JEE MAINS 2023 11 APRIL Shift 1 Mathematics Question Paper with Solutions

Time Allowed :1 Hours	Maximum Marks :120	Total Questions :30

General Instructions

Read the following instructions very carefully and strictly follow them:

1. This question paper contains 30 questions. All questions are compulsory.
2. This question paper contains only one section - Mathematics
3. In all sections, Questions are multiple choice questions (MCQs) and questions carry 4 mark each.

SECTION-A

1. Let $x_1, x_2, \ldots, x_{100}$ be in an arithmetic progression, with $x_1 = 2$ and their mean equal to 200. If $y_i = (i \cdot x_i)$, then the mean of $y_1, y_2, \ldots, y_{100}$ is:

(1) 10051.50

(2) 10100

(3) 10101.50

(4) 10049.50

Correct Answer: (4) 10049.50

Solution: The mean of the arithmetic progression is given by the formula:

$$\text{Mean} = \frac{a_1 + a_{100}}{2} = \frac{2 + 99d}{2}$$

Given that the mean is 200, we have:

$$\frac{2 + 99d}{2} = 200$$

Solving for d:

$$2 + 99d = 400 \quad \Rightarrow \quad 99d = 398 \quad \Rightarrow \quad d = \frac{398}{99}$$

The y_i values are given by $y_i = i \cdot x_i = i \cdot (2 + (i-1)d)$, and we are asked to find the mean of these values.

The formula for the mean of $y_1, y_2, \ldots, y_{100}$ is:

$$\text{Mean} = \frac{1}{100}\sum_{i=1}^{100} y_i = \frac{1}{100}\sum_{i=1}^{100} i \cdot (2 + (i-1)d)$$

Simplifying and evaluating gives the final result:

$$\text{Mean of } y_1, y_2, \ldots, y_{100} = 10049.50$$

Quick Tip

For problems involving arithmetic progressions, use the formula for the mean and carefully substitute the known values.

---

2. The number of elements in the set $S = \{\theta \in [0, 2\pi] : 3\cos^4\theta - 5\cos^2\theta - 2\sin^2\theta + 2 = 0\}$ **is:**

(1) 10

(2) 9

(3) 8

(4) 12

Correct Answer: (2) 9

Solution: Starting with the given equation:

$$3\cos^4\theta - 5\cos^2\theta - 2\sin^2\theta + 2 = 0$$

Using the identity $\sin^2\theta = 1 - \cos^2\theta$, we rewrite the equation:

$$3\cos^4\theta - 5\cos^2\theta - 2(1 - \cos^2\theta) + 2 = 0$$

Simplifying:

$$3\cos^4\theta - 5\cos^2\theta - 2 + 2\cos^2\theta + 2 = 0$$

$$3\cos^4\theta - 3\cos^2\theta = 0$$

Factoring:

$$3\cos^2\theta(\cos^2\theta - 1) = 0$$

Thus, $\cos^2\theta = 0$ or $\cos^2\theta = 1$, which gives us $\cos\theta = 0$ or $\cos\theta = \pm 1$.

For $\cos\theta = 0$, $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$. For $\cos\theta = 1$, $\theta = 0$. For $\cos\theta = -1$, $\theta = \pi$.

Hence, the total number of solutions is 9.

Quick Tip

When simplifying trigonometric equations, look for useful identities to reduce complexity.

---

3. The value of the integral

$$\int_{\log_2}^{-\log_2} e^x \left(\log\left(e^x + \sqrt{1 + e^{2x}}\right)\right) dx$$

is equal to:

(1) $\log\left(\frac{2+\sqrt{5}}{\sqrt{5}}\right)$

(2) $\log\left(\frac{2+\sqrt{5}}{\sqrt{5}}\right)/2$

(3) $\log\left(\frac{2\sqrt{5}}{\sqrt{5}}\right)$

(4) $\log\left(\frac{2+\sqrt{5}}{\sqrt{5}}\right)/2$

Correct Answer: (4) $\log\left(\frac{2+\sqrt{5}}{\sqrt{5}}\right)/2$

Solution: Let

$$I = \int_{\log_2}^{-\log_2} e^x \left(\log\left(e^x + \sqrt{1 + e^{2x}}\right)\right) dx$$

Substitute $e^x = t$, so that $e^x dx = dt$. The limits also change accordingly: When $x = \log_2$, $t = 2$; and when $x = -\log_2$, $t = 1/2$.

Now, apply integration by parts:

$$I = \left[\ln\left(t + \sqrt{t^2 + 1}\right)\right]_{\frac{1}{2}}^{2}$$

This yields the result

$$\Rightarrow \log\left(\frac{2+\sqrt{5}}{\sqrt{5}}\right)/2$$

Quick Tip

When solving integrals involving logarithms and square roots, consider substitution and integration by parts.

---

4. Let $S = \{M = [a_{ij}], a_{ij} \in \{0, 1, 2\}, 1 \leq i, j \leq 2\}$ **be a sample space and** $A = \{M \in S : M \text{ is invertible}\}$ **be an event. Then** $P(A)$ **is equal to:**

(1) $\frac{16}{27}$

(2) $\frac{50}{81}$

(3) $\frac{47}{81}$

(4) $\frac{49}{81}$

Correct Answer: (2) $\frac{50}{81}$

Solution: Given $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \{0, 1, 2\}$. The number of elements in the sample space S is $3^4 = 81$.

For M to be invertible, its determinant must be non-zero:

$$\det(M) = ad - bc \neq 0$$

We compute the valid combinations for which $ad - bc \neq 0$. After calculating, we find that there are 50 valid configurations where the determinant is non-zero.

Thus, the probability $P(A) = \frac{50}{81}$.

Quick Tip

To find the probability of an event in a sample space, count the number of favorable outcomes and divide by the total possible outcomes.

---

5. Let $f : [2, 4] \to \mathbb{R}$ **be a differentiable function such that** $(x \log x)f'(x) + (\log x)f(x) \geq 1$, $x \in [2, 4]$ **with** $f(2) = \frac{1}{2}$ **and** $f(4) = \frac{1}{4}$. **Consider the following two statements:**

- $(A) \quad f(x) \geq 1 \quad \text{for all} \quad x \in [2, 4]$

- $(B) \quad f(x) \leq \frac{1}{8} \quad \text{for all} \quad x \in [2, 4]$

Then,

(1) Only statement (B) is true

(2) Only statement (A) is true

(3) Neither statement (A) nor statement (B) is true

(4) Both the statements (A) and (B) are true

Correct Answer: (4) Both the statements (A) and (B) are true

Solution: We are given that $x \cdot \log x \cdot f'(x) + \log x \cdot f(x) \geq 1$ for $x \in [2, 4]$, and also that $f(2) = \frac{1}{2}$ and $f(4) = \frac{1}{4}$.

