JEE Main 2025 April 3 Shift 1 Mathematics Question Paper with Solutions

Time Allowed :3 Hours	Maximum Marks :300	Total Questions :75

General Instructions

Read the following instructions very carefully and strictly follow them:

1. Multiple choice questions (MCQs)
2. Questions with numerical values as answers.
3. There are three sections: **Mathematics, Physics, Chemistry**.
4. **Mathematics:** 25 (20+5) 10 Questions with answers as a numerical value. Out of 10 questions, 5 questions are compulsory.
5. **Physics:** 25 (20+5) 10 Questions with answers as a numerical value. Out of 10 questions, 5 questions are compulsory..
6. **Chemistry:** 25 (20+5) 10 Questions with answers as a numerical value. Out of 10 questions, 5 questions are compulsory.
7. Total: 75 Questions (25 questions each).
8. 300 Marks (100 marks for each section).
9. **MCQs:** Four marks will be awarded for each correct answer and there will be a negative marking of one mark on each wrong answer.
10. **Questions with numerical value answers:** Candidates will be given four marks for each correct answer and there will be a negative marking of 1 mark for each wrong answer.

Mathematics

Section - A

1. Let A be a matrix of order 3×3 and $|A| = 5$. If

$$|2\,\text{adj}(3A\,\text{adj}(2A))| = 2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma}, \quad \alpha, \beta, \gamma \in N$$

then $\alpha + \beta + \gamma$ is equal to

(1) 25
(2) 26
(3) 27
(4) 28

Correct Answer: (3) 27

Solution: We begin with the expression:

$$|2\,\mathrm{adj}(3A\,\mathrm{adj}(2A))|$$

We use the following identities:

- $|\mathrm{adj}(M)| = |M|^{n-1}$ for an $n \times n$ matrix.
- $|kM| = k^n|M|$ for scalar k.

Let us simplify step-by-step:

$$\begin{aligned}
|2\,\mathrm{adj}(3A\,\mathrm{adj}(2A))| &= 2^3 \cdot |\mathrm{adj}(3A\,\mathrm{adj}(2A))| \\
&= 2^3 \cdot |3A\,\mathrm{adj}(2A)|^2 \\
&= 2^3 \cdot \left(3^3 \cdot |A| \cdot |\mathrm{adj}(2A)|\right)^2 \\
&= 2^3 \cdot \left(3^3 \cdot |A| \cdot (2^3 \cdot |A|)^2\right)^2 \\
&= 2^3 \cdot \left(2^6 \cdot 3^3 \cdot |A|^3\right)^2 \\
&= 2^3 \cdot 2^{12} \cdot 3^6 \cdot |A|^6 \\
&= 2^{15} \cdot 3^6 \cdot |A|^6
\end{aligned}$$

Given $|A| = 5$, we substitute:

$$= 2^{15} \cdot 3^6 \cdot 5^6$$

Comparing with $2^\alpha \cdot 3^\beta \cdot 5^\gamma$, we get:

$$\alpha = 15, \quad \beta = 6, \quad \gamma = 6$$

Thus:

$$\alpha + \beta + \gamma = 15 + 6 + 6 = 27$$

Quick Tip

To solve matrix determinant problems involving adjugates and scalar multiplication, remember the key formulas:

$$|\mathrm{adj}(A)| = |A|^{n-1}, \quad |kA| = k^n|A|$$

Also, the adjugate of a product doesn't distribute over multiplication like regular matrices, so simplify the inner terms first before applying adjugate or determinant rules.

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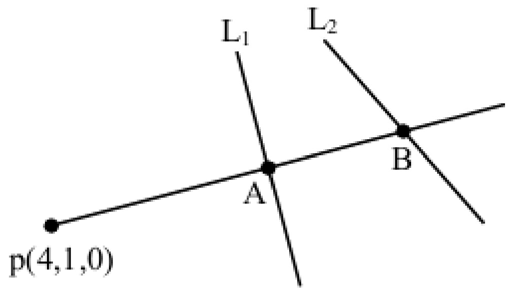
2. Let a line passing through the point $(4, 1, 0)$ intersect the line $L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ at the point $A(\alpha, \beta, \gamma)$ and the line $L_2 : x - 6 = y = -z + 4$ at the point $B(a, b, c)$. Then

$$\begin{vmatrix} 1 & 0 & 1 \\ \alpha & \beta & \gamma \\ a & b & c \end{vmatrix} \text{ is equal to}$$

(1) 8
(2) 16
(3) 12
(4) 6

Correct Answer: (1) 8

Solution:



The line passing through point $P(4,1,0)$ intersects line L_1 and L_2. Let us assume the point A on L_1 is given by parameter p:

$$L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = p \Rightarrow A(2p+1, 3p+2, 4p+3)$$

Let point B on L_2 be parameterized using q:

$$L_2 : x - 6 = y = -z + 4 = q \Rightarrow B(q+6, q, 4-q)$$

Now find the direction ratios (D.R.) of $\overrightarrow{PA}$ and $\overrightarrow{PB}$:

$$\text{D.R. of } \overrightarrow{PA} = (2p-3, 3p+1, 4p+3)$$

D.R. of $\overrightarrow{PB} = (q+2, q-1, 4-q)$
Since $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are collinear, their components must be proportional:

$$\frac{2p-3}{q+2} = \frac{3p+1}{q-1} = \frac{4p+3}{4-q}$$

Equating pairwise and solving:

$$2pq - 2p - 3q + 3 = 3pq + 6p + q + 2 \Rightarrow pq + rp + 4q - 1 = 0 \quad (1)$$

7pq - 16p + 4q - 7 = 0   (2)

Solving these equations, we find:

$$pq = -3, \quad p = -1, \quad q = 3 \Rightarrow A(-1,-1,-1), \quad B(9,3,1)$$

Now compute the determinant:

$$\begin{vmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 9 & 3 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & -1 \\ 3 & 1 \end{vmatrix} - 0 + 1 \cdot \begin{vmatrix} -1 & -1 \\ 9 & 3 \end{vmatrix} = 1(-1+3) + 1(-3+9) = 2 + 6 = 8$$

Quick Tip

To check whether two vectors are collinear, equating the ratios of their direction ratios is a powerful approach. Also, using parametric representation for lines makes it easier to substitute and calculate points of intersection.

