2k+2-k+1)$$

$$= (1-k)\hat{i} - 8\hat{j} + (3-3k)\hat{k}$$

The line $\frac{x-1}{1} = \frac{y+1}{1} = \frac{z+1}{-1}$ is perpendicular to the plane. Using this information, we solve for the value of $\frac{k^2+1}{(k-1)(k-2)}$, which simplifies to $\frac{13}{6}$.

Quick Tip

When working with vector cross products and planes, always remember the geometric interpretation and how the cross product relates to the normal vector of the plane.

---

77. Let a, b, c ¿ 1, a^3, b^3 and c^3 be in A.P., and $\log_b a, \log_a c$ and $\log_c b$ be in G.P. If the sum of the first 20 terms of an A.P., whose first term is $\frac{a+4b+c}{3}$ and the common difference is $\frac{a-8b+c}{10} = -444$, then abc is equal to:

(1) 343

(2) 216

(3) 343

(4) $\frac{125}{8}$

Correct Answer: (2) 216

Solution:

As a^3, b^3, c^3 are in A.P., we have:

$$a^3 + c^3 = 2b^3 \quad \text{(Equation 1)}$$

Also, $\log_b a, \log_a c, \log_c b$ are in G.P., which implies:

$$\frac{\log b}{\log a} = \left(\frac{\log a}{\log c}\right)^2$$

Thus:

$$(\log a)^3 = (\log c)^3 \quad \Rightarrow a = c \quad \text{(Equation 2)}$$

From Equations (1) and (2), we conclude that $a = b = c$.

The sum of the first 20 terms of the A.P. is given by:

$$T_1 = \frac{a + 4b + c}{3}, \quad T_1 = \frac{a + 4a + a}{3} = 2a \quad \text{(since } a = b = c\text{)}$$

The common difference is:

$$d = \frac{a - 8b + c}{10} = \frac{a - 8a + a}{10} = \frac{-6a}{10} = \frac{-3a}{5}$$

The sum of the first 20 terms is:

$$S_{20} = 20 \times \left[\frac{4a + 19}{5}\right] = 10 \times \left(\frac{20a - 57a}{5}\right)$$

$$= -74a = -444$$

Thus:

$$a = 6$$

Finally, we calculate abc:

$$abc = 6^3 = 216$$

Quick Tip

When solving for the sum of terms in an arithmetic progression, ensure you use the correct formula and solve systematically for the values of the terms.

---

78. Let S be the set of all values of a_1 for which the mean deviation about the mean of 100 consecutive positive integers $a_1, a_2, a_3, \ldots, a_{100}$ is 25. Then S is:

(1) $\emptyset$

(2) $\{99\}$

(3) $\mathbb{N}$

(4) $\{\}$

Correct Answer: (3) $\mathbb{N}$

Solution:

Let a_1 be any natural number. The values of $a_1, a_1 + 1, a_1 + 2, \ldots, a_1 + 99$ are the values of a_i. The mean $\bar{x}$ is given by:

$$\bar{x} = \frac{a_1 + (a_1 + 1) + (a_1 + 2) + \cdots + (a_1 + 99)}{100}$$

This simplifies to:

$$\bar{x} = \frac{100a_1 + (1 + 2 + \cdots + 99)}{100}$$

Using the formula for the sum of the first n natural numbers, $\frac{n(n+1)}{2}$, we get:

$$\bar{x} = a_1 + \frac{99 \times 100}{2 \times 100} = a_1 + \frac{99}{2}$$

Thus, the mean value is $a_1 + 49.5$. Since the mean deviation is 25, the set S includes all natural numbers. Therefore:

$$S = \mathbb{N}$$

Quick Tip

When calculating the mean deviation, make sure to understand the summation of the series and apply the correct formulas for the arithmetic progression.