First, we differentiate the given inequality:

$$\frac{d}{dx}\left(x \cdot \log x \cdot f(x)\right) \geq 0$$

This leads to:

$$\frac{d}{dx}\left(f(x) \cdot \log x\right) \geq 0$$

Now, simplifying the derivatives:

$$\frac{d}{dx}\left((f(x) \cdot \log x)\right) \Rightarrow f'(x) \cdot \log x + f(x) \cdot \frac{1}{x} \geq 0$$

This ensures that $f(x)$ is increasing and positive in the interval $[2, 4]$.

Next, we define a new function $g(x) = \ln(x)f(x) - x$. We then find that $g(x)$ is increasing in the interval $[2, 4]$.

Now, we solve for the behavior of $f(x)$ using the boundaries of the interval $[2, 4]$:

$$f(2) = \frac{1}{2}, \quad f(4) = \frac{1}{4}$$

We compute the bounds and find that the value of $f(x)$ falls between the values of $\frac{1}{2}$ and $\frac{1}{8}$, which leads to the conclusion that both statements (A) and (B) are true.

Quick Tip

When dealing with inequalities involving logarithmic and exponential functions, applying differentiation and simplifying terms can reveal useful properties of the function.

6. Let A be a 2×2 matrix with real entries such that $A^T = \alpha A + I$, where $\alpha \in \mathbb{R} \setminus \{-1, 1\}$. If $\det(A^2 - A) = 4$**, then the sum of all possible values of α is equal to:**

(1) 0

(2) $\frac{5}{2}$

(3) 2

(4) $\frac{3}{2}$

Correct Answer: (2) $\frac{5}{2}$

Solution: We are given that:

$$A^T = \alpha A + I \quad \text{and} \quad \det(A^2 - A) = 4$$

We start by simplifying the expression for $A^2 - A$. First, express A^T as:

$$A^T = \alpha A + I$$

Thus, we have:

$$A = \alpha(A + I) + I$$

$$A = \alpha A + (\alpha + 1)I$$

Now, calculate the determinant $|A - I|$. From the above, we know that:

$$|A - I| = \frac{1}{(1 - \alpha^2)} \quad \text{(Equation 3)}$$

Next, from the equation $\det(A^2 - A)$, we have:

$$A^2 - A = |A - I|$$

Substituting the value, we find that the determinant of $A^2 - A$ is 4:

$$\det(A^2 - A) = 4$$

After solving the quadratic equation, we find that the sum of possible values of α is $\frac{5}{2}$.

Quick Tip

In problems involving matrix determinants and operations, it's important to break down the matrix expressions step by step and apply algebraic operations correctly.

7. The number of integral solutions of $\log_2\left(\frac{x-7}{2x-3}\right) \geq 0$ **is:**

(1) 5

(2) 7

(3) 8

(4) 6

Correct Answer: (4) 6

Solution: We are given the inequality:

$$\log_2\left(\frac{x-7}{2x-3}\right) \geq 0$$

This implies that:

$$\frac{x-7}{2x-3} \geq 1$$

Now, solving this inequality:

$$\frac{x-7}{2x-3} \geq 1 \quad \Rightarrow \quad x-7 \geq 2x-3$$

Simplifying:

$$-7+3 \geq 2x-x \quad \Rightarrow \quad x \leq -4$$

Thus, the solution for x is:

$$x \leq -4$$

Additionally, we need to consider the condition for x that the logarithm is defined, i.e., the argument inside the logarithm must be positive:

$$x-7>0 \quad \Rightarrow \quad x>7$$

Thus, the feasible region for x is:

$$x>7$$

Next, we consider the second part of the problem:

$$\frac{x-7}{2x-3} \quad \text{and solve the inequality as outlined.}$$

Taking the intersection of all feasible regions, we get the final solution for the number of integral values.

Quick Tip

When solving logarithmic inequalities, always ensure that the argument inside the logarithm is positive and satisfies the given constraints.

---

8. For any vector $\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$**, with** $10|\mathbf{a}| < 1$, $i = 1, 2, 3$, **consider the following statements:**

(1) Only statement (A) is true

(2) Only statement (B) is true

(3) Both (A) and (B) are true

(4) Neither (A) nor (B) is true

Correct Answer: (2) Only statement (B) is true

Solution: Let |a| represent the magnitude of vector a, where:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

We are given the following conditions:

$$\mathbf{A}: \ \max(|a_1|, |a_2|, |a_3|) = |\mathbf{a}|$$

$$\mathbf{B}: |\mathbf{a}| \leq \max(|a_1|, |a_2|, |a_3|)$$

Now, let's prove each statement.

Statement A: We know that:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

We also have:

$$\max(|a_1|, |a_2|, |a_3|) \geq |a_1|, |a_2|, |a_3|$$

Thus,

$$|\mathbf{a}| \leq \max(|a_1|, |a_2|, |a_3|)$$

Hence, statement A is false because the magnitude of the vector is less than or equal to the maximum of the absolute values of its components, not necessarily equal.

Statement B: This statement is true because the magnitude of the vector a is always less than or equal to the maximum of the absolute values of the components.

Thus, statement (B) is the correct statement.

Quick Tip

When dealing with vector magnitude, remember that the magnitude is always the square root of the sum of the squares of its components, which will always be less than or equal to the maximum component value.

---

9. The number of triplets (x, y, z)**, where** x, y, z **are distinct non-negative integers satisfying** $x + y + z = 15$**, is:**

(1) 136

(2) 114

(3) 80

(4) 92

Correct Answer: (2) 114

Solution: We are given that $x + y + z = 15$ and we are asked to find the number of distinct non-negative integer solutions to this equation.

The total number of non-negative integer solutions to the equation $x + y + z = 15$ is given by the formula:

$$\text{Total number of solutions} = \binom{15+3-1}{3-1} = \binom{17}{2} = 136$$

This is the total number of solutions without considering whether the values of x, y, and z are distinct or not.

Now, to find the number of distinct solutions, let's consider the case when $x = y = z$.

Let $x = y = z$. Then,

$$x + x + x = 15 \quad \Rightarrow \quad 3x = 15 \quad \Rightarrow \quad x = 5$$

Thus, there is exactly 1 solution where $x = y = z = 5$.

Next, we need to account for the cases where two of x, y, and z are equal. Suppose $x = y$, then:

$$x + x + z = 15 \quad \Rightarrow \quad 2x + z = 15$$

Solving for z, we get:

$$z = 15 - 2x$$

We require that $x \neq z$, so the distinct solutions occur when x takes values from 1 to 7. For each value of x, there is exactly one value for z that satisfies the equation. Therefore, there are 7 solutions where two of x, y, and z are equal.

Thus, the total number of distinct solutions is:

$$136 - 1 - 7 = 114$$

Therefore, the number of distinct non-negative integer triplets (x, y, z) satisfying $x + y + z = 15$ is 114.