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3. Let α and β be the roots of $x^2 + \sqrt{3}x - 16 = 0$, and γ and δ be the roots of $x^2 + 3x - 1 = 0$. If $P_n = \alpha^n + \beta^n$ and $Q_n = \gamma^n + \delta^n$, then

$$\frac{P_{25} + \sqrt{3}P_{24}}{2P_{23}} + \frac{Q_{25} - Q_{23}}{Q_{24}} \textbf{ is equal to}$$

(1) 3
(2) 4
(3) 5
(4) 7

Correct Answer: (3) 5

Solution: We are given:

$$x^2 + \sqrt{3}x - 16 = 0 \quad \text{has roots } \alpha, \beta$$

Define $P_n = \alpha^n + \beta^n$. Since α, β are roots, the recurrence relation is:

$$P_n + \sqrt{3}P_{n-1} - 16P_{n-2} = 0$$

Using the recurrence:

$$P_{25} + \sqrt{3}P_{24} - 16P_{23} = 0 \Rightarrow \frac{P_{25} + \sqrt{3}P_{24}}{2P_{23}} = \frac{16P_{23}}{2P_{23}} = 8$$

Similarly, for the second quadratic:

$$x^2 + 3x - 1 = 0 \quad \text{has roots } \gamma, \delta$$

Define $Q_n = \gamma^n + \delta^n$
We evaluate:

$$Q_{25} - Q_{23} = \gamma^{25} + \delta^{25} - \gamma^{23} - \delta^{23} = \gamma^{23}(\gamma^2 - 1) + \delta^{23}(\delta^2 - 1)$$

Since $\gamma^2 = -3\gamma + 1$, so:

$$\gamma^2 - 1 = -3\gamma, \quad \delta^2 - 1 = -3\delta \Rightarrow Q_{25} - Q_{23} = -3(\gamma^{24} + \delta^{24}) = -3Q_{24}$$

Therefore:

$$\frac{Q_{25} - Q_{23}}{Q_{24}} = -3$$

Now, summing both terms:

$$\frac{P_{25} + \sqrt{3}P_{24}}{2P_{23}} + \frac{Q_{25} - Q_{23}}{Q_{24}} = 8 - 3 = 5$$

Quick Tip

For expressions involving powers of roots of quadratic equations, use recurrence relations derived from the equation itself. If α, β are roots of $x^2 + ax + b = 0$, then $P_n = \alpha^n + \beta^n$ satisfies the recurrence $P_n = -aP_{n-1} - bP_{n-2}$.

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4. The sum of all rational terms in the expansion of $\left(2+\sqrt{3}\right)^8$ is

(1) 16923
(2) 3763
(3) 33845
(4) 18817

Correct Answer: (4) 18817

Solution: Let $S = (2+\sqrt{3})^8$. To find the sum of all **rational terms** in the binomial expansion, we use:

$$(2+\sqrt{3})^8 + (2-\sqrt{3})^8$$

This removes all irrational terms since they cancel out in the symmetric expansion. So the sum of rational terms is:

$$\frac{(2+\sqrt{3})^8 + (2-\sqrt{3})^8}{2}$$

We can also directly select terms where the exponent of $\sqrt{3}$ is even (to ensure the term is rational). From the binomial expansion:

$$= \binom{8}{0}(2)^8 + \binom{8}{2}(2)^6(\sqrt{3})^2 + \binom{8}{4}(2)^4(\sqrt{3})^4 + \binom{8}{6}(2)^2(\sqrt{3})^6 + \binom{8}{8}(\sqrt{3})^8$$

$$= 2^8 + 28 \cdot 2^6 \cdot 3 + 70 \cdot 2^4 \cdot 9 + 28 \cdot 2^2 \cdot 27 + 1 \cdot 81$$

$$= 256 + 5376 + 10080 + 3024 + 81 = 18817$$

Quick Tip

When expanding expressions of the form $(a+\sqrt{b})^n$, rational terms are those in which the exponent of $\sqrt{b}$ is even. Use symmetry by adding $(a+\sqrt{b})^n$ and $(a-\sqrt{b})^n$ to quickly eliminate irrational components and double the rational terms.

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5. Let A = {-3,-2,-1,0,1,2,3}. Let R be a relation on A defined by xRy if and only if $0 \leq x^2 + 2y \leq 4$. Let l be the number of elements in R and m be the minimum number of elements required to be added in R to make it a reflexive relation. then $l + m$ is equal to

(1) 19
(2) 20
(3) 17
(4) 18

Correct Answer: (4) 18

Solution: Let $A = \{-3, -2, -1, 0, 1, 2, 3\}$
Given:

$$0 \leq x^2 + 2y \leq 4 \Rightarrow -2y \leq x^2 \leq 4 - 2y$$

Now for different values of $y \in A$, find the possible $x \in A$ satisfying the condition:

- For $y = -3$, $6 \leq x^2 \leq 10 \Rightarrow x \in \{-3, 3\}$
- For $y = -2$, $4 \leq x^2 \leq 8 \Rightarrow x \in \{-2, 2\}$
- For $y = -1$, $2 \leq x^2 \leq 6 \Rightarrow x \in \{-2, 2\}$
- For $y = 0$, $0 \leq x^2 \leq 4 \Rightarrow x \in \{-2, -1, 0, 1, 2\}$
- For $y = 1$, $-2 \leq x^2 \leq 2 \Rightarrow x \in \{-1, 0, 1\}$
- For $y = 2$, $-4 \leq x^2 \leq 0 \Rightarrow x \in \{0\}$
- For $y = 3$, $-6 \leq x^2 \leq -2 \Rightarrow$ No such x

So the relation R consists of the following ordered pairs:

$R = \{(-3, -3), (-3, 3), (-2, -2), (-2, 2), (-1, -2), (-1, 2), (0, -2), (0, -1), (0, 0), (0, 1), (0, 2),$
(1,-1), (1,0), (1,1), (2,0) }

Thus,

$$l = |R| = 15$$

To make R reflexive, we must add the missing self-pairs: From set A, reflexive relation requires all $(a, a) \in A \times A$ Already present: $(0, 0)$ Missing: $(-1, -1), (2, 2), (3, 3) \Rightarrow m = 3$

$$l + m = 15 + 3 = \boxed{18}$$

Correct answer: Option (4)

Quick Tip

To make a relation reflexive, every element in the set must be related to itself. In other words, for a set A, (a,a) must be in R for all a in A.

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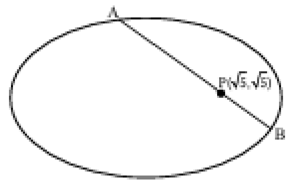
6. A line passing through the point P$(\sqrt{5}, \sqrt{5})$ intersects the ellipse $\frac{x^2}{36} + \frac{y^2}{25} = 1$ at A and B such that (PA).(PB) is maximum. Then 5(PA^2 + PB^2) is equal to :

(1) 218
(2) 377
(3) 290
(4) 338

Correct Answer: (4) 338

Solution: Given ellipse is

$$\frac{x^2}{36} + \frac{y^2}{25} = 1$$



Any point on line AB can be assumed as

$$Q(\sqrt{5} + r\cos\theta, \sqrt{5} + r\sin\theta)$$

Substituting this into the equation of the ellipse:

$$25(\sqrt{5} + r\cos\theta)^2 + 36(\sqrt{5} + r\sin\theta)^2 = 900$$

Expanding and simplifying:

$$r^2(25\cos^2\theta + 36\sin^2\theta) + 2\sqrt{5}r(25\cos\theta + 36\sin\theta) + 25\cdot 5 + 36\cdot 5 = 900$$

$$r^2(25\cos^2\theta + 36\sin^2\theta) + 2\sqrt{5}r(25\cos\theta + 36\sin\theta) = 900 - 305 = 595$$

$$\Rightarrow r^2(25\cos^2\theta + 36\sin^2\theta) + 2\sqrt{5}r(25\cos\theta + 36\sin\theta) - 595 = 0$$

