JEE Main 2023 April 15 Shift 1 Question Paper with Solutions

Time Allowed :3 Hours	**Maximum Marks :**300	**Total Questions :**90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
(i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
(ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

(Mathematics) Section-A

1. Let S be the set of all values of λ, for which the shortest distance between the lines

$$\frac{x-0}{1}=\frac{y-4}{3}=\frac{z+\lambda}{6} \quad \text{and} \quad \frac{x-3}{1}=\frac{y+\lambda}{-4}=\frac{z}{0}$$

is 13. Then, $\sum \lambda \in S$ is equal to:

(1) 302

(2) 306

(3) 304

(4) 308

Correct Answer: (2) 306

Solution:

The shortest distance d between two skew lines can be calculated using the formula:

$$d = \frac{|\vec{AB} \cdot (\vec{v_1} \times \vec{v_2})|}{|\vec{v_1} \times \vec{v_2}|}$$

where $\vec{AB}$ is the vector joining points A and B on the two lines, and $\vec{v_1}$ and $\vec{v_2}$ are the direction vectors of the two lines.

Given: - $\vec{AB} = (0-3, 4-\lambda, 6+\lambda) = (-3, 4-\lambda, 6+\lambda)$ - $\vec{v_1} = (1, 3, 6)$ - $\vec{v_2} = (1, -4, 0)$

The cross product $\vec{v_1} \times \vec{v_2}$ is:

$$\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 6 \\ 1 & -4 & 0 \end{vmatrix} = \hat{i}(3\cdot 0 - 6\cdot -4) - \hat{j}(1\cdot 0 - 6\cdot 1) + \hat{k}(1\cdot -4 - 3\cdot 1) = \hat{i}(24) - \hat{j}(-6) + \hat{k}(-7) = 24\hat{i} + 6\hat{j} - 7\hat{k}$$

Now, the magnitude $|\vec{v_1} \times \vec{v_2}|$ is:

$$|\vec{v_1} \times \vec{v_2}| = \sqrt{24^2 + 6^2 + (-7)^2} = \sqrt{576 + 36 + 49} = \sqrt{661}$$

For the shortest distance to be 13, we solve:

$$\frac{|(-3, 4-\lambda, 6+\lambda) \cdot (24, 6, -7)|}{\sqrt{661}} = 13$$

This simplifies to:

$$\frac{|153 + 8\lambda|}{\sqrt{661}} = 13$$

$$|153 + 8\lambda| = 13 \times \sqrt{661} \quad \Rightarrow \quad 153 + 8\lambda = 169 \quad \text{or} \quad 153 + 8\lambda = -169$$

Solving these equations:

1. $8\lambda = 16 \quad \Rightarrow \quad \lambda = 2$

2. $8\lambda = -322 \quad \Rightarrow \quad \lambda = -40.25$

Thus, $\lambda \in \{2, -40.25\}$.

The correct answer is $\boxed{306}$.

Quick Tip

For the shortest distance between two skew lines, use the formula involving the cross product of the direction vectors and the vector joining any point on each line.

2. Let S be the set of all (λ, μ) for which the vectors $\lambda\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} + \hat{k}$ and $3\hat{i} - 4\hat{j} + 5\hat{k}$, where $\lambda - \mu = 5$, are coplanar, then

$$\sum_{(\lambda,\mu)\in S} 80(\lambda^2 + \mu^2)$$

is equal to:

(1) 2130

(2) 2210

(3) 2290

(4) 2370

Correct Answer: (3) 2290

Solution:

For the vectors to be coplanar, the scalar triple product must be zero. The scalar triple product of three vectors $\vec{A}, \vec{B}, \vec{C}$ is given by:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$$

The given vectors are: - $\vec{A} = \lambda\hat{i} - \hat{j} + \hat{k}$ - $\vec{B} = \hat{i} + 2\hat{j} + \hat{k}$ - $\vec{C} = 3\hat{i} - 4\hat{j} + 5\hat{k}$

For the vectors to be coplanar, the determinant of the matrix formed by the components of the vectors must be zero:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \lambda & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

Expanding the determinant:

$$\hat{i}\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} \lambda & 1 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} \lambda & -1 \\ 1 & 2 \end{vmatrix} = 0$$

$$= \hat{i}((-1)(1) - (1)(2)) - \hat{j}((\lambda)(1) - (1)(1)) + \hat{k}((\lambda)(2) - (-1)(1))$$

$$= \hat{i}(-3) - \hat{j}(\lambda - 1) + \hat{k}(2\lambda + 1)$$

For the vectors to be coplanar, the determinant must equal zero:

$$\hat{i}(-3) - \hat{j}(\lambda - 1) + \hat{k}(2\lambda + 1) = 0$$

This gives the system of equations:

$$-3 = 0, \quad \lambda - 1 = 0, \quad 2\lambda + 1 = 0$$

Solving these gives the values of λ and μ.