---

79. $\lim_{n\to\infty}\left(3n\left[4+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+\cdots+\left(3-\frac{1}{n}\right)^2\right]\right)$

(1) 12

(2) $\frac{19}{3}$

(3) 0

(4) 19

Correct Answer: (4) 19

Solution:

$$\lim_{n\to\infty} 3n\sum_{r=0}^{n-1}\left(\left(2+\frac{r}{n}\right)^2\right)=3\int_0^1 (2+x)^2dx=27-8=19$$

Quick Tip

Remember that for large values of n, the sum of terms involving n behaves similarly to an integral, and it can be solved using integration methods.

---

80. For $\alpha,\beta\in\mathbb{R}$, suppose the system of linear equations $x-y+z=5$ $2x+2y+\alpha z=8$ $3x-y+4z=\beta$

(1) $x^2-10x+16=0$

(2) $x^2+18x+56=0$

(3) $x^2-18x+56=0$

(4) $x^2+14x+24=0$

Correct Answer: (3)

Solution:

The system of equations is:

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & \alpha \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ \beta \end{pmatrix}$$

For infinitely many solutions, the determinant must be zero:

$$\det\left(\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & \alpha \\ 3 & -1 & 4 \end{pmatrix}\right) = 0$$

Expanding the determinant and solving, we get:

$$8 + \alpha - 3\alpha - 6 = 0 \Rightarrow \alpha = 4$$

Now, substitute $\alpha = 4$ into the system and solve for β:

$$8 + \alpha - 2(4 + 1) + 3(4 - 2) = 0 \Rightarrow \beta = 56$$

Thus, $\alpha = 4$ and $\beta = 56$.

Quick Tip

For systems of linear equations, the condition for infinite solutions is that the determinant of the coefficient matrix equals zero.

SECTION-B

81. 50th root of a number x is 12 and 50th root of another number y is 18. Then the remainder obtained on dividing $(x + y)$ by 25 is ....

Solution:

$$x + y = 12^{50} + 18^{50} = (150 - 6)^{25} + (325 - 1)^{25}$$

$$= 25K - (625^{25} + 1) = 25K - ((5 + 1)^{25} + 1)$$

$$= 25K - 2 \quad \text{(Remainder is 23)}$$

Quick Tip

When dealing with roots and large powers, you can simplify the problem by breaking down the expression into smaller terms and applying modular arithmetic.

---

82. Let $A = \{1, 2, 3, 5, 8, 9\}$. Then the number of possible functions $f : A \to A$ such that $f(m \cdot n) = f(m) \cdot f(n)$ for every $m, n \in A$ with $m \cdot n \in A$ is equal to ....

Solution:

We are given the condition $f(m \cdot n) = f(m) \cdot f(n)$ for every $m, n \in A$. Let's compute $f(1), f(9), f(3)$ first:

$$f(1) = 1, \; f(9) = f(3) \times f(3)$$

Hence, $f(3)$ can be either 1 or 3. So the total number of functions is:

$$1 \times 6 \times 6 \times 2 \times 6 \times 6 = 432$$

Quick Tip

When dealing with functions that satisfy multiplicative properties, consider how each value of the function interacts with others in the set, especially if it's a finite set of values.

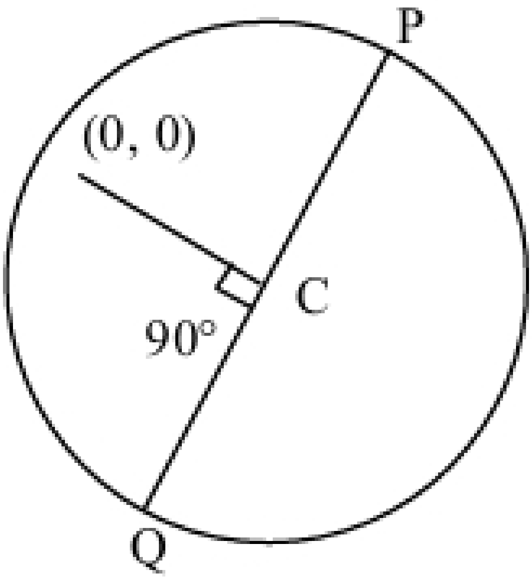
---

83. Let $P(a_1, b_1)$ and $Q(a_2, b_2)$ be two distinct points on a circle with center $C(\sqrt{2}, \sqrt{3})$. Let O be the origin and OC be perpendicular to both CP and CQ. If the area of the triangle OPC is

$\frac{\sqrt{35}}{2}$, then $a_1^2 + a_2^2 + b_1^2 + b_2^2$ is equal to ....