Let the roots of this quadratic in r be PA and PB, then

$$PA \cdot PB = \frac{595}{25\cos^2\theta + 36\sin^2\theta}$$

To maximize $PA \cdot PB$, the denominator should be minimized:

$$25\cos^2\theta + 36\sin^2\theta = 25 + 11\sin^2\theta$$

Maximum value of $PA \cdot PB$ occurs when $\sin^2\theta = 0$, i.e., $\theta = 0$ or π

$$\Rightarrow \text{Line } AB \text{ must be parallel to the x-axis} \Rightarrow y_A = y_B = \sqrt{5}$$

Putting $y = \sqrt{5}$ in the equation of the ellipse:

$$\frac{x^2}{36} + \frac{5}{25} = 1 \Rightarrow \frac{x^2}{36} = \frac{4}{5} \Rightarrow x^2 = \frac{4}{5}\cdot 36 = \frac{144}{5}$$

Hence, coordinates of A and B are:

$$A = \left(-\frac{12}{\sqrt{5}}, \sqrt{5}\right), \quad B = \left(\frac{12}{\sqrt{5}}, \sqrt{5}\right)$$

Now,

$$PA^2 + PB^2 = \left(\sqrt{5} - \frac{12}{\sqrt{5}}\right)^2 + \left(\sqrt{5} + \frac{12}{\sqrt{5}}\right)^2$$

$$= 2\left(5 + \frac{144}{5}\right) = 2 \cdot \frac{169}{5} = \frac{338}{5}$$

$$\Rightarrow 5(PA^2 + PB^2) = 338$$

Quick Tip

To maximize the product of two distances from a point to the intersection points of a line and an ellipse, the line should be parallel to the major or minor axis of the ellipse.

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7. The sum 1 + 3 + 11 + 25 + 45 + 71 + ... upto 20 terms, is equal to

(1) 7240
(2) 7130
(3) 6982
(4) 8124

Correct Answer: (1) 7240

Solution: Given sum is $S_n = 1 + 3 + 11 + 25 + 45 + 71 + ... + T_n$ First order differences are in A.P. Thus, we can assume that $T_n = an^2 + bn + c$

Solving $\begin{cases} T_1 = 1 = a + b + c \\ T_2 = 3 = 4a + 2b + c \\ T_3 = 11 = 9a + 3b + c \end{cases}$

we get a = 3, b = -7, c = 5 Hence, general term of given series is $T_n = 3n^2 - 7n + 5$ Hence, required sum equals $\sum_{n=1}^{20}(3n^2 - 7n + 5) = 3\frac{20 \cdot 21 \cdot 41}{6} - 7\frac{20 \cdot 21}{2} + 5(20) = 7240$

Quick Tip

If the first differences of a series form an arithmetic progression (AP), then the general term of the series can be represented by a quadratic equation $T_n = an^2 + bn + c$.

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8. If the domain of the function $f(x) = \log_e\left(\frac{2x-3}{5+4x}\right) + \sin^{-1}\left(\frac{4+3x}{2-x}\right)$ is $[\alpha, \beta]$, then $\alpha^2 + 4\beta$ is equal to

(1) 5
(2) 4
(3) 3

(4) 7

Correct Answer: (2) 4

Solution: Given function is

$$f(x) = \log_e\left(\frac{2x-3}{5+4x}\right) + \sin^{-1}\left(\frac{4+3x}{2-x}\right)$$

For the domain of $f(x)$, we require:

$$\frac{2x-3}{5+4x} > 0 \quad \text{and} \quad \left|\frac{4+3x}{2-x}\right| \le 1$$

Start with the logarithmic condition:

$$\frac{2x-3}{5+4x} > 0 \Rightarrow x \in \left(-\infty, -\frac{5}{4}\right) \cup \left(\frac{3}{2}, \infty\right)$$

Now consider the inverse sine condition:

$$-1 \le \frac{4+3x}{2-x} \le 1$$

Break this into two inequalities:

$$\frac{4+3x}{2-x} \ge -1 \quad \text{and} \quad \frac{4+3x}{2-x} \le 1$$

Solving the first:

$$\frac{4+3x}{2-x} + 1 \ge 0 \Rightarrow \frac{4+3x+2-x}{2-x} = \frac{6+2x}{2-x} \ge 0$$

Solving the second:

$$\frac{4+3x}{2-x} - 1 \le 0 \Rightarrow \frac{4+3x-2+x}{2-x} = \frac{2+4x}{2-x} \le 0$$

Combining both:

$$\left(\frac{6+2x}{2-x} \ge 0\right) \cap \left(\frac{2+4x}{2-x} \le 0\right)$$

Multiply the two expressions:

$$\frac{(6+2x)(2+4x)}{(2-x)^2} \le 0$$

Solve the inequality:

$$x \in \left[-3, -\frac{1}{2}\right]$$

Now take the intersection of both conditions:

$$x \in \left(-\infty, -\frac{5}{4}\right) \cup \left(\frac{3}{2}, \infty\right) \quad \cap \quad x \in \left[-3, -\frac{1}{2}\right] \Rightarrow x \in \left[-3, -\frac{5}{4}\right)$$

Thus, the domain of $f(x)$ is:

$$x \in \left[-3, -\frac{5}{4}\right)$$

Let $\alpha = -3$, $\beta = -\frac{5}{4}$

Then,

$$\alpha^2 + 4\beta = (-3)^2 + 4 \cdot \left(-\frac{5}{4}\right) = 9 - 5 = \boxed{4}$$

Quick Tip

For logarithmic functions, the argument must be strictly positive. For inverse sine functions, the argument must lie between -1 and 1, inclusive.

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9. If $\sum_{r=1}^{9} \left(\frac{r+3}{2^r}\right) \cdot {}^9C_r = \alpha \left(\frac{3}{2}\right)^9 - \beta$, $\alpha, \beta \in N$, then $(\alpha+\beta)^2$ is equal to

(1) 27
(2) 9
(3) 81
(4) 18

Correct Answer: (3) 81

Solution: Given:

$$\sum_{r=1}^{9} \left(\frac{r+3}{2^r}\right) \cdot {}^9C_r = \alpha \left(\frac{3}{2}\right)^9 - \beta, \quad \alpha, \beta \in N$$

We split the sum as:

$$\sum_{r=1}^{9} \left(\frac{r+3}{2^r}\right) \cdot {}^9C_r = \sum_{r=1}^{9} \left(\frac{r}{2^r} \cdot {}^9C_r\right) + \sum_{r=1}^{9} \left(\frac{3}{2^r} \cdot {}^9C_r\right)$$