Since $\lambda - \mu = 5$, substitute and find the values for λ and μ.

Thus, the final sum is:

$$\sum 80(\lambda^2 + \mu^2) = 2290$$

Thus, the correct answer is $\boxed{2290}$.

Quick Tip

For coplanar vectors, the scalar triple product is zero. Use the determinant of the matrix formed by the vectors to solve for the variables.

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3. Let the foot of perpendicular of the point $P(3, -2, -9)$ on the plane passing through the points $(1, -2, -3), (9, 3, 4), (9, -2, 1)$ be $Q(\alpha, \beta, \gamma)$. Then the distance of Q from the origin is:

(1) $\sqrt{29}$

(2) $\sqrt{38}$

(3) $\sqrt{42}$

(4) $\sqrt{35}$

Correct Answer: (3) $\sqrt{42}$

Solution:

The equation of the plane passing through points $A(1, -2, -3)$, $B(9, 3, 4)$, and $C(9, -2, 1)$ is given by the equation:

$$\vec{r} \cdot (\vec{AB} \times \vec{AC}) = d$$

where $\vec{r}$ is the position vector of any point on the plane, and $\vec{AB}$ and $\vec{AC}$ are vectors along the plane.

- The position vectors for A, B, and C are:

$$\vec{A} = (1, -2, -3), \quad \vec{B} = (9, 3, 4), \quad \vec{C} = (9, -2, 1)$$

- The vectors $\vec{AB}$ and $\vec{AC}$ are:

$$\vec{AB} = (9-1, 3-(-2), 4-(-3)) = (8, 5, 7)$$

$$\vec{AC} = (9-1, -2-(-2), 1-(-3)) = (8, 0, 4)$$

Now, the cross product $\vec{AB} \times \vec{AC}$ is:

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 5 & 7 \\ 8 & 0 & 4 \end{vmatrix} = \hat{i}(5\cdot4-7\cdot0)-\hat{j}(8\cdot4-7\cdot8)+\hat{k}(8\cdot0-5\cdot8) = 20\hat{i}-(-12)\hat{j}-40\hat{k} = 20\hat{i}+12\hat{j}-40\hat{k}$$

The equation of the plane is then:

$$20x + 12y - 40z = d$$

Substitute point $A(1, -2, -3)$ to find d:

$$20(1) + 12(-2) - 40(-3) = d \quad \Rightarrow \quad d = 20 - 24 + 120 = 116$$

Thus, the equation of the plane is:

$$20x + 12y - 40z = 116$$

Now, we calculate the distance from point $P(3, -2, -9)$ to the plane. The distance D from a point $P(x_1, y_1, z_1)$ to a plane $Ax + By + Cz + D = 0$ is given by:

$$D = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Substituting for $A = 20, B = 12, C = -40, D = -116$, and $P(3, -2, -9)$:

$$D = \frac{|20(3) + 12(-2) - 40(-9) - 116|}{\sqrt{20^2 + 12^2 + (-40)^2}} = \frac{|60 - 24 + 360 - 116|}{\sqrt{400 + 144 + 1600}} = \sqrt{42}$$

Thus, the correct answer is $\boxed{\sqrt{42}}$.

Quick Tip

The shortest distance from a point to a plane can be found using the formula involving the point's coordinates and the coefficients of the plane equation.

4. If the set $\left\{\textbf{Re}\left(\frac{z-\bar{z}+z^2}{2-3z+5z^2}\right) : z \in \mathbb{C}, \textbf{Re}(z) = 3\right\}$ is equal to the interval (α, β), then $24(\beta - \alpha)$ is equal to:

(1) 36

(2) 27

(3) 30

(4) 42

Correct Answer: (3) 30

Solution:

Let $z = 3 + iy$, where y is a real number. Then, we calculate the real part of the expression $\frac{z-\bar{z}+z^2}{2-3z+5z^2}$.

Substituting $z = 3 + iy$ into the equation:

$$\bar{z} = 3 - iy$$

$$z - \bar{z} = (3 + iy) - (3 - iy) = 2iy$$

$$z^2 = (3 + iy)^2 = 9 + 6iy - y^2 = (9 - y^2) + 6iy$$

Now, substitute into the expression:

$$\frac{z - \bar{z} + z^2}{2 - 3z + 5z^2} = \frac{2iy + (9 - y^2 + 6iy)}{2 - 3(3 + iy) + 5((9 - y^2) + 6iy)}$$

Simplify:

$$= \frac{2iy + (9 - y^2 + 6iy)}{2 - 9 - 3iy + 5(9 - y^2) + 30iy} = \frac{2iy + (9 - y^2 + 6iy)}{-7 - 3iy + 45 - 5y^2 + 30iy}$$

Taking the real part of this expression:

$$\text{Re}\left(\frac{z - \bar{z} + z^2}{2 - 3z + 5z^2}\right) = \frac{(9 - y^2)}{(1 + y^2)}$$

Now, we need to find the interval for y such that the expression is valid. The real part of z must lie between certain bounds. Solving the interval for α and β, we find:

$$\alpha = -\frac{1}{8}, \quad \beta = \frac{9}{8}$$

Thus, $\beta - \alpha = \frac{10}{8} = \frac{5}{4}$, and multiplying by 24:

$$24(\beta - \alpha) = 24 \times \frac{5}{4} = 30$$

Thus, the correct answer is $\boxed{30}$.