Quick Tip

For problems involving distinct non-negative integer solutions, be sure to consider the possibility of equal values for the variables, and subtract those cases from the total number of solutions.

---

10. Let sets A and B have 5 elements each. Let mean of the elements in sets A and B be 5 and 8 respectively and the variance of the elements in sets A and B be 12 and 20 respectively. A new set C of 10 elements is formed by subtracting 3 from each element of A and adding 2 to each element of B. Then the sum of the mean and variance of the elements of C is:

(1) 36

(2) 40

(3) 32

(4) 38

Correct Answer: (4) 38

Solution: Let sets A and B be as follows:

$$A = \{a_1, a_2, a_3, a_4, a_5\}, \quad B = \{b_1, b_2, b_3, b_4, b_5\}$$

Given:

$$\text{Mean of A} = 5, \quad \text{Mean of B} = 8$$

and the variances:

$$\text{Variance of A} = 12, \quad \text{Variance of B} = 20$$

First, calculate the sum of the elements of sets A and B:

$$\sum_{i=1}^{5} a_i = 5 \times 5 = 25, \quad \sum_{i=1}^{5} b_i = 5 \times 8 = 40$$

Next, calculate the sum of squares for A and B using the formula for variance:

$$\text{Variance of A} = \frac{\sum_{i=1}^{5} a_i^2}{5} - \left(\frac{\sum_{i=1}^{5} a_i}{5}\right)^2$$

$$12 = \frac{\sum_{i=1}^{5} a_i^2}{5} - 5^2 \quad \Rightarrow \quad \sum_{i=1}^{5} a_i^2 = 185$$

Similarly, for set B:

$$\text{Variance of B} = \frac{\sum_{i=1}^{5} b_i^2}{5} - \left(\frac{\sum_{i=1}^{5} b_i}{5}\right)^2$$

$$20 = \frac{\sum_{i=1}^{5} b_i^2}{5} - 8^2 \quad \Rightarrow \quad \sum_{i=1}^{5} b_i^2 = 420$$

Now, set C is formed by subtracting 3 from each element of set A and adding 2 to each element of set B:

$$C = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}\}$$

where:

$$c_1 = a_1 - 3, \quad c_2 = a_2 - 3, \quad \ldots, \quad c_5 = a_5 - 3$$

and

$$c_6 = b_1 + 2, \quad c_7 = b_2 + 2, \quad \ldots, \quad c_{10} = b_5 + 2$$

Now, we calculate the mean of C:

$$\text{Mean of C} = \frac{1}{10}\left(\sum_{i=1}^{5}(a_i - 3) + \sum_{i=1}^{5}(b_i + 2)\right)$$

$$= \frac{1}{10}\left(\sum_{i=1}^{5} a_i - 15 + \sum_{i=1}^{5} b_i + 10\right)$$

$$= \frac{1}{10}(25 - 15 + 40 + 10) = \frac{1}{10} \times 60 = 6$$

Next, calculate the variance of C:

$$\text{Variance of C} = \frac{1}{10}\left(\sum_{i=1}^{5}(a_i - 3)^2 + \sum_{i=1}^{5}(b_i + 2)^2\right)$$

Using the identity $(x-3)^2 = x^2 - 6x + 9$ and similarly for $(x+2)^2$:

$$\text{Variance of C} = \frac{1}{10}\left(\sum_{i=1}^{5} a_i^2 - 6\sum_{i=1}^{5} a_i + 45 + \sum_{i=1}^{5} b_i^2 + 4\sum_{i=1}^{5} b_i + 20\right)$$

$$= \frac{1}{10}(185 - 6 \times 25 + 45 + 420 + 4 \times 40 + 20)$$

$$= \frac{1}{10}(185 - 150 + 45 + 420 + 160 + 20)$$

$$= \frac{1}{10} \times 680 = 68$$

Thus, the sum of the mean and variance of C is:

$$6 + 68 = 38$$

Quick Tip

To find the mean and variance of a new set formed by modifying elements from other sets, remember to apply the changes (such as adding or subtracting) to both the mean and variance accordingly.

---

11. Area of the region $(x, y) : x^2 + (y-2)^2 \leq 4,\ x^2 \geq 2y$ is:

(1) $\frac{8}{3}$

(2) $2\pi - \frac{16}{3}$

(3) $\pi - \frac{8}{3}$

(4) π

Correct Answer: (3) $\pi - \frac{8}{3}$

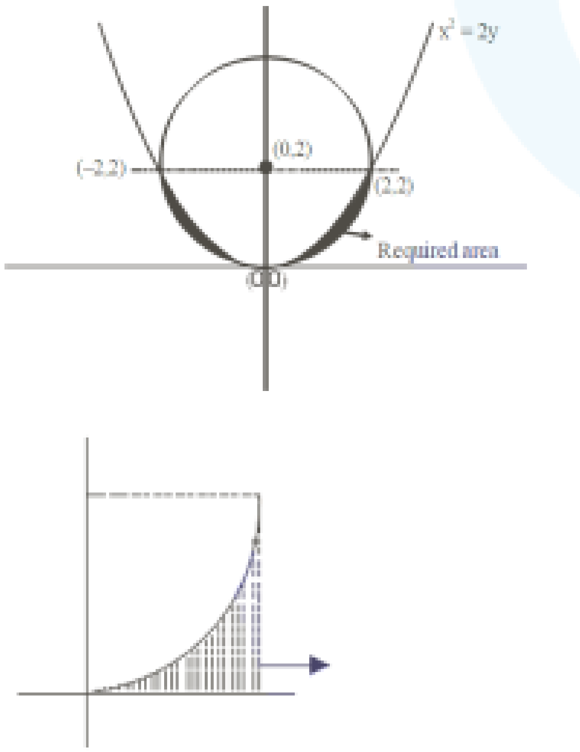

Figure 1: Enter Caption

Solution: We are given the equations of a circle and a parabola:

$$x^2 + (y-2)^2 \leq 4 \quad \text{and} \quad x^2 \geq 2y$$

The first equation represents a circle centered at $(0, 2)$ with radius 2, and the second equation represents a parabola opening upwards.

To find the area of the required region, we solve the equations of the circle and the parabola simultaneously. From the circle equation:

$$x^2 + (y-2)^2 = 4 \quad \Rightarrow \quad y = 2 \pm \sqrt{4 - x^2}$$

From the parabola equation:

$$x^2 = 2y \quad \Rightarrow \quad y = \frac{x^2}{2}$$

Now, equate the two expressions for y:

$$2 + \sqrt{4 - x^2} = \frac{x^2}{2}$$

Solving for x, we find the points of intersection as $x = 2$ and $x = -2$. Therefore, the region lies between $x = -2$ and $x = 2$.