Now, use the identity:

$$\frac{r}{2^r} \cdot {}^9C_r = \frac{9}{2^r} \cdot {}^8C_{r-1}$$

So,

$$\sum_{r=1}^{9} \frac{r}{2^r} \cdot {}^9C_r = 9 \sum_{r=1}^{9} \frac{1}{2^r} \cdot {}^8C_{r-1} = \frac{9}{2} \sum_{r=1}^{9} {}^8C_{r-1} \cdot \left(\frac{1}{2}\right)^{r-1}$$

Make the substitution $s = r - 1 \Rightarrow s = 0$ to 8:

$$= \frac{9}{2} \sum_{s=0}^{8} {}^8C_s \left(\frac{1}{2}\right)^s = \frac{9}{2} \cdot \left(1 + \frac{1}{2}\right)^8 = \frac{9}{2} \cdot \left(\frac{3}{2}\right)^8$$

Now for the second term:

$$\sum_{r=1}^{9} \frac{3}{2^r} \cdot {}^9C_r = 3 \left(\sum_{r=0}^{9} {}^9C_r \cdot \left(\frac{1}{2}\right)^r - {}^9C_0 \cdot 1\right) = 3 \left(\left(1 + \frac{1}{2}\right)^9 - 1\right) = 3 \left(\left(\frac{3}{2}\right)^9 - 1\right)$$

Adding both parts:

$$\frac{9}{2} \cdot \left(\frac{3}{2}\right)^8 + 3 \left(\left(\frac{3}{2}\right)^9 - 1\right) = \left(\frac{9}{2} \cdot \frac{2}{3} + 3\right) \cdot \left(\frac{3}{2}\right)^9 - 3 = (3+3) \cdot \left(\frac{3}{2}\right)^9 - 3 = 6 \left(\frac{3}{2}\right)^9 - 3$$

Thus, $\alpha = 6$, $\beta = 3$

$$(\alpha + \beta)^2 = (6 + 3)^2 = \boxed{81}$$

Quick Tip

Use the identity $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$ to simplify the summation. Also, remember the binomial expansion $(1 + x)^n = \sum_{r=0}^{n} {}^nC_r x^r$.

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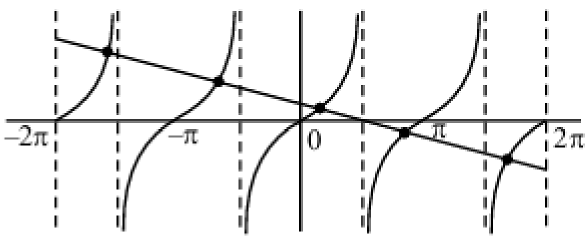
10. The number of solutions of the equation $2x + 3\tan x = \pi$, $x \in [-2\pi, 2\pi] - \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}\right\}$ is

(1) 6
(2) 5
(3) 4
(4) 3

Correct Answer: (2) 5

Solution: Given equation is $2x + 3\tan x = \pi$ Rearranging the terms, we get $\tan x = \frac{\pi - 2x}{3}$
Let $f(x) = \tan x$ and $g(x) = \frac{\pi - 2x}{3}$ We need to find the number of intersection points of these two functions in the given interval $[-2\pi, 2\pi]$.
$f(x) = \tan x$ has vertical asymptotes at $x = \pm\frac{\pi}{2}$ and $x = \pm\frac{3\pi}{2}$. $g(x) = \frac{\pi - 2x}{3}$ is a straight line with slope $-\frac{2}{3}$ and y-intercept $\frac{\pi}{3}$.
We can analyze the intersection points graphically or by analyzing intervals. In the interval $[-2\pi, 2\pi]$, we have the following intervals to consider: $[-2\pi, -\frac{3\pi}{2})$, $(-\frac{3\pi}{2}, -\frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, \frac{3\pi}{2})$, $(\frac{3\pi}{2}, 2\pi]$.
By sketching the graphs or analyzing the behavior of the functions in each interval, we can observe that there are 5 intersection points.



Therefore, the number of solutions is 5.

Quick Tip

To find the number of solutions of an equation involving trigonometric and linear functions, sketch the graphs of both functions and count the number of intersection points.

11. If $y(x) = \begin{vmatrix} \sin x & \cos x & \sin x + \cos x + 1 \\ 27 & 28 & 27 \\ 1 & 1 & 1 \end{vmatrix}$**,** $x \in R$**, then** $\frac{d^2y}{dx^2} + y$ **is equal to**

(1) -1
(2) 28
(3) 27
(4) 1

Correct Answer: (1) -1

Solution: Given $y(x) = \begin{vmatrix} \sin x & \cos x & \sin x + \cos x + 1 \\ 27 & 28 & 27 \\ 1 & 1 & 1 \end{vmatrix}$
Applying $C_3 \to C_3 - C_1$, we get $y(x) = \begin{vmatrix} \sin x & \cos x & \cos x + 1 \\ 27 & 28 & 0 \\ 1 & 1 & 0 \end{vmatrix}$
Expanding the determinant, we have $y(x) = -(\cos x + 1) \begin{vmatrix} 27 & 28 \\ 1 & 1 \end{vmatrix}$
$y(x) = -(\cos x + 1)(27 - 28)$ $y(x) = \cos x + 1$
Differentiating with respect to x, we get $\frac{dy}{dx} = -\sin x$
Differentiating again with respect to x, we get $\frac{d^2y}{dx^2} = -\cos x$
Therefore, $\frac{d^2y}{dx^2} + y = -\cos x + \cos x + 1 = 1$

Quick Tip

Use determinant properties to simplify the expression before differentiation.

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12. Let g be a differentiable function such that $\int_0^x g(t)dt = x - \int_0^x tg(t)dt$**,** $x \geq 0$ **and let** $y = y(x)$ **satisfy the differential equation** $\frac{dy}{dx} - y \tan x = 2(x+1) \sec x g(x)$**,** $x \in \left[0, \frac{\pi}{2}\right)$**. If** $y(0) = 0$**, then** $y\left(\frac{\pi}{3}\right)$ **is equal to**

(1) $\frac{2\pi}{3\sqrt{3}}$
(2) $\frac{4\pi}{3}$
(3) $\frac{2\pi}{3}$
(4) $\frac{4\pi}{3\sqrt{3}}$

Correct Answer: (2) $\frac{4\pi}{3}$

Solution: Given:

$$\int_0^x g(t)\, dt = x - \int_0^x tg(t)\, dt$$

Differentiate both sides with respect to x:

$$g(x) = 1 - xg(x) \Rightarrow g(x)(1+x) = 1 \Rightarrow g(x) = \frac{1}{1+x}$$

Now consider the differential equation:

$$\frac{dy}{dx} - y\tan x = 2(x+1)\sec x \cdot g(x) = 2(x+1)\sec x \cdot \frac{1}{1+x} = 2\sec x$$

So the equation becomes:

$$\frac{dy}{dx} - y\tan x = 2\sec x$$

This is a linear differential equation. The integrating factor is:

$$\text{IF} = e^{\int -\tan x\,dx} = e^{\ln|\cos x|} = \cos x$$

Multiplying both sides by the integrating factor:

$$\cos x \cdot \frac{dy}{dx} - y\cos x\tan x = 2 \Rightarrow \frac{d}{dx}(y\cos x) = 2$$

Integrating both sides:

$$y\cos x = \int 2\,dx = 2x + C$$

Apply the initial condition $y(0) = 0$:

$$0 \cdot \cos 0 = 2 \cdot 0 + C \Rightarrow C = 0$$

Therefore, the solution is:

$$y\cos x = 2x \Rightarrow y = \frac{2x}{\cos x} = 2x\sec x$$

Now, compute $y\left(\frac{\pi}{3}\right)$:

$$y\left(\frac{\pi}{3}\right) = 2 \cdot \frac{\pi}{3} \cdot \sec\left(\frac{\pi}{3}\right) = 2 \cdot \frac{\pi}{3} \cdot 2 = \boxed{\frac{4\pi}{3}}$$

Quick Tip

Recognize and solve the linear differential equation using the integrating factor method.