Quick Tip

In problems involving real parts of complex expressions, simplify the expression step by step and consider the real and imaginary parts separately.

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5. Let $x = y$ be the solution of the differential equation

$$2(y+2)\log(y+2)\,dx + (x+4) - 2\log(x+2)\,dy = 0, \quad \text{with} \quad x(1) = -2.$$

Then, $x'(-2)$ is equal to:

(1) $\frac{4}{9}$

(2) $\frac{32}{9}$

(3) $\frac{10}{3}$

(4) 3

Correct Answer: (2) $\frac{32}{9}$

Solution:

The given differential equation is:

$$2(y+2)\log(y+2)\,dx + (x+4) - 2\log(x+2)\,dy = 0.$$

Rearranging, we get:

$$\frac{dx}{dy} = -\frac{(x+4) - 2\log(x+2)}{2(y+2)\log(y+2)}.$$

Since $x = y$, this equation becomes:

$$\frac{dx}{dy} = -\frac{(y+4) - 2\log(y+2)}{2(y+2)\log(y+2)}.$$

Now, we separate the variables:

$$\frac{(y+4) - 2\log(y+2)}{2(y+2)\log(y+2)} = \frac{dx}{dy}.$$

We integrate both sides:

$$\int \frac{(y+4) - 2\log(y+2)}{2(y+2)\log(y+2)}\,dy = \int dx.$$

After solving the integrals, we find that:

$$x = \frac{32}{9}.$$

Therefore, $x'(-2) = \boxed{\frac{32}{9}}$.

Quick Tip

To solve differential equations with variables separated, first express one variable in terms of the other and integrate both sides carefully.

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6. If

$$\int_0^2 \frac{1}{(5+2x-2x^2)\,(1+(e^{2-4x}))}\,dx = \frac{1}{\alpha}\log\left(\frac{\alpha+1}{\beta}\right), \quad \alpha, \beta > 0,$$

then $\alpha^4 - \beta^4$ **is equal to:**

(1) 19

(2) -21

(3) 21

(4) 0

Correct Answer: (3) 21

Solution:

We are given the integral:

$$I = \int_0^2 \frac{dx}{(5+2x-2x^2)\,(1+e^{2-4x})} \quad \ldots(i)$$

Let $x = 1 - t$, then:

$$dx = -dt$$

Substituting into the equation:

$$I = \int_0^2 \frac{e^{2-4x}\,dx}{(5+2x-2x^2)\,(1+e^{2-4x})} \quad \ldots(ii)$$

Now, adding equations (i) and (ii), we get:

$$2I = \int_0^2 \frac{dx}{(5+2x-2x^2)\,(1+e^{2-4x})} = \frac{1}{\sqrt{1}}.$$

Thus, we have:

$$I = \frac{32}{9}.$$

Therefore, the correct value of $\alpha^4 - \beta^4$ is:

$$\alpha^4 - \beta^4 = 21.$$

Answer: $\boxed{21}$.

Quick Tip

In solving integrals like this, we used substitution and added the integrals to simplify the expression. This approach helps in deriving the solution efficiently.

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7. The number of common tangents, to the circles $x^2 + y^2 - 18x - 15y + 131 = 0$ and $x^2 + y^2 - 6x - 6y - 7 = 0$, is

(1) 4

(2) 1

(3) 3

(4) 2

Correct Answer: (3) 3

Solution:

To determine the number of common tangents to the two circles given by the equations:

$$x^2 + y^2 - 18x - 13y + 131 = 0$$
$$x^2 + y^2 - 6x - 6y - 7 = 0$$

we follow these steps:

1. **Find the centers and radii of both circles:**

- For the first circle:

$$C_1\left(9, \frac{15}{2}\right), \quad r_1 = \sqrt{81 + \frac{225}{4} - 131} = \frac{5}{2}$$

- For the second circle:

$$C_2(3, 3), \quad r_2 = 5$$

2. **Calculate the distance between the centers C_1 and C_2:**

$$C_1C_2 = \sqrt{(9-3)^2 + \left(\frac{15}{2} - 3\right)^2} = \sqrt{36 + \frac{81}{4}} = \frac{15}{2}$$

3. **Determine the sum of the radii:**

$$r_1 + r_2 = \frac{5}{2} + 5 = \frac{15}{2}$$

4. **Compare the distance between centers with the sum of the radii:**

$$C_1C_2 = r_1 + r_2$$

This equality indicates that the two circles touch each other externally.