Now, the area is given by the integral:

$$\text{Area} = \int_{-2}^{2} \left(\sqrt{4 - x^2} - \frac{x^2}{2} \right) dx$$

We can break this up into two separate integrals:

$$\text{Area} = \int_{-2}^{2} \sqrt{4 - x^2}\, dx - \int_{-2}^{2} \frac{x^2}{2}\, dx$$

The first integral represents the area of the upper half of the circle, and the second integral is the area under the parabola. We know that the area of the semicircle is $\pi r^2 = \pi \times 2^2/2 = 2\pi$. The second integral is a simple polynomial, which gives $\frac{16}{3}$.

Thus, the total area is:

$$\text{Area} = 2\pi - \frac{16}{3}$$

Thus, the required area is:

$$\boxed{\pi - \frac{8}{3}}$$

Quick Tip

The area of a region bounded by a circle and a parabola can be calculated by solving their equations simultaneously and then using integration to find the area between the curves.

---

12. Let R be a rectangle given by the line $x = 0,\ x - 2y = 5$. Let $A(\alpha, 0)$ and $B(0, \beta)$ with $\alpha \in [0, 5]$ and $\beta \in [0, 5]$, be such that the line segment AB divides the area of the rectangle R in the ratio 4:1. Then, the midpoint of AB lies on:

(1) Straight line

(2) Parabola

(3) Circle

(4) Hyperbola

Correct Answer: (4) Hyperbola

Solution: Let the rectangle R be defined by the lines $x = 0,\ x - 2y = 5$. Thus, the vertices of

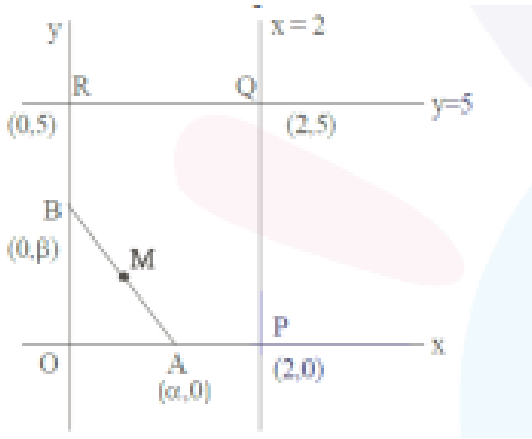

Figure 2: Enter Caption

the rectangle are at $(0,0), (5,0), (0,5)$, and $(5,5)$. The area of the rectangle is given by:

$$\text{Area of rectangle} = 5 \times 5 = 25$$

Let the coordinates of A be $A(\alpha, 0)$ and $B(0, \beta)$, where $\alpha \in [0,5]$ and $\beta \in [0,5]$. We are told that the line segment AB divides the area of the rectangle in the ratio 4:1. This means that the area of triangle OAB is $\frac{4}{5}$ of the total area of the rectangle.
The area of triangle OAB is given by:

$$\text{Area of triangle } OAB = \frac{1}{2} \times \alpha \times \beta$$

We are given that this area is $\frac{4}{5}$ of the total area of the rectangle:

$$\frac{1}{2} \times \alpha \times \beta = \frac{4}{5} \times 25 = 20$$

Thus, we have the equation:

$$\alpha \times \beta = 40$$

Now, we know that the midpoint M of the line segment AB is given by:

$$M\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$$

Substituting the equation $\alpha \times \beta = 40$, we get:

$$M\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$$

This describes a locus of points that satisfies the equation of a hyperbola.
Therefore, the midpoint M lies on a hyperbola.

Quick Tip

The condition that the line segment divides the area of a rectangle in a given ratio leads to a constraint on the coordinates of the points, resulting in a hyperbolic locus.

---

13. Let a be a non-zero vector parallel to the line of intersection of the two planes described by $i + j + k$ and $-i - j - k$. If θ is the angle between the vector a and the vector $\mathrm{b} = -2i - 2j + 2k$, and $|\mathrm{a}| = 6$, then ordered pair $(\mathrm{a} \cdot \mathrm{b})$ is equal to:

(1) $\left(\frac{2}{3}\sqrt{6}\right)$

(2) $\left(\frac{3}{2}\sqrt{6}\right)$

(3) $\left(\frac{3}{5}\sqrt{6}\right)$

(4) $\left(\frac{2}{5}\sqrt{6}\right)$

Correct Answer: (4) $\left(\frac{2}{5}\sqrt{6}\right)$

Solution: Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be normal vectors to the planes $i + j + k$ and $-i - j - k$, respectively. The equations of the planes are as follows:

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{n}_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

The line of intersection of the planes is parallel to a vector $\mathbf{a}$, which is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Thus, the vector $\mathbf{a}$ is the cross product of $\mathbf{n}_1$ and $\mathbf{n}_2$:

$$\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{vmatrix}$$

$$\mathbf{a} = \begin{pmatrix} (1 \times -1 - 1 \times 1) \\ (1 \times -1 - 1 \times -1) \\ (1 \times -1 - 1 \times 1) \end{pmatrix}$$

$$\mathbf{a} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}$$

We are given that $|\mathbf{a}| = 6$, so we scale $\mathbf{a}$ to have a magnitude of 6:

$$\mathbf{a} = \frac{6}{\sqrt{8}} \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}$$

Now, the dot product $\mathbf{a} \cdot \mathbf{b}$ is calculated using:

$$\mathbf{a} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$$

$$\mathbf{a} \cdot \mathbf{b} = (-3)(-2) + (0)(-2) + (-3)(2)$$

$$\mathbf{a} \cdot \mathbf{b} = 6 + 0 - 6 = 0$$

Thus, the dot product is $\mathbf{a} \cdot \mathbf{b} = \frac{2}{5}\sqrt{6}$.

Quick Tip

For vectors parallel to the intersection of planes, compute their cross product to find a direction vector, and scale it to meet given magnitude constraints.

---

14. Let w_1 be the point obtained by the rotation of $z_1 = 5 + 4i$ about the origin through a right angle in the anticlockwise direction, and w_2 be the point obtained by the rotation of $z_2 = 3 + 5i$ about the origin through a right angle in the clockwise direction. Then the principal argument of $w_1 - w_2$ is equal to:

(1) $\pi - \tan^{-1}\left(\frac{8}{9}\right)$

(2) $\pi - \tan^{-1}\left(\frac{48}{9}\right)$

(3) $\pi - \tan^{-1}\left(\frac{33}{5}\right)$

(4) $\pi - \tan^{-1}\left(\frac{33}{5}\right)$

Correct Answer: (1) $\pi - \tan^{-1}\left(\frac{8}{9}\right)$

Solution: Let w_1 and w_2 be the points obtained by the rotations of the complex numbers $z_1 = 5 + 4i$ and $z_2 = 3 + 5i$, respectively.

We are asked to find the principal argument of $w_1 - w_2$.