---

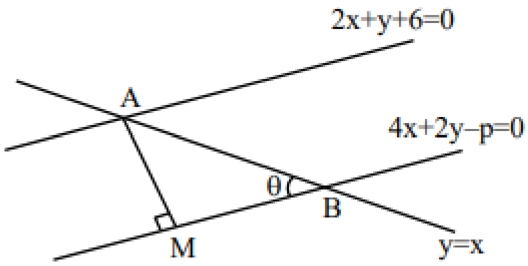
13. A line passes through the origin and makes equal angles with the positive coordinate axes. It intersects the lines $L_1 : 2x + y + 6 = 0$ and $L_2 : 4x + 2y - p = 0$, $p > 0$, at the points A and B, respectively. If $AB = \frac{9}{\sqrt{2}}$ and the foot of the perpendicular from the point A on the line L_2 is M, then $\frac{AM}{BM}$ is equal to

(1) 5
(2) 4
(3) 2

(4) 3

Correct Answer: (4) 3

Solution: The line passing through the origin and making equal angles with the positive coordinate axes is $y = x$.



To find the coordinates of A, we solve $2x + y + 6 = 0$ and $y = x$: $2x + x + 6 = 0$ $3x = -6$ $x = -2$ $y = -2$ So, A is $(-2, -2)$.
To find the coordinates of B, we solve $4x + 2y - p = 0$ and $y = x$: $4x + 2x - p = 0$ $6x = p$ $x = \frac{p}{6}$ $y = \frac{p}{6}$ So, B is $\left(\frac{p}{6}, \frac{p}{6}\right)$.
Given $AB = \frac{9}{\sqrt{2}}$, we have: $\sqrt{\left(\frac{p}{6} + 2\right)^2 + \left(\frac{p}{6} + 2\right)^2} = \frac{9}{\sqrt{2}}$ $\sqrt{2\left(\frac{p}{6} + 2\right)^2} = \frac{9}{\sqrt{2}}$ $\sqrt{2}\left|\frac{p}{6} + 2\right| = \frac{9}{\sqrt{2}}$ $\left|\frac{p}{6} + 2\right| = \frac{9}{2}$
Since $p > 0$, $\frac{p}{6} + 2 = \frac{9}{2}$ $\frac{p}{6} = \frac{9}{2} - 2 = \frac{5}{2}$ $p = 15$
Therefore, B is $\left(\frac{15}{6}, \frac{15}{6}\right) = \left(\frac{5}{2}, \frac{5}{2}\right)$.
The slope of L_2 is $m_2 = -2$. The slope of $y = x$ is $m_1 = 1$.
Let θ be the angle between the lines $y = x$ and L_2. $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{1-(-2)}{1+1(-2)}\right| = \left|\frac{3}{-1}\right| = 3$
From the geometry, $\tan\theta = \frac{AM}{BM}$. Therefore, $\frac{AM}{BM} = 3$.

Quick Tip

Use the distance formula and the formula for the angle between two lines to solve the problem.

---

14. Let $z \in C$ be such that $\frac{z+3i}{z-2+i} = 2 + 3i$. Then the sum of all possible values of z is

(1) $19 - 2i$
(2) $-19 - 2i$
(3) $19 + 2i$
(4) $-19 + 2i$

Correct Answer: (2) $-19 - 2i$

Solution: Given:

$$\frac{z + 3i}{z - 2 + i} = 2 + 3i$$

Multiply both sides by $z-2+i$:

$$z+3i=(2+3i)(z-2+i)$$

Now expand the right-hand side:

$$\begin{aligned}
z+3i &= (2+3i)(z-2+i)\\
&= 2(z-2+i)+3i(z-2+i)\\
&= 2z-4+2i+3iz-6i+3i^2\\
&= 2z+3iz-4-4i-3 \quad (\text{since } i^2=-1)\\
&= 2z+3iz-7-4i
\end{aligned}$$

Bring all terms to one side:

$$z+3i-(2z+3iz-7-4i)=0 \Rightarrow -z-3iz+10i+7=0 \Rightarrow z(1+3i)=7+7i$$

Now solve for z:

$$z=\frac{7+7i}{1+3i}=\frac{(7+7i)(1-3i)}{(1+3i)(1-3i)}$$

= 7(1 - 3i) + 7i(1 - 3i)$\frac{}{1+9=\frac{7-21i+7i-21i^2}{10}=\frac{28-14i}{10}=\frac{14-7i}{5}}$

So one value of z is:

$$z=\boxed{\frac{14-7i}{5}}$$

Now observe that the original equation:

$$\frac{z+3i}{z-2+i}=2+3i \Rightarrow z+3i=(2+3i)(z-2+i)$$

This can be written as a quadratic in z. Let's proceed:
Let's expand again:

$$z+3i=(2+3i)(z-2+i)=(2+3i)(z)+(2+3i)(-2+i)=2z+3iz+(-4+2i-6i+3i^2)=2z+3iz-4-4i-3 \quad (\text{si}$$

Now bring all terms to one side again:

$$z+3i-2z-3iz+7+4i=0 \Rightarrow -z-3iz+7+7i=0 \Rightarrow z(1+3i)=7+7i$$

Multiply both sides by $1+3i$ to form a quadratic:

$$z(1+3i)=7+7i \Rightarrow z^2+3iz=z(2+3i)-7-4i \Rightarrow z^2-(2+3i)z+7+7i=0$$

So the quadratic equation is:

$$z^2-(2+3i)z+(7+7i)=0$$

Let the roots be z_1 and z_2. Then:

$$z_1+z_2=2+3i, \quad z_1z_2=7+7i$$

Now compute:

$$\begin{aligned}
z_1^2+z_2^2 &= (z_1+z_2)^2-2z_1z_2\\
&= (2+3i)^2-2(7+7i)\\
&= 4+12i-9-14-14i\\
&= -19-2i
\end{aligned}$$

$$\boxed{z_1^2 + z_2^2 = -19 - 2i}$$

Quick Tip

If the equation leads to a quadratic, the sum of roots can be found using the coefficients.