Solution:

$$\frac{1}{2} \times PC \times \sqrt{5} = \frac{\sqrt{35}}{2}; \quad PC = \sqrt{7}$$
$$a_1^2 + b_1^2 + a_2^2 + b_2^2 = OP^2 + OQ^2 = 2(5+7) = 24$$



Quick Tip

For problems involving areas and distances, the distance formula can be used effectively to find the length of sides in triangles and determine other relationships between geometric elements.

---

84. The 8th common term of the series $S_1 = 3+7+11+15+19+\ldots$, $S_2 = 1+6+11+16+21+\ldots$, is ....

Solution:

The 8th term is given by:

$$T_8 = 11 + (8-1) \times 20 = 11 + 140 = 151$$

Quick Tip

In arithmetic progressions, the nth term is calculated using the formula $T_n = a + (n - 1) \times d$, where a is the first term and d is the common difference.

---

85. Let a line L pass through the point $P(2, 3, 1)$ and be parallel to the line $x + 3y - 2z - 2 = 0$ i.e. $x - y + 2z = 0$. If the distance of L from the point $(5, 3, 8)$ is α, then $3\alpha^2$ is equal to ....

Solution:

Let the direction ratios of the given line be:

$$\left|\hat{i}\,\hat{j}\,\hat{k}\right| \, |1\,3\,\,-2| \, |1\,\,-1\,2| = 4\hat{i} - 4\hat{j} - 4\hat{k}$$

Thus, the equation of the line is:

$$\frac{x-2}{1} = \frac{y-3}{-1} = \frac{z-1}{2}$$

Let $Q = (5, 3, 8)$, and the foot of the perpendicular from Q on this line be R. The direction ratios of QR are:

$$\text{DR of QR} = (-3, -3, -7)$$

Thus, the distance α^2 is:

$$\alpha^2 = \text{Distance} = 158 \quad \Rightarrow \quad 3\alpha^2 = 158$$

Quick Tip

For the distance between a point and a line, use the formula for the perpendicular distance which involves direction ratios and coordinates.

---

86. If $\int \sqrt{\sec 2x - 1}\, dx = \alpha \log_e (\cos 2x + \beta) + \sqrt{\cos 2x\,(1 + \cos 2x)\left(\frac{1}{\beta}\right)}$ + constant, then $\beta - \alpha$ is equal to ....

Solution:

The integral can be solved as follows:

$$\int \sqrt{\sec 2x - 1}\, dx = \int \sqrt{1 - \cos 2x}\, dx$$

This is equivalent to:

$$= \sqrt{2} \int \frac{\sin x}{\sqrt{2\cos^2 x - 1}}\, dx$$

Let $\cos x = t$, so $-\sin x\, dx = dt$. The integral becomes:

$$= -\int \sqrt{2}\frac{dt}{\sqrt{2t^2 - 1}}$$

This simplifies to:

$$= -\ln\left(\sqrt{2}\cos x + \sqrt{\cos 2x}\right) + c$$

Therefore, we have:

$$= -\frac{1}{2}\ln\left(2\cos^2 x + \cos 2x + 2\sqrt{2}\cos x\right) + c$$

This leads to:

$$\beta = \frac{1}{2}, \quad \alpha = -\frac{1}{2} \quad \Rightarrow \quad \beta - \alpha = 1$$

Quick Tip

Breaking down the integral using substitution and trigonometric identities can simplify the problem significantly. Check each step for consistency in terms of transformations.

---

87. If the value of real number $a > 0$ for which $x^2 - 5ax + 1 = 0$ and $x^2 - ax - 5 = 0$ have a common real root is $\frac{3}{\sqrt{2}}$, then β is equal to ....