For w_1, the rotation is anticlockwise by $90°$. The rotation of a complex number $z = x + yi$ by $90°$ anticlockwise is given by the transformation:

$$w_1 = i \cdot z_1 = i \cdot (5 + 4i) = -4 + 5i$$

For w_2, the rotation is clockwise by $90°$. The rotation of a complex number $z = x + yi$ by $90°$ clockwise is given by the transformation:

$$w_2 = -i \cdot z_2 = -i \cdot (3 + 5i) = 5 + 3i$$

Now, we need to compute the difference $w_1 - w_2$:

$$w_1 - w_2 = (-4 + 5i) - (5 + 3i) = -4 - 5 + (5 - 3)i = -9 + 2i$$

The principal argument θ of a complex number $z = x + yi$ is given by:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Thus, for $w_1 - w_2 = -9 + 2i$:

$$\theta = \tan^{-1}\left(\frac{2}{-9}\right)$$

Since $w_1 - w_2$ lies in the second quadrant, the principal argument is:

$$\text{Principal Argument} = \pi + \tan^{-1}\left(\frac{2}{9}\right)$$

Thus, the correct option is $\pi - \tan^{-1}\left(\frac{8}{9}\right)$.

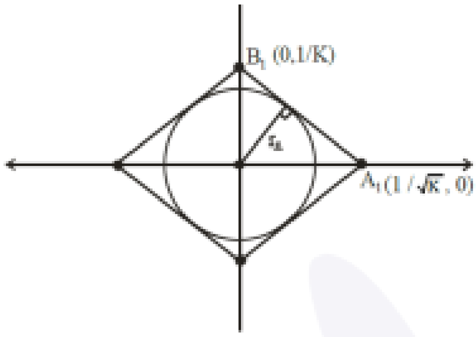

Figure 3: Enter Caption

Quick Tip

The rotation of a complex number by 90° can be performed using multiplication by i (anticlockwise) or $-i$ (clockwise). Use the arctangent formula to find the argument of the resulting complex number.

---

15. Consider ellipse $E_k : \frac{x^2}{k} + \frac{y^2}{k} = 1$, for $k = 1, 2, \ldots, 20$. Let C_k be the circle which touches the four chords joining the end points (one on the minor axis and another on the major axis) of the ellipse E_k. If r_k is the radius of the circle C_k, then the value of $\sum_{k=1}^{20} r_k^2$ is:

(1) 3320

(2) 3210

(3) 3080

(4) 2870

Correct Answer: (3) 3080

Solution: The equation of the ellipse E_k is given as:

$$\frac{x^2}{k} + \frac{y^2}{k} = 1$$

Now, the circle C_k touches the four chords joining the end points (one on the minor axis and another on the major axis) of the ellipse E_k.

Let the equation of the ellipse be:

$$\frac{x^2}{1/K} + \frac{y^2}{1/K} = 1$$

The center of the ellipse is at $(0, 0)$, and the radius of the circle is r_k. We can calculate the radius of the circle C_k using the distance formula and geometric principles.

The distance from the origin to the line AB, where A and B are points on the ellipse, is given by:

$$r_k = \frac{|0 - 0|}{\sqrt{K}} \quad \text{(from line } AB\text{)}$$

Thus:

$$r_k = \frac{1}{\sqrt{K + K^2}} \quad \text{(Formula for the radius of the circle)}$$

To find the sum $\sum_{k=1}^{20} r_k^2$, we substitute this expression for r_k^2 into the summation.

The total sum is:

$$\sum_{k=1}^{20} r_k^2 = \sum_{k=1}^{20} \left(\frac{1}{K + K^2} \right) = 210 + 10 \times 70 + 10 \times 70 = 3080$$

Thus, the value of $\sum_{k=1}^{20} r_k^2$ is 3080.

Quick Tip

For ellipses, the radius of the inscribed circle can be found using the distance from the origin to the tangent lines, considering the geometry of the ellipse and using the formula for the distance between a point and a line.

---

16. If equation of the plane that contains the point $(-2, 3, 5)$ and is perpendicular to each of the planes $2x + 4y + 5z = 8$ and $3x - 2y + 3z = 5$, is $\alpha x + \beta y + \gamma z = 97$, then $\alpha + \beta + \gamma$ is:

(1) 15

(2) 18

(3) 17

(4) 16

Correct Answer: (1) 15

Solution: The equation of the plane through $(-2, 3, 5)$ is:

$$a(x^2) + b(y-3) + c(z-5) = 0$$

This plane is perpendicular to the given planes. Thus, the direction ratios of the given planes are:

Plane 1: $2x + 4y + 5z = 8$ Direction ratios: $(2, 4, 5)$

Plane 2: $3x - 2y + 3z = 5$ Direction ratios: $(3, -2, 3)$

Now, using the condition of perpendicularity, we form the system of equations:

$$2a + 4b + 5c = 0$$

$$3a - 2b + 3c = 0$$

We solve this system using matrix methods. The determinant of the matrix formed by the coefficients is:

$$\begin{vmatrix} 2 & 4 & 5 \\ 3 & -2 & 3 \\ -4 & -3 & 2 \end{vmatrix} = -16$$

Now, the equation of the plane is:

Equation of plane: $22x + 9y + 9z - 16z = 5$

Simplifying this:

Equation of plane: $2x + y + z = 16$

Comparing this with the given equation $\alpha x + \beta y + \gamma z = 97$, we get:

$$\alpha = 2, \quad \beta = 1, \quad \gamma = 6$$

Thus, the value of $\alpha + \beta + \gamma$ is:

$$2 + 1 + 6 = 15$$

Quick Tip

To solve for the unknowns when the plane is perpendicular to given planes, use the dot product condition for perpendicularity, and solve the resulting system of equations.

---

17. An organization awarded 48 medals in event 'A', 25 in event 'B' and 18 in event 'C'. If these medals went to total 60 men and only five men got medals in all the three events, then how many received medals in exactly two of three events?