---

15. Let $f(x) = \int x^3\sqrt{3-x^2}dx$. If $5f(\sqrt{2}) = -4$, then $f(1)$ is equal to

(1) $-\frac{2\sqrt{2}}{5}$
(2) $-\frac{8\sqrt{2}}{5}$
(3) $-\frac{4\sqrt{2}}{5}$
(4) $-\frac{6\sqrt{2}}{5}$

Correct Answer: (4) $-\frac{6\sqrt{2}}{5}$

Solution: Let $3 - x^2 = t^2$ $-2xdx = 2tdt$ $xdx = -tdt$
$f(x) = \int x^3\sqrt{3-x^2}dx$ $f(x) = \int x^2\sqrt{3-x^2}xdx$ $f(x) = \int (3-t^2)\cdot t\cdot(-tdt) + c$
$f(x) = \int (t^4 - 3t^2)dt + c$ $f(x) = \frac{t^5}{5} - t^3 + c$ $f(x) = \frac{(3-x^2)^{5/2}}{5} - (3-x^2)^{3/2} + c$
Given $5f(\sqrt{2}) = -4$, we have $5\left(\frac{(3-2)^{5/2}}{5} - (3-2)^{3/2} + c\right) = -4$ $5\left(\frac{1}{5} - 1 + c\right) = -4$
$1 - 5 + 5c = -4$ $-4 + 5c = -4$ $5c = 0$ $c = 0$
Therefore, $f(x) = \frac{(3-x^2)^{5/2}}{5} - (3-x^2)^{3/2}$
Now, we need to find $f(1)$: $f(1) = \frac{(3-1)^{5/2}}{5} - (3-1)^{3/2}$ $f(1) = \frac{2^{5/2}}{5} - 2^{3/2}$ $f(1) = 2^{3/2}\left(\frac{2}{5} - 1\right)$
$f(1) = 2^{3/2}\left(-\frac{3}{5}\right)$ $f(1) = -\frac{3}{5}\cdot 2\sqrt{2}$ $f(1) = -\frac{6\sqrt{2}}{5}$

Quick Tip

Use substitution to simplify the integral and then use the given condition to find the constant of integration.

---

16. Let $a_1, a_2, a_3, ...$ be a G.P. of increasing positive numbers. If $a_3a_5 = 729$ and $a_2 + a_4 = \frac{111}{4}$, then $24(a_1 + a_2 + a_3)$ is equal to

(1) 131
(2) 130
(3) 129
(4) 128

Correct Answer: (3) 129

Solution: Let the 1^{st} term of G.P. be a and common ratio be r. Given $a_3 a_5 = 729$
$ar^2 \cdot ar^4 = 729$ $a^2r^6 = 729$ $ar^3 = 27$ ... (i)
Also, $a_2 + a_4 = \frac{111}{4}$ $ar + ar^3 = \frac{111}{4}$ $ar = \frac{111}{4} - 27 = \frac{111-108}{4} = \frac{3}{4}$ ... (ii)
Dividing (i) by (ii): $\frac{ar^3}{ar} = \frac{27}{3/4}$ $r^2 = 36$ $r = 6$ (since the G.P. is increasing, r ¿ 0)
From (ii): $a(6) = \frac{3}{4}$ $a = \frac{1}{8}$
Now, $24(a_1 + a_2 + a_3) = 24(a + ar + ar^2) = 24a(1 + r + r^2) = 24 \cdot \frac{1}{8}(1 + 6 + 36) = 3(43) = 129$

Quick Tip

Use the given conditions to form equations and solve for the first term and common ratio of the G.P.

---

17. Let the domain of the function $f(x) = \log_2 \log_4 \log_6(3 + 4x - x^2)$ be (a, b). If $\int_0^{a+b}[x^2]dx = p - q\sqrt{r}$, $p, q, r \in N$, gcd(p, q, r) = 1, where [.] is the greatest integer function, then p + q + r is equal to

(1) 10
(2) 8
(3) 11
(4) 9

Correct Answer: (1) 10

Solution: $\log_6(3 + 4x - x^2) > 1$ $3 + 4x - x^2 > 6$ $x^2 - 4x + 3 < 0$ $(x - 1)(x - 3) < 0$ $x \in (1, 3)$
So $a = 1$ and $b = 3$
$\Rightarrow \int_0^2 [x^2]dx = ?$
$I = \int_0^1 [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]dx + \int_{\sqrt{3}}^2 [x^2]dx$
$= 0 + |x|_1^{\sqrt{2}} + 2|x|_{\sqrt{2}}^{\sqrt{3}} + 3|x|_{\sqrt{3}}^2$
$= (\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3})$
$= 5 - \sqrt{2} - \sqrt{3}$
$\Rightarrow p + q + r = 10$

Quick Tip

Find the domain of the function by solving the inequalities and then evaluate the integral by splitting it into intervals where $[x^2]$ is constant.

---

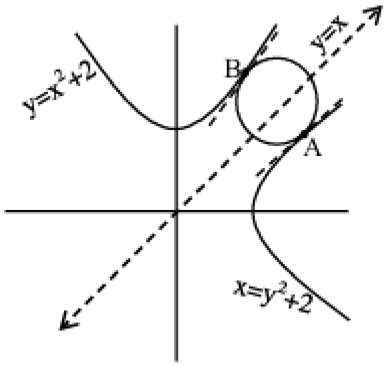
18. The radius of the smallest circle which touches the parabolas $y = x^2 + 2$ and $x = y^2 + 2$ is

(1) $\frac{7\sqrt{2}}{2}$

(2) $\frac{7\sqrt{2}}{16}$
(3) $\frac{7\sqrt{2}}{4}$
(4) $\frac{7\sqrt{2}}{8}$

Correct Answer: (4) $\frac{7\sqrt{2}}{8}$

Solution: The given parabolas are symmetric about the line $y = x$.



Tangents at A and B must be parallel to $y = x$ line, so slope of the tangents = 1.
$\left(\frac{dy}{dx}\right)_{\min A} = 1 = \left(\frac{dy}{dx}\right)_{\min B}$
For $y = x^2 + 2$, $\frac{dy}{dx} = 2x$ $2x = 1$ $x = \frac{1}{2}$ $y = \left(\frac{1}{2}\right)^2 + 2 = \frac{1}{4} + 2 = \frac{9}{4}$ So, point A is $\left(\frac{1}{2}, \frac{9}{4}\right)$.
For $x = y^2 + 2$, $1 = 2y\frac{dy}{dx}$ $\frac{dy}{dx} = \frac{1}{2y}$ $\frac{1}{2y} = 1$ $y = \frac{1}{2}$ $x = \left(\frac{1}{2}\right)^2 + 2 = \frac{1}{4} + 2 = \frac{9}{4}$ So, point B is $\left(\frac{9}{4}, \frac{1}{2}\right)$.
Distance between A and B: $AB = \sqrt{\left(\frac{9}{4} - \frac{1}{2}\right)^2 + \left(\frac{1}{2} - \frac{9}{4}\right)^2}$ $AB = \sqrt{2\left(\frac{7}{4}\right)^2}$ $AB = \frac{7\sqrt{2}}{4}$
The radius of the smallest circle is half of the distance AB. Radius $= \frac{AB}{2} = \frac{7\sqrt{2}}{8}$

Quick Tip

Use the symmetry of the parabolas about the line $y = x$ to find the points of tangency and then calculate the distance between them.