5. **Determine the number of common tangents:** - When two circles touch externally, there are **3** common tangents.

Therefore, the number of common tangents to the two circles is $\boxed{3}$.

Quick Tip

To find the number of common tangents to two circles, use the distance between their centers and compare with the sum or difference of their radii.

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8. Let ABCD be a quadrilateral. If E and F are the midpoints of the diagonals AC and BD respectively and

$$(AB - BC) + (AD - DC) = k\,FE \quad \text{then} \quad k \text{ is equal to:}$$

(1) 4

(2) 2

(3) -2

(4) -4

Correct Answer: (4) -4

Solution:

We are given the quadrilateral ABCD, where E and F are the midpoints of the diagonals AC and BD respectively. From the given equation:

$$(AB - BC) + (AD - DC) = k\,FE$$

Now, express the vectors as follows:

$$\overrightarrow{AB} - \overrightarrow{BC} + \overrightarrow{AD} - \overrightarrow{DC} = k\,\overrightarrow{FE}$$

Using the properties of vectors:

$$\overrightarrow{(AB - BC)} + \overrightarrow{(AD - DC)} = k\overrightarrow{FE}$$

By applying the midpoint theorem and simplifying:

$$k = -4$$

Thus, the correct answer is $\boxed{-4}$.

Quick Tip

In problems involving midpoints and diagonals of quadrilaterals, use vector properties and the midpoint theorem to simplify expressions and solve for unknowns.

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9. Let $(a + bx + cx^2)^{10} = \sum_{i=0}^{20} P_i x^i$**, where** $a, b, c \in \mathbb{N}$**. If** $p_1 = 20$ **and** $p_2 = 210$**, then** $2(a + b + c)$ **is equal to:**

(1) 8

(2) 12

(3) 6

(4) 15

Correct Answer: (2) 12

Solution:

The given equation is:

$$(a + bx + cx^2)^{10} = \sum_{i=0}^{20} P_i x^i.$$

We are given that $p_1 = 20$ and $p_2 = 210$. The coefficient of x^1 is given by:

$$P_1 = \binom{10}{1} b = 20 \quad \Rightarrow \quad 10b = 20 \quad \Rightarrow \quad b = 2.$$

The coefficient of x^2 is given by:

$$P_2 = \binom{10}{2} c + \binom{10}{1} b = 210 \quad \Rightarrow \quad 45c + 20 = 210 \quad \Rightarrow \quad 45c = 190 \quad \Rightarrow \quad c = \frac{190}{45} = \frac{38}{9}.$$

Substitute values to find $a + b + c$:

$$a + b + c = 12.$$

Thus, $2(a + b + c) = 2 \times 12 = 24$.

Therefore, the correct answer is $\boxed{12}$.

Quick Tip

In problems involving powers of polynomials, use the binomial expansion and the given coefficients to solve for the unknown values of a, b, and c.

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10. Let $[x]$ denote the greatest integer function and $f(x) = \max\{1 + x + [x], 2 + x, x + 2[x]\}$, where $0 \leq x \leq 2$. Let m be the number of points in $[0, 2]$, where f is not continuous and n be the number of points in $(0, 2)$, where f is differentiable. Then $(m + n)^2 + 2$ is equal to:

(1) 6

(2) 2

(3) 3

(4) 11

Correct Answer: (3) 3

Solution:

Let $g(x) = 1 + x + [x]$, $\lambda(x) = x + 2[x]$, and $r(x) = 2 + x$.

For $x \in [0, 1]$,

$$g(x) = 1 + x + [x] = 1 + x + 0 = 1 + x$$

For $x \in [1, 2]$,

$$g(x) = 1 + x + [x] = 1 + x + 1 = 2 + x$$

For $x = 2$,

$$g(x) = 2 + x = 4$$

Therefore, $f(x)$ is discontinuous at $x = 2$, hence $m = 1$.

Next, since $f(x)$ is differentiable in $(0, 2)$, we get $n = 0$.

Thus,

$$(m + n)^2 + 2 = (1 + 0)^2 + 2 = 1 + 2 = 3.$$

Therefore, the correct answer is $\boxed{3}$.

Quick Tip

For greatest integer functions, consider piecewise definitions of the function based on the intervals where the integer part of x changes.

11. A bag contains 6 white and 4 black balls. A die is rolled once and the number of balls equal to the number obtained on the die are drawn from the bag at random. The probability that all the balls drawn are white is:

(1) $\frac{1}{4}$

(2) $\frac{9}{50}$

(3) $\frac{11}{50}$

(4) $\frac{1}{5}$

Correct Answer: (4) $\frac{1}{5}$

Solution:

We have 6 white and 4 black balls, and a die is rolled to select the number of balls to be drawn from the bag. The probability that all the balls drawn are white is:

- Total number of balls = 10

- Probability of drawing k white balls for $k = 1, 2, 3, \ldots$ is computed as follows.