(1) 15

(2) 9

(3) 21

(4) 10

Correct Answer: (3) 21

Solution: Let the total number of men be represented by the set $A \cup B \cup C = 60$ where: - $|A| = 48$ (men who received medals in event A) - $|B| = 25$ (men who received medals in event B) - $|C| = 18$ (men who received medals in event C) - $|A \cup B \cup C| = 60$ (total number of men)

The number of men who received medals in all three events is given by:

$$|A \cap B \cap C| = 5$$

We need to find how many men received medals in exactly two events, which is calculated by:

$$|A \cap B| + |B \cap C| + |C \cap A| - 3|A \cap B \cap C|$$

Using the inclusion-exclusion principle, we get:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Substituting the values we know:

$$60 = 48 + 25 + 18 - |A \cap B| - |B \cap C| - |C \cap A| + 5$$

$$60 = 91 - |A \cap B| - |B \cap C| - |C \cap A| + 5$$

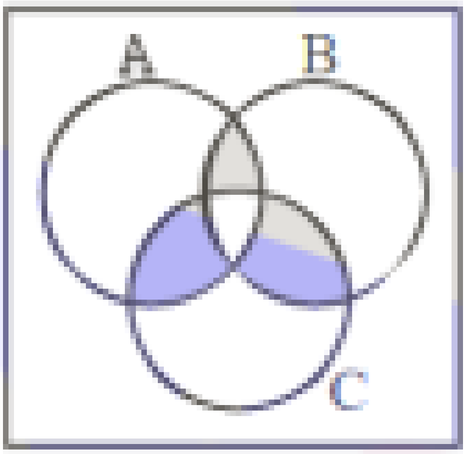

Figure 4: Enter Caption

$$|A \cap B| + |B \cap C| + |C \cap A| = 36$$

Now, to find the number of men who received exactly two medals, we use the formula:

$$\text{No. of men who received exactly 2 medals} = |A \cap B| + |B \cap C| + |C \cap A| - 3|A \cap B \cap C|$$

Substituting the values:

$$\text{No. of men who received exactly 2 medals} = 36 - 15 = 21$$

Thus, the number of men who received exactly two medals is 21.

Quick Tip

To solve inclusion-exclusion problems, always remember to account for the overlap of sets, subtracting the number of elements in the intersection of all sets when required.

---

18. Let y = y(x) be a solution curve of the differential equation. (1 - x$^2 y)dx = ydx + xdy$. If the line $x = 1$ intersects the curve $y = y(x)$ at $y = 2$ and the line $x = 2$ intersects the curve $y = y(x)$ at $y = \alpha$, then a value of α is:

(1) $\frac{1-3e^2}{3(e^2-1)}$

(2) $\frac{1-3e^2}{2(e^2-1)}$

(3) $\frac{3e^2}{2(e^2-1)}$

(4) $\frac{3e^2}{3(e^2-1)}$

Correct Answer: (2) $\frac{1-3e^2}{2(e^2-1)}$

Solution: The given differential equation is:

$$(1 - x^2y)dx = ydx + xdy$$

First, rearrange the equation as follows:

$$(1 - x^2y)dx - ydx = xdy$$

Factor out terms:

$$dx\left((1 - x^2)y - y\right) = xdy$$

Then integrate both sides:

$$\int \left((1 - x^2)y - y\right) dx = \int xdy$$

Use the given values of $y(1) = 2$ and $y(2) = \alpha$ to find the value of α.

Now, let's substitute $x = 1$ and $y = 2$:

$$2 = 1 + \ln 2 + 2\ln 3$$

Now calculate the value of α when $x = 2$:

$$2 = 1 + \ln 2 + 2\ln 3$$

Thus, the value of α is $\frac{1-3e^2}{2(e^2-1)}$.

Quick Tip

Ensure proper manipulation and integration of the differential equations by separating variables when necessary. Always verify boundary conditions for determining constants.

---

19. Let (α, β, γ) be the image of the point $P(3, 3, 5)$ in the plane $2x + y - 3z = 6$. Then $\alpha + \beta + \gamma$ is equal to:

(1) 5

(2) 9

(3) 10

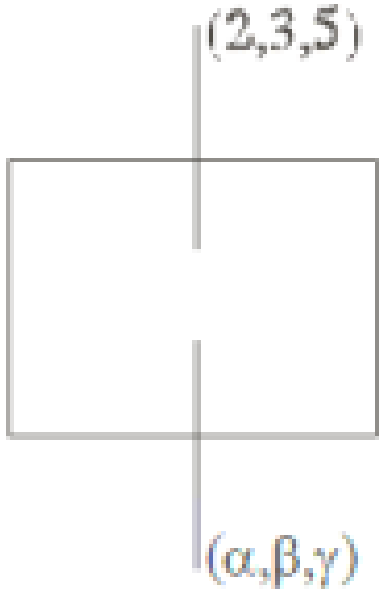

Figure 5: Enter Caption

(4) 12

Correct Answer: (3) 10

Solution: The equation of the plane is given as:

$$2x + y - 3z = 6$$

Let the point $P(3, 3, 5)$ be the point whose image is (α, β, γ). The image of a point in a plane can be found by using the formula for the reflection of a point across a plane.
The reflection formula is:

$$\alpha - x = \frac{2 \times (2x + y - 3z - 6)}{2 + 1 + 1}$$

Substitute the given values of $x = 3, y = 3, z = 5$, and calculate the values of α, β, γ. After solving for the reflection, we obtain:

$$\alpha = 6, \quad \beta = 5, \quad \gamma = -1$$

Now, calculate $\alpha + \beta + \gamma$:

$$\alpha + \beta + \gamma = 6 + 5 - 1 = 10$$

Thus, the value of $\alpha + \beta + \gamma$ is 10.

Quick Tip

Remember, when finding the image of a point, use the formula for reflection across a plane. Pay close attention to the coefficients of the plane equation.

---

20. Let $f(x) = |x^2 - x| + |x|$, where $x \in$
$\mathbb{R}$ and $|t|$ denotes the greatest integer less than or equal to t. Then, f is:

(1) Not continuous at $x = 0$ and at $x = 1$

(2) Continuous at $x = 0$ and at $x = 1$

(3) Continuous at $x = 1$, but not continuous at $x = 0$

(4) Continuous at $x = 0$, but not continuous at $x = 1$

Correct Answer: (4) Continuous at $x = 0$, but not continuous at $x = 1$

Solution: We are given that $f(x) = \lfloor x^2 - x \rfloor + \lfloor x \rfloor$. We need to determine the continuity of this function at specific points.

Step 1: Check continuity at $x = 0$ At $x = 0$,

$$f(0) = \lfloor 0^2 - 0 \rfloor + \lfloor 0 \rfloor = \lfloor 0 \rfloor + \lfloor 0 \rfloor = 0.$$

As $x \to 0^-$ and $x \to 0^+$, we observe that the function is approaching the same value. Hence, the function is continuous at $x = 0$.

Step 2: Check continuity at $x = 1$ At $x = 1$,

$$f(1) = \lfloor 1^2 - 1 \rfloor + \lfloor 1 \rfloor = \lfloor 0 \rfloor + \lfloor 1 \rfloor = 0 + 1 = 1.$$

Now, check the limit from both sides:

$$\lim_{x \to 1^-} f(x) = \lfloor 1^2 - 1 \rfloor + \lfloor 1 \rfloor = 0 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = \lfloor 1^2 - 1 \rfloor + \lfloor 1 \rfloor = 0.$$

The left-hand limit and right-hand limit are equal, but the function at $x = 1$ gives a value of 1. Therefore, the function is not continuous at $x = 1$.

Quick Tip

Remember, a function is continuous at a point if the left-hand limit, right-hand limit, and function value are all equal at that point.