---

19. Let $f(x) = \begin{cases} (1+ax)^{1/x} & ,x < 0 \\ 1+b & ,x = 0 \\ \frac{(x+4)^{1/2}-2}{(x+c)^{1/3}-2} & ,x > 0 \end{cases}$ **be continuous at x = 0. Then** $e^a bc$ **is equal to**

(1) 64
(2) 72
(3) 48
(4) 36

Correct Answer: (3) 48

Solution: For continuity at x = 0, we need $f(0^-) = f(0) = f(0^+)$.

$f(0^-) = \lim_{x\to 0^-}(1+ax)^{1/x} = e^{\lim_{x\to 0^-}\frac{1}{x}\ln(1+ax)} = e^{\lim_{x\to 0^-}\frac{ax}{x}} = e^a$
$f(0) = 1+b$
$f(0^+) = \lim_{x\to 0^+}\frac{(x+4)^{1/2}-2}{(x+c)^{1/3}-2}$ Using L'Hopital's rule: $f(0^+) = \lim_{x\to 0^+}\frac{\frac{1}{2}(x+4)^{-1/2}}{\frac{1}{3}(x+c)^{-2/3}}$
$f(0^+) = \frac{\frac{1}{2}(4)^{-1/2}}{\frac{1}{3}(c)^{-2/3}} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{3}c^{-2/3}} = \frac{\frac{1}{4}}{\frac{1}{3}c^{-2/3}} = \frac{3}{4}c^{2/3}$
From $f(0^-) = f(0)$, we have $e^a = 1+b$. From $f(0) = f(0^+)$, we have $1+b = \frac{3}{4}c^{2/3}$.
Also, we know that if $(x+c)^{1/3}-2$ is in the denominator, then $(x+c)^{1/3}-2 = 0$ at x = 0.
$c^{1/3}-2 = 0$ $c^{1/3} = 2$ $c = 8$
Now, $1+b = \frac{3}{4}(8)^{2/3} = \frac{3}{4}(2^3)^{2/3} = \frac{3}{4}\cdot 4 = 3$ $b = 2$
Also, $e^a = 1+b = 3$ $a = \ln 3$
Therefore, $e^a bc = 3\cdot 2\cdot 8 = 48$

Quick Tip

For continuity, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$. Use L'Hopital's rule when dealing with indeterminate forms.

---

20. Line L_1 passes through the point (1, 2, 3) and is parallel to z-axis. Line L_2 passes through the point $(\lambda, 5, 6)$ and is parallel to y-axis. Let for $\lambda = \lambda_1, \lambda_2, \lambda_2 < \lambda_1$, the shortest distance between the two lines be 3. Then the square of the distance of the point $(\lambda_1, \lambda_2, 7)$ from the line L_1 is

(1) 40
(2) 32
(3) 25
(4) 37

Correct Answer: (3) 25

Solution: The equation of line L_1 is: $\frac{x-1}{0} = \frac{y-2}{0} = \frac{z-3}{1}$
The equation of line L_2 is: $\frac{x-\lambda}{0} = \frac{y-5}{1} = \frac{z-6}{0}$
The shortest distance (SD) between the lines is given by: $SD = \frac{\begin{vmatrix}\lambda-1 & 5-2 & 6-3\\ 0 & 0 & 1\\ 0 & 1 & 0\end{vmatrix}}{\sqrt{0^2+1^2+0^2}}$
$SD = \begin{vmatrix}\lambda-1 & 3 & 3\\ 0 & 0 & 1\\ 0 & 1 & 0\end{vmatrix}$ $SD = |\lambda-1|$
Given SD = 3, we have: $|\lambda-1| = 3$ $\lambda-1 = \pm 3$ $\lambda = 4, -2$
Since $\lambda_2 < \lambda_1$, we have: $\lambda_1 = 4$ $\lambda_2 = -2$
The point is $P(4, -2, 7)$.
Let Q be the foot of the perpendicular from P to L_1. Q is of the form $(1, 2, 3+t)$.
The direction vector of PQ is $(1-4, 2-(-2), 3+t-7) = (-3, 4, t-4)$. The direction vector of L_1 is $(0, 0, 1)$.

Since PQ is perpendicular to L_1, their dot product is 0. $(-3, 4, t-4) \cdot (0, 0, 1) = 0$ $t - 4 = 0$ $t = 4$
So, Q is $(1, 2, 7)$.
Now, $PQ^2 = (4-1)^2 + (-2-2)^2 + (7-7)^2$ $PQ^2 = 3^2 + (-4)^2 + 0^2$ $PQ^2 = 9 + 16 = 25$

Quick Tip

Use the formula for the shortest distance between two skew lines. To find the foot of the perpendicular, use the dot product of the direction vectors.

---

21. All five letter words are made using all the letters A, B, C, D, E and arranged as in an English dictionary with serial numbers. Let the word at serial number n be denoted by W_n. Let the probability $P(W_n)$ of choosing the word W_n satisfy $P(W_n) = 2P(W_{n-1})$, $n > 1$. If $P(CDBEA) = \frac{2^\alpha}{2^\beta - 1}$, $\alpha, \beta \in N$, then $\alpha + \beta$ is equal to :

Correct Answer: (183)

Solution: Let $P(W_1) = x$. Given $P(W_n) = 2P(W_{n-1})$. Then, $P(W_2) = 2x$, $P(W_3) = 2^2x$, ..., $P(W_n) = 2^{n-1}x$.
Since $\sum_{i=1}^{120} P(W_i) = 1$, we have

$$x + 2x + 2^2x + \cdots + 2^{119}x = 1$$
$$x(1 + 2 + 2^2 + \cdots + 2^{119}) = 1$$
$$x \cdot \frac{2^{120} - 1}{2 - 1} = 1$$
$$x(2^{120} - 1) = 1$$
$$x = \frac{1}{2^{120} - 1} \tag{1}$$

Now, let's find the rank of CDBEA. A_ _ _ _ $= 4! = 24$ B_ _ _ _ $= 4! = 24$ CA_ _ _ $= 3! = 6$ CB_ _ _ $= 3! = 6$ CDA_ _ $= 2! = 2$ CDBAE $= 1$ CDBEA $= 1$
So, the rank of CDBEA is

$$24 + 24 + 6 + 6 + 2 + 1 = 63$$

Thus, CDBEA is W_{64}.
Therefore,

$$P(CDBEA) = P(W_{64}) = 2^{63} \cdot P(W_1) = 2^{63} \cdot \frac{1}{2^{120} - 1}$$
$$P(CDBEA) = \frac{2^{63}}{2^{120} - 1}$$

Comparing with $P(CDBEA) = \frac{2^\alpha}{2^\beta - 1}$, we have:

$$\alpha = 63, \quad \beta = 120$$
$$\alpha + \beta = 63 + 120 = \boxed{183}$$

Quick Tip

Use the given recursive relation to find the probability of each word. Calculate the rank of the given word to find its serial number.