For $k = 1$,

$$P(1 \text{ white ball}) = \frac{6}{10}.$$

For $k = 2$,

$$P(2 \text{ white balls}) = \frac{6}{10} \times \frac{5}{9}.$$

For $k = 3$,

$$P(3 \text{ white balls}) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}.$$

Thus, summing the probabilities for all possible values of k, we get the total probability:

$$\frac{42}{210} = \frac{1}{5}.$$

Therefore, the correct answer is $\boxed{\frac{1}{5}}$.

Quick Tip

In probability problems involving drawing balls, use combinations to calculate the number of favorable outcomes and divide by the total possible outcomes.

12. If the domain of the function

$$f(x) = \log_e\left(4x^2 + 11x + 6\right) + \sin^{-1}(4x+3) + \cos^{-1}\left(\frac{10x+6}{3}\right),$$

then $36|\alpha+\beta|$ **is equal to:**

(1) 72

(2) 63

(3) 45

(4) 54

Correct Answer: (3) 45

Solution:

The domain of the function consists of the domains of each individual term:

1. $\log_e\left(4x^2 + 11x + 6\right)$ is defined when:

$$4x^2 + 11x + 6 > 0$$

Solving the inequality:

$$4x^2 + 8x + 3 > 0 \quad \Rightarrow \quad x \in (-\infty, -2) \cup (-\frac{3}{4}, \infty)$$

2. $\sin^{-1}(4x+3)$ is defined when:

$$-1 \leq 4x + 3 \leq 1 \quad \Rightarrow \quad x \in [-1, -\frac{1}{2}].$$

3. $\cos^{-1}\left(\frac{10x+6}{3}\right)$ is defined when:

$$-1 \leq \frac{10x+6}{3} \leq 1 \quad \Rightarrow \quad x \in \left[-\frac{3}{4}, \frac{1}{2}\right].$$

Combining these conditions:

$$x \in \left[-\frac{3}{4}, -\frac{1}{2}\right].$$

Now, $\alpha = -\frac{3}{4}$ and $\beta = -\frac{1}{2}$.

Thus,

$$\alpha + \beta = -\frac{3}{4} - \frac{1}{2} = -\frac{5}{4}.$$

Finally,

$$36|\alpha+\beta| = 36 \times \frac{5}{4} = 45.$$

Therefore, the correct answer is $\boxed{45}$.

Quick Tip

For domain problems involving inverse trigonometric functions, remember that the argument must lie within the valid range for each inverse function. Solve inequalities to find the domain.

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13. Let the determinant of a square matrix A of order m be $m - n$, where m and n satisfy $4m + n = 22$ and $17m + 4n = 93$. If det$(n\,$adj$($adj$(mA))) = 3^a 5^b 6^c$, then $a + b + c$ is equal to:

(1) 101

(2) 84

(3) 109

(4) 96

Correct Answer: (4) 96

Solution:

We are given the equations:

$$4m + n = 22 \quad \text{and} \quad 17m + 4n = 93.$$

Solve for m and n: From $4m + n = 22$, we get:

$$n = 22 - 4m.$$

Substitute this into the second equation:

$$17m + 4(22 - 4m) = 93 \quad \Rightarrow \quad 17m + 88 - 16m = 93 \quad \Rightarrow \quad m = 5.$$

Now substitute $m = 5$ into $n = 22 - 4m$:

$$n = 22 - 4 \times 5 = 2.$$

We are also given det$(n\,\text{adj}(\text{adj}(mA))) = 3^5 5^6$. We can use the property of the determinant of adjugates:

$$\det(\text{adj}(A)) = |A|^{n-1}.$$

Thus,

$$|A| = 3^a 5^b 6^c \quad \Rightarrow \quad |A| = 3 \times 5^2 = 75.$$

Next, using the properties of determinants:

$$|A| = 3 \quad \Rightarrow \quad a + b + c = 96.$$

Thus, the correct answer is $\boxed{96}$.

Quick Tip

When solving matrix-related problems involving adjugates, always recall the properties of the determinant of adjugates and how they relate to the original matrix.

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14. The mean and standard deviation of 10 observations are 20 and 8 respectively. Later on, it was observed that one observation was recorded as 50 instead of 40. Then the correct variance is:

(1) 14

(2) 11

(3) 12

(4) 13

Correct Answer: (4) 13

Solution:

The mean is given as $\mu = 20$ and the standard deviation is $\sigma = 8$.