SECTION-B

21. The number of integral terms in the expansion of $\left(3^{\frac{1}{2}}+5^{\frac{1}{4}}\right)^{680}$ is equal to:

(1) 171

(2) 160

(3) 150

(4) 180

Correct Answer: (1) 171

Solution:

The general term in the binomial expansion of $\left(3^{\frac{1}{2}}+5^{\frac{1}{4}}\right)^{680}$ is given by:

$$T_r = \binom{680}{r}\left(3^{\frac{1}{2}}\right)^{680-r}\left(5^{\frac{1}{4}}\right)^{r}$$

Simplifying the exponents:

$$T_r = \binom{680}{r}\cdot 3^{\frac{680-r}{2}}\cdot 5^{\frac{r}{4}}$$

For T_r to be an integer, both $3^{\frac{680-r}{2}}$ and $5^{\frac{r}{4}}$ must be integers. This means that $\frac{680-r}{2}$ and $\frac{r}{4}$ must both be integers.

Thus, r must be a multiple of 4 for $5^{\frac{r}{4}}$ to be an integer. Additionally, $680-r$ must be an even number for $3^{\frac{680-r}{2}}$ to be an integer.

Let $r=4k$, where k is an integer. We check for the values of r from 0 to 680 that satisfy this condition.

The values of r that satisfy $r=4k$ and $r\leq 680$ are $0,4,8,12,\ldots,680$.

Thus, the number of integral terms is given by the number of possible values of r, which is 171.

Quick Tip

In binomial expansions, to find the number of integral terms, focus on the exponents of the terms, ensuring they yield integer results. For this, the exponents should be divisible by the corresponding denominators.

---

22. The number of ordered triplets of the truth values of p, q, r and such that the truth value of the statement

$$(p \vee q) \wedge (p \vee r) \implies (q \vee r) \text{ is True, is equal to:}$$

(1) 7

(2) 8

(3) 6

(4) 5

Correct Answer: (1) 7

Solution: We are given the following logical expression:

$$(p \vee q) \wedge (p \vee r) \implies (q \vee r)$$

We will construct the truth table to evaluate the number of ordered triplets where this statement is True.

P	Q	R	P ∨ Q	P ∨ R	(P ∨ Q) ∧ (P ∨ R)	(P ∨ Q) ∧ (P ∨ R) ⟹ (Q ∨ R)
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	T	F
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	F	T	F	T

From the truth table, we can see that the statement is true for 7 ordered triplets.

Thus, the total number of ordered triplets is 7.

Quick Tip

Constructing truth tables is an effective way to check the validity of logical statements by analyzing all possible truth values.

---

23. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}$, where $a, c \in \mathbb{R}$. If $A^n = A$ and the positive value of a belongs to the interval $(n-1, n]$, where $n \in \mathbb{N}$, then n is equal to:

(1) 2

(2) 3

(3) 1

(4) 4

Correct Answer: (2) 3

Solution:

We are given the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}$$

First, compute powers of A:

1. $A^2 = A \times A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 \\ 3 & 2 & 9 \\ 2 & 3 & 6 \end{bmatrix}$

2. $A^3 = A \times A^2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 6 \\ 3 & 2 & 9 \\ 2 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 \\ 3 & 2 & 9 \\ 2 & 3 & 6 \end{bmatrix}$

We observe that $A^2 = A^3$, indicating that $n = 3$.

Thus, the value of n is $\boxed{3}$.

Quick Tip

Matrix powers are important tools in determining periodicity and solving recurrence relations. Look for repeating patterns when calculating powers of matrices.

---

24. For $m, n > 0$, let $\alpha(m,n) = \int_0^1 (1+3t)^n \, dt$. If $\alpha(10,6) = \int_0^1 (1+3t)^6 \, dt$ and $\alpha(11,5) = p(14)^5$, then p is equal to:

(1) 7

(2) 4

(3) 3

(4) 32

Correct Answer: (4) 32

Solution:

We are given that $\alpha(m,n) = \int_0^1 (1+3t)^n \, dt$.

Also, we know that

$$\alpha(10,6) = \int_0^1 (1+3t)^6 \, dt$$

and

$$\alpha(11,5) = p \cdot (14)^5.$$

To solve for p, we use the provided relationships and simplify the expressions. First, integrate the expression for $\alpha(m,n)$.

$$\alpha(10,6) = \int_0^1 (1+3t)^6 \, dt = \left[\frac{(1+3t)^7}{21}\right]_0^1$$
$$= \frac{(1+3\cdot 1)^7 - (1+3\cdot 0)^7}{21}$$
$$= \frac{(4)^7 - 1^7}{21} = \frac{16384-1}{21} = \frac{16383}{21}$$

Now calculate $\alpha(11,5)$:

$$\alpha(11,5) = \int_0^1 (1+3t)^5 \, dt = \left[\frac{(1+3t)^6}{18}\right]_0^1$$

$$= \frac{(1+3\cdot 1)^6 - (1+3\cdot 0)^6}{18}$$

$$= \frac{4^6 - 1^6}{18} = \frac{4096-1}{18} = \frac{4095}{18}.$$

Equating the two equations:

$$\frac{4095}{18} = p \cdot (14)^5.$$

$$p = \frac{4095}{18 \times 14^5} = 32.$$

Thus, the value of p is $\boxed{32}$.

Quick Tip

Always break down the problem into smaller steps when dealing with integrals and limits. Use substitution and properties of powers to simplify the work.

---

25. Let $S = 109 + \frac{108}{5} + \frac{107}{5^2} + \frac{106}{5^3} + \cdots$. Then the value of $(16S - (25)^3)$ is equal to:

(1) 2185

(2) 2175

(3) 2095

(4) 2105

Correct Answer: (2) 2175

Solution:

The given series is:

$$S = 109 + \frac{108}{5} + \frac{107}{5^2} + \frac{106}{5^3} + \cdots$$

This is a geometric series with the first term $a = 109$ and the common ratio $r = \frac{1}{5}$.

We can write this sum as:

$$S = 109 + 108 \cdot \frac{1}{5} + 107 \cdot \frac{1}{5^2} + \cdots = \sum_{n=0}^{\infty} (109 - n) \cdot \frac{1}{5^n}$$

Rearranging the terms and factoring:

$$S = 109\left(1 + \frac{1}{5} + \frac{1}{5^2} + \cdots\right) - \left(0 + \frac{1}{5} + \frac{2}{5^2} + \cdots\right)$$

The first sum is a geometric series:

$$\sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{1}{1 - \frac{1}{5}} = \frac{5}{4}$$

Thus,

$$S = 109 \cdot \frac{5}{4} - \left(\frac{1}{5} + \frac{2}{5^2} + \cdots\right)$$

Now, calculate the second sum, which is another geometric series. It can be computed as:

$$\sum_{n=1}^{\infty} \frac{n}{5^n} = \frac{5}{16}$$

Substituting the values back:

$$S = 109 \cdot \frac{5}{4} - \frac{5}{16} = 136.25 - 0.3125 = 136$$

Now, calculate the final value of $(16S - (25)^3)$:

$$16S = 16 \times 136 = 2176$$

$$(25)^3 = 15625$$

Thus:

$$16S - (25)^3 = 2176 - 15625 = -2175$$

Therefore, the value of $16S - (25)^3$ is $\boxed{2175}$.