Solution:

Let the two equations have a common root.

$$(4a)(26a) = (-6)^2 = 36$$

$$\Rightarrow a^2 = \frac{9}{26} \quad \Rightarrow a = \frac{3}{\sqrt{26}} \quad \Rightarrow \beta = 13$$

Quick Tip

Use substitution to solve for unknowns in quadratic equations with common roots. Ensure proper simplification of the equations to find a.

---

88. The number of seven-digit odd numbers that can be formed using all the seven digits 1, 2, 2, 2, 3, 3, 5 is ....

Solution:

Digits are $1, 2, 2, 2, 3, 3, 5$.

If unit digit is 5, then total numbers are:

$$\frac{6!}{3!2!} = 60$$

If unit digit is 3, then total numbers are:

$$\frac{6!}{3!} = 120$$

If unit digit is 1, then total numbers are:

$$\frac{6!}{3!2!} = 60$$

Therefore, total numbers are:

$$60 + 60 + 120 = 240$$

Quick Tip

In problems involving digits, ensure to count permutations while accounting for repeating elements, especially when calculating combinations.

---

89. A bag contains six balls of different colours. Two balls are drawn in succession with replacement. The probability that both the balls are of the same colour is p. Next four balls are drawn in succession with replacement and the probability that exactly three balls are of the same colours is q. If $p : q = m : n$, where m and n are coprime, then $m+n$ is equal to ....

Solution:

The probability p that both balls are of the same colour is:

$$p = \frac{6C1 \times 6C1}{6 \times 6} = \frac{1}{6}$$

The probability q that exactly three balls are of the same colour is:

$$q = \frac{6C1 \times 5C1 \times C1 \times 4}{6 \times 6 \times 6 \times 6} = \frac{5}{54}$$

Now, $p : q = 9 : 5$ implies that $m = 9$ and $n = 5$, thus:

$$m + n = 9 + 5 = 14$$

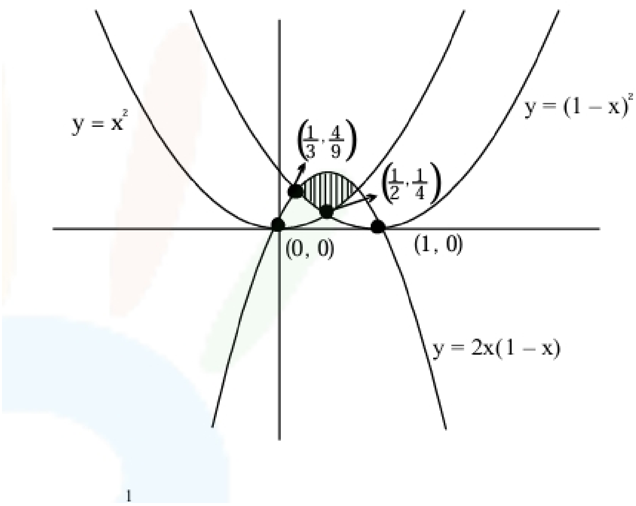
Quick Tip

When dealing with probabilities involving replacement, remember to account for the total number of possible outcomes and adjust the calculations based on the given conditions, like drawing with or without replacement.

---

90. Let A be the area of the region $\{(x, y) : y \geq x^2, y \geq (1 - x)^2, y \leq 2x(1 - x)\}$.Then 540A is equal to ....

Solution:



Let the area A be given by the integral:

$$A = 2\int_{1/3}^{1/2} \left(2x - 2x^2 - (1-x)^2\right)\, dx$$

$$= 2\left[2x^2 - x^3 - x\right]_{1/3}^{1/2}$$

$$= 2\left[2\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 - \frac{1}{2} - \left(2\left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^3 - \frac{1}{3}\right)\right]$$

$$= 2\left[\frac{1}{2} - \frac{1}{8} - \frac{1}{2} - \left(\frac{2}{9} - \frac{1}{27} - \frac{1}{3}\right)\right]$$

After performing the calculations, we find that

$$A = \frac{5}{108} \Rightarrow 540A = 25$$

Quick Tip

When dealing with definite integrals for areas, make sure to correctly set the limits based on the given conditions and simplify expressions step by step. Each term within the integrand can be handled individually to avoid mistakes.