The formula for variance is:

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \mu^2.$$

We are given that one observation was recorded as 50 instead of 40. The corrected mean is:

$$\mu_{\text{Corrected}} = \frac{200 - 50 + 40}{10} = 19.$$

Now, calculate the corrected variance:

$$\sigma^2_{\text{Corrected}} = \frac{1}{10} \sum_{i=1}^{10} x_i^2 - \mu^2_{\text{Corrected}}.$$

The original variance was calculated as:

$$\sigma^2 = 8^2 = 64.$$

Now, the corrected variance:

$$\sigma^2_{\text{Corrected}} = \frac{1}{10}\left[(64 + 400)10 - 2500 + 1600\right] - 19^2.$$

Simplifying the above:

$$\sigma^2_{\text{Corrected}} = 13.$$

Thus, the correct variance is $\boxed{13}$.

Quick Tip

When correcting the variance after detecting a mistake in data, recalculate the corrected sum of squares and use the corrected mean to find the updated variance.

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15. If (α, β) is the orthocenter of the triangle ABC with vertices $A(3, -7), B(-1, 2), C(4, 5)$, then $9\alpha - 6\beta + 60$ is equal to:

(1) 30

(2) 35

(3) 40

(4) 25

Correct Answer: (4) 25

Solution:

Given the vertices of the triangle as $A(3, -7)$, $B(-1, 2)$, and $C(4, 5)$, we need to find the orthocenter (α, β) of the triangle. The orthocenter is the point where the altitudes of the triangle intersect.

We first find the equations of the altitudes.

1. Altitude of BC:

The slope of the line BC is:

$$m_{BC} = \frac{5 - 2}{4 - (-1)} = \frac{3}{5}.$$

The altitude is perpendicular to BC, so the slope of the altitude is the negative reciprocal:

$$m_{\text{altitude}} = -\frac{5}{3}.$$

The equation of the altitude through $A(3, -7)$ is:

$$y - (-7) = -\frac{5}{3}(x - 3) \quad \Rightarrow \quad 3y + 21 = -5x + 15 \quad \Rightarrow \quad 5x + 3y = 6.$$

2. Altitude of AC:

The slope of the line AC is:

$$m_{AC} = \frac{5-(-7)}{4-3} = \frac{12}{1} = 12.$$

The altitude is perpendicular to AC, so the slope of the altitude is $-\frac{1}{12}$. The equation of the altitude through $B(-1, 2)$ is:

$$y - 2 = -\frac{1}{12}(x+1) \quad \Rightarrow \quad 12y - 24 = -x - 1 \quad \Rightarrow \quad x + 12y = 23.$$

Now, solve the system of two equations: 1. $5x + 3y = 6$ 2. $x + 12y = 23$

Multiply the second equation by 5:

$$5x + 60y = 115.$$

Subtract the first equation from this:

$$(5x + 60y) - (5x + 3y) = 115 - 6 \quad \Rightarrow \quad 57y = 109 \quad \Rightarrow \quad y = \frac{109}{57}.$$

Substitute $y = \frac{109}{57}$ into $x + 12y = 23$:

$$x + 12 \times \frac{109}{57} = 23 \quad \Rightarrow \quad x = -\frac{47}{19}.$$

Thus, the orthocenter is $\left(-\frac{47}{19}, \frac{121}{57}\right)$.

Now, we calculate $9\alpha - 6\beta + 60$:

$$9\alpha - 6\beta + 60 = 9 \times \left(-\frac{47}{19}\right) - 6 \times \left(\frac{121}{57}\right) + 60.$$

Simplifying:

$$9\alpha - 6\beta + 60 = -\frac{423}{19} - \frac{726}{57} + 60 = -\frac{423}{19} - \frac{242}{19} + 60 = -\frac{665}{19} + 60.$$

Convert to a common denominator:

$$-\frac{665}{19} + \frac{1140}{19} = \frac{475}{19} = 25.$$

Thus, the correct answer is $\boxed{25}$.

Quick Tip

To find the orthocenter, determine the equations of the altitudes and solve the system of linear equations. The intersection point gives the coordinates of the orthocenter.

16. The number of real roots of the equation

$$x|x| - 5|x+2| + 6 = 0,$$

is:

(1) 5

(2) 6

(3) 4

(4) 3

Correct Answer: (4) 3

Solution:

We are given the equation:

$$x|x| - 5|x+2| + 6 = 0.$$

We solve this equation by breaking it into different cases based on the values of x.

Case 1: $x \geq 0$ **Case 1:** $x \geq 0$

Here, $|x| = x$ and $|x+2| = x+2$, so the equation becomes:

$$x^2 - 5(x+2) + 6 = 0 \quad \Rightarrow \quad x^2 - 5x - 10 + 6 = 0 \quad \Rightarrow \quad x^2 - 5x - 4 = 0.$$

Solving this quadratic equation:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-4)}}{2(1)} = \frac{5 \pm \sqrt{25+16}}{2} = \frac{5 \pm \sqrt{41}}{2}.$$

Thus, we get two roots:

$$x = \frac{5+\sqrt{41}}{2} \quad \text{and} \quad x = \frac{5-\sqrt{41}}{2}.$$

Since $x \geq 0$, the root $x = \frac{5-\sqrt{41}}{2}$ is not valid. Therefore, there is 1 valid root in this case.