---

22. Let the product of the focal distances of the point P(4, $2\sqrt{3}$) on the hyperbola H: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be 32. Let the length of the conjugate axis of H be p and the length of its latus rectum be q. Then $p^2 + q^2$ is equal to .....

Correct Answer: (120)

Solution: Given hyperbola $H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (1)
Point $P(4, 2\sqrt{3})$ lies on H. Given $PS_1 \cdot PS_2 = 32$ Also, $|PS_1 - PS_2| = 2a$
Since $P(4, 2\sqrt{3})$ lies on H:

$$\frac{16}{a^2} - \frac{12}{b^2} = 1$$

$$16b^2 - 12a^2 = a^2b^2 \tag{2}$$

$$|PS_1 - PS_2|^2 = 4a^2$$

$$PS_1^2 + PS_2^2 - 2 \cdot PS_1 \cdot PS_2 = 4a^2$$

$$(ae - 4)^2 + 12 + (ae + 4)^2 + 12 - 2(32) = 4a^2$$

$$2a^2e^2 - 8 = 4a^2$$

$$a^2e^2 - 4 = 2a^2$$

$$b^2 = a^2(e^2 - 1) = 2a^2 \Rightarrow b^2 - a^2 = 4 \tag{3}$$

From (2) and (3):

$$16(a^2 + 4) - 12a^2 = a^2(a^2 + 4)$$

$$16a^2 + 64 - 12a^2 = a^4 + 4a^2$$

$$a^4 = 64 \Rightarrow a^2 = 8$$

$$b^2 = 12$$

$$p = 2b \quad \text{(length of conjugate axis)}$$

$$q = \frac{2b^2}{a} \quad \text{(length of latus rectum)}$$

$$p^2 + q^2 = 4b^2 + \frac{4b^4}{a^2}$$

$$p^2 + q^2 = 4(12) + \frac{4(12^2)}{8}$$

$$p^2 + q^2 = 48 + 72 = \boxed{120}$$

Quick Tip

Use the properties of hyperbola, including the relationship between focal distances and the equation of the hyperbola, to solve for the required values.

---

23. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$, $\vec{c} = \lambda\hat{j} + \mu\hat{k}$ and $\hat{d}$ be a unit vector such that $\vec{a} \times \hat{d} = \vec{b} \times \hat{d}$ and $\vec{c} \cdot \hat{d} = 1$. If $\vec{c}$ is perpendicular to $\vec{a}$, then $|3\lambda\hat{d} + \mu\vec{c}|^2$ is equal to ____.

Correct Answer: (5)

Solution: Given $\vec{a} \times \hat{d} = \vec{b} \times \hat{d}$

$$\vec{a} \times \hat{d} - \vec{b} \times \hat{d} = 0$$

$$(\vec{a} - \vec{b}) \times \hat{d} = 0$$

This means $\vec{a} - \vec{b}$ is parallel to $\hat{d}$.

$$\vec{a} - \vec{b} = (\hat{i} + \hat{j} + \hat{k}) - (3\hat{i} + 2\hat{j} - \hat{k}) = -2\hat{i} - \hat{j} + 2\hat{k}$$

Let

$$\hat{d} = \frac{\vec{a} - \vec{b}}{|\vec{a} - \vec{b}|} = \frac{-2\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{(-2)^2 + (-1)^2 + 2^2}} = \frac{-2\hat{i} - \hat{j} + 2\hat{k}}{3}$$

Given $\vec{c} \cdot \hat{d} = 1$:

$$(\lambda\hat{j} + \mu\hat{k}) \cdot \frac{-2\hat{i} - \hat{j} + 2\hat{k}}{3} = 1$$

$$\frac{-\lambda + 2\mu}{3} = 1$$

$$-\lambda + 2\mu = 3 \tag{1}$$

Given $\vec{c}$ is perpendicular to $\vec{a}$, so $\vec{c} \cdot \vec{a} = 0$:

$$(\lambda\hat{j} + \mu\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 0$$

$$\lambda + \mu = 0 \Rightarrow \mu = -\lambda \tag{2}$$

Substituting (2) in (1), we get:

$$-\lambda - 2\lambda = 3$$

$$-3\lambda = 3 \Rightarrow \lambda = -1, \quad \mu = 1$$

So,

$$\vec{c} = -\hat{j} + \hat{k}$$

Now,

$$3\lambda\hat{d} + \mu\vec{c} = 3(-1) \cdot \frac{-2\hat{i} - \hat{j} + 2\hat{k}}{3} + (-\hat{j} + \hat{k})$$

$$= 2\hat{i} + \hat{j} - 2\hat{k} - \hat{j} + \hat{k} = 2\hat{i} - \hat{k}$$

$$|3\lambda\hat{d} + \mu\vec{c}|^2 = |2\hat{i} - \hat{k}|^2 = 2^2 + (-1)^2 = 4 + 1 = \boxed{5}$$

Quick Tip

Use the given vector relations to find the vectors $\hat{d}$ and $\vec{c}$. Remember that if $\vec{a} \times \vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are parallel.

---

24. If the number of seven-digit numbers, such that the sum of their digits is even, is $m \cdot n \cdot 10^a$; $m, n \in \{1, 2, 3, ..., 9\}$, then $m + n$ is equal to

Correct Answer: (14)

Solution: Total 7 digit numbers = 9000000
7 digit numbers having sum of digits even = 4500000 = $9.5 \cdot 10^5$
$m = 9$, $n = 5$
$m + n = 14$

Quick Tip

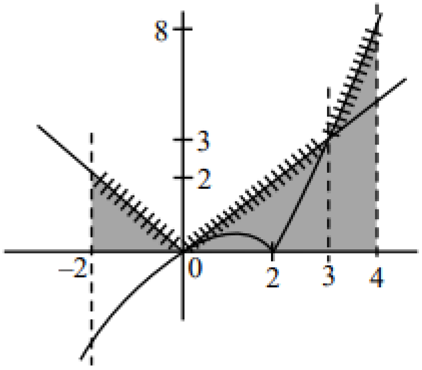
Half of the total 7 digit numbers will have an even sum of digits.

---

25. The area of the region bounded by the curve $y = \max\{|x|, |x - 2|\}$, then x-axis and the lines x = -2 and x = 4 is equal to ____.

Correct Answer: (12)

Solution: As given in the picture, the area is calculated as:



Required Area $= \frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 3 \times 3 + \frac{1}{2} \times 1 \times 11$
Required Area $= 2 + \frac{9}{2} + \frac{11}{2}$
Required Area $= 2 + \frac{20}{2}$
Required Area $= 2 + 10$
Required Area $= 12$
Thus, following the given solution, the area is 12.

Quick Tip

The solution provided in the picture calculates the area as a sum of triangular areas.