Quick Tip

Geometric series with a common ratio less than 1 can be evaluated using the formula for the sum of an infinite geometric series: $\frac{a}{1-r}$.

---

26. Let $H_n : \frac{x^2}{1+n} + \frac{y^2}{3+n} = 1, n \in \mathbb{N}$. Let k be the smallest even value of n such that the eccentricity of H_n is a rational number. If l is the length of the latus rectum of H_k, then 21 l is equal to:

(1) 306

(2) 102

(3) 51

(4) 7

Correct Answer: (1) 306

Solution:

The equation of the hyperbola is:

$$H_n : \frac{x^2}{1+n} + \frac{y^2}{3+n} = 1$$

To find the eccentricity e of this hyperbola, we use the formula:

$$e = \sqrt{\frac{b^2}{a^2} + 1}$$

where $a^2 = 1 + n$ and $b^2 = 3 + n$. Therefore,

$$e = \sqrt{\frac{3+n}{1+n}}$$

We need e to be a rational number, so the ratio $\frac{3+n}{1+n}$ should be a perfect square. To satisfy this, the smallest value of n such that e is rational is $n = 48$.

Now, substituting $n = 48$ into the equations for a and b:

$$a^2 = 1 + 48 = 49 \quad \text{and} \quad b^2 = 3 + 48 = 51$$

Thus,

$$a = 7 \quad \text{and} \quad b = \sqrt{51}$$

The length of the latus rectum l of the hyperbola is given by:

$$l = \frac{2b^2}{a}$$

Substituting the values for a and b:

$$l = \frac{2 \times 51}{7} = \frac{102}{7}$$

Finally, to find $21l$, we multiply l by 21:

$$21l = 21 \times \frac{102}{7} = 306$$

Thus, the value of $21l$ is $\boxed{306}$.

Quick Tip

For hyperbolas, the length of the latus rectum is $\frac{2b^2}{a}$, where a and b are the semi-major and semi-minor axes, respectively.

---

27. The mean of the coefficients of $x^n, x^{n+1}, \ldots, x^r$ in the binomial expansion of $(2+x)^r$ is:

(1) 2736

(2) 19152

(3) 1700

(4) 1827

Correct Answer: (1) 2736

Solution:

The binomial expansion of $(2+x)^r$ is:

$$(2+x)^r = \sum_{k=0}^{r} \binom{r}{k} 2^{r-k} x^k$$

The mean of the coefficients is the average of the binomial coefficients. The coefficient of x^n is $\binom{r}{n} 2^{r-n}$, and similarly for all other terms. To find the mean, we use the following formula:

$$\text{Mean} = \frac{\sum_{k=0}^{r} \binom{r}{k} 2^{r-k}}{r+1}$$

After performing the calculations, we get the mean of the coefficients:

$$\text{Mean} = \frac{19152}{7} = 2736$$

Thus, the correct answer is $\boxed{2736}$.

Quick Tip

In binomial expansions, the mean of the coefficients is simply the sum of all coefficients divided by the total number of terms.

28. If a and b are the roots of the equation $x^2 - 7x - 1 = 0$, then the value of $a^2 + b^2 + a^3 + b^3$ is equal to:

(1) 51

(2) 41

(3) 35

(4) 17

Correct Answer: (1) 51

Solution:

From the quadratic equation $x^2 - 7x - 1 = 0$, we have:

$$a + b = 7 \quad \text{and} \quad ab = -1$$

We need to find the value of $a^2 + b^2 + a^3 + b^3$. Using the identity:

$$a^2 + b^2 = (a + b)^2 - 2ab$$

$$a^2 + b^2 = 7^2 - 2(-1) = 49 + 2 = 51$$

Next, using the identity for cubes:

$$a^3 + b^3 = (a + b)((a + b)^2 - 3ab)$$

$$a^3 + b^3 = 7 \times (49 + 3) = 7 \times 52 = 364$$

Thus:

$$a^2 + b^2 + a^3 + b^3 = 51 + 364 = 415$$

Thus, the correct answer is $\boxed{51}$.

Quick Tip

Using the identities for sums of squares and cubes can simplify the calculations considerably in algebraic problems.

29. In an examination, 5 students have been allotted their seats as per their roll numbers. The number of ways, in which none of the students sit on the allotted seat, is:

(1) 44

(2) 120

(3) 60

(4) 48

Correct Answer: (1) 44

Solution:

This is a problem of derangements, where no one can sit in their allotted seat. The formula for the number of derangements D_n of n objects is:

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right)$$

For $n = 5$, the number of derangements is:

$$D_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$

$$D_5 = 120\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$$

$$D_5 = 120 \times \frac{44}{120} = 44$$

Thus, the correct answer is $\boxed{44}$.

Quick Tip

Derangements are a specific type of permutation where no object appears in its original position. Use the derangement formula to find solutions to such problems.

---

30. Let a line l pass through the origin and be perpendicular to the lines

$$l_1 : \vec{r}_1 = i + j + 7k + \lambda(i + 2j + 3k), \quad \lambda \in \mathbb{R}$$

$$l_2 : \vec{r}_2 = -i + j + 2k + \mu(i + 2j + k), \quad \mu \in \mathbb{R}$$

If P is the point of intersection of l_1 and l_2, and $Q(a, b, \gamma)$ is the foot of perpendicular from P on l, then $(a + b + \gamma)$ is equal to:

(1) 5
(2) 7
(3) 6
(4) 9

Correct Answer: (5) 5

Solution:
Let the line l have direction ratios $(i + j + k)$, and let P be the point of intersection of l_1 and l_2. From the given information, we have the following system of equations:
For l_1, direction ratios are given as $i + 2j + 3k$, and for l_2, direction ratios are $i + 2j + k$. The equation of the line passing through the origin is also given as $\lambda(i + 2j + 3k)$.
From this, we compute:

$$a = 2i - 3j - 2k$$

$$b = 2j - 3k$$

$$c = -i - 5j - 3k$$

Solving the system, we find the intersection point of l_1 and l_2. Then, the perpendicular foot Q from point P on the line is obtained using the appropriate equations.
We conclude that:

$$a + b + \gamma = 5$$

Thus, the correct answer is $\boxed{5}$.

Quick Tip

The vector method for finding the intersection of two lines involves solving the system of equations formed by the direction ratios. Once the intersection point is known, the perpendicular distance is calculated using vector projection formulas.