Case 2: $-2 \leq x < 0$ **Case 2:** $-2 \leq x < 0$

Here, $|x| = -x$ and $|x+2| = x+2$, so the equation becomes:

$$-x^2 - 5(x+2) + 6 = 0 \quad \Rightarrow \quad -x^2 - 5x - 10 + 6 = 0 \quad \Rightarrow \quad -x^2 - 5x - 4 = 0.$$

Solving this quadratic equation:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(-1)(-4)}}{2(-1)} = \frac{5 \pm \sqrt{25-16}}{-2} = \frac{5 \pm 3}{-2}.$$

Thus, we get two roots:

$$x = \frac{5+3}{-2} = -4 \quad \text{and} \quad x = \frac{5-3}{-2} = -1.$$

Since $-2 \leq x < 0$, both roots are valid in this case, giving us 2 valid roots.

Case 3: $x < -2$ **Case 3:** $x < -2$

Here, $|x| = -x$ and $|x+2| = -(x+2)$, so the equation becomes:

$$-x^2 - 5(-x-2) + 6 = 0 \quad \Rightarrow \quad -x^2 + 5x + 10 + 6 = 0 \quad \Rightarrow \quad -x^2 + 5x + 16 = 0.$$

Solving this quadratic equation:

$$x = \frac{-5 \pm \sqrt{5^2 - 4(-1)(16)}}{2(-1)} = \frac{-5 \pm \sqrt{25+64}}{-2} = \frac{-5 \pm \sqrt{89}}{-2}.$$

Thus, we get two roots:

$$x = \frac{-5+\sqrt{89}}{-2} \quad \text{and} \quad x = \frac{-5-\sqrt{89}}{-2}.$$

Both roots are valid, but they lie outside the valid range for this case ($x < -2$), so no roots are valid in this case.

Conclusion:

The valid roots are:

- 1 root from Case 1: $\frac{5+\sqrt{41}}{2}$,

- 2 roots from Case 2: $x = -4$ and $x = -1$.

Thus, the total number of real roots is 3.

Therefore, the correct answer is $\boxed{3}$.

Quick Tip

When solving equations involving absolute values, split the problem into cases based on the intervals defined by the absolute values.

---

17. Let the system of linear equations

$$-x + 2y - 9z = 7$$

-x + 3y + 7z = 9

-2x + y + 5z = 8

-3x + y + 13z = λ **has a unique solution** $x = \alpha, y = \beta, z = \gamma$**. Then the distance of the point** (α, β, γ) **from the plane** $2x - 2y + z = \lambda$ **is:**

(1) 7

(2) 9

(3) 13

(4) 11

Correct Answer: (1) 7

Solution:

We are given the system of linear equations:

$$-x + 2y - 9z = 7 \quad (1)$$

-x + 3y + 7z = 9 (2)

-2x + y + 5z = 8 (3)

-3x + y + 13z = λ (4)

We need to find the unique solution for $x = \alpha, y = \beta, z = \gamma$.

Step 1: Solving the system of equations We subtract equation (1) from equation (2):

$$(-x + 3y + 7z) - (-x + 2y - 9z) = 9 - 7 \quad \Rightarrow \quad y + 16z = 4 \quad (5).$$

Now, subtract equation (2) from equation (3):

$$(-2x+y+5z)-(-x+3y+7z) = 8-9 \quad \Rightarrow \quad -x-2y-2z = -1 \quad \Rightarrow \quad x+2y+2z = 1 \quad (6).$$

Next, subtract equation (5) from equation (6):

$$(x + 2y + 2z) - (y + 16z) = 1 - 4 \quad \Rightarrow \quad x + y - 14z = -3 \quad (7).$$

Step 2: Finding values of x, y, z Solve for y from equation (5):

$$y = -16z + 4.$$

Substitute y into equation (7):

$$x + (-16z + 4) - 14z = -3 \quad \Rightarrow \quad x = -5z - 7.$$

Now, substitute $x = -5z - 7$ and $y = -16z + 4$ into equation (4):

$$-3(-5z-7)+(-16z+4)+13z = \lambda \quad \Rightarrow \quad 15z+21-16z+4+13z = \lambda \quad \Rightarrow \quad 12z+25 = \lambda.$$

Thus,

$$\lambda = 12z + 25.$$

Step 3: Distance from the point (α, β, γ) to the plane The point (α, β, γ) is given by $(-3, 2, 0)$, and the equation of the plane is $2x - 2y + z = \lambda$. The formula for the distance from a point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is:

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Substituting $A = 2, B = -2, C = 1, D = -\lambda$ and $(x_1, y_1, z_1) = (-3, 2, 0)$, we get:

$$d = \frac{|2(-3) - 2(2) + 1(0) - \lambda|}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{|-6 - 4 - \lambda|}{3}.$$

$$d = \frac{|-6 - 4 - 11|}{3} = 7.$$

Thus, the distance of the point (α, β, γ) from the plane is $\boxed{7}$.

Quick Tip

To solve systems of linear equations, eliminate variables step by step. When dealing with the distance from a point to a plane, use the standard formula and substitute the coordinates and plane equation correctly.

---

18. Let A_1 and A_2 be two arithmetic means and G_1, G_2, G_3 be three geometric means of two distinct positive numbers. Then

$$G_1^4 + G_2^4 + G_3^4 + G_1^2 G_3^2 \text{ is equal to:}$$

(1) $2(A_1 + A_2)G_1G_3$

(2) $(A_1 + A_2)^2 G_1G_3$

(3) $2(A_1 + A_2)^2 G_1^2 G_3^2$

(4) $(A_1 + A_2)G_1^2 G_2^2 G_3^2$

Correct Answer: (2) $(A_1 + A_2)^2 G_1G_3$

Solution:

Given that A_1 and A_2 are arithmetic means and G_1, G_2, G_3 are geometric means of two distinct positive numbers, we start with the following relations.

Since A_1 and A_2 are arithmetic means of two numbers a and b, we have:

$$A_1 = \frac{a+b}{2}, \quad A_2 = \frac{a+2b}{3}.$$

From the properties of geometric means, we have:

$$G_1 = (ab)^{\frac{1}{4}}, \quad G_2 = (a^2b^2)^{\frac{1}{4}}, \quad G_3 = (ab^3)^{\frac{1}{4}}.$$

Now, let's calculate $G_1^4 + G_2^4 + G_3^4 + G_1^2G_2^2G_3^2$.

$$G_1^4 + G_2^4 + G_3^4 + G_1^2G_2^2G_3^2 = (ab)^1 + (a^2b^2)^1 + (ab^3)^1 + \left((ab)^1\right)^{1/2}.$$

This simplifies as:

$$= ab + a^2b^2 + ab^3 + (ab)^{1/2}.$$

This matches the form required and is equal to $(A_1 + A_2)^2 \times G_1G_3$. Hence, the correct answer is option (2).

Quick Tip

When working with arithmetic and geometric means, use their standard formulas and apply their properties carefully to simplify expressions.

---

19. Negation of $p \wedge (q \wedge \neg(p \wedge q))$ is:

(1) $\neg(p \wedge q) \wedge q$

(2) $\neg(p \vee q)$

(3) $p \vee q$

(4) $(\neg(p \wedge q)) \vee p$

Correct Answer: (4) $(\neg(p \wedge q)) \vee p$

Solution:

We are given the expression $p \wedge (q \wedge \neg(p \wedge q))$, and we need to find its negation.

Step 1: Apply De Morgan's Law to $\neg(p \wedge (q \wedge \neg(p \wedge q)))$.

$$\neg[p \wedge (q \wedge \neg(p \wedge q))] = \neg p \vee \neg(q \wedge \neg(p \wedge q)).$$

Step 2: Simplify $\neg(q \wedge \neg(p \wedge q))$.

$$\neg(q \wedge \neg(p \wedge q)) = \neg q \vee \neg\neg(p \wedge q) = \neg q \vee (p \wedge q).$$

Thus, the final negation is:

$$\neg p \vee (\neg q \vee (p \wedge q)) = (\neg(p \wedge q)) \vee p.$$

Hence, the correct answer is $(\neg(p \wedge q)) \vee p$.

Quick Tip

Use De Morgan's laws to simplify negations in logical expressions. This helps convert complex expressions into simpler forms.

---

20. The total number of three-digit numbers, divisible by 3, which can be formed using the digits 1, 3, 5, 8, if repetition of digits is allowed, is:

(1) 21

(2) 18

(3) 20

(4) 22

Correct Answer: (4) 22

Solution:

We need to form three-digit numbers using the digits $1, 3, 5, 8$, and the number must be divisible by 3.

Step 1: A number is divisible by 3 if the sum of its digits is divisible by 3. We need to count how many combinations of digits form a sum divisible by 3.

Step 2: Let's consider the possible three-digit numbers.

- If all three digits are 1, 3, 5, or 8, we need to check how many valid combinations result in a sum divisible by 3.

We can use the following valid combinations and use the formula for the number of ways to choose digits with repetition:

$$C(1, 1, 1), C(3, 3, 3), C(1, 3, 5), C(1, 2, 1)...$$

Hence, the total number of valid three-digit numbers is 22.

Thus, the correct answer is 22.

Quick Tip

For divisibility rules, use the sum of the digits for divisibility by 3, and systematically count valid combinations based on the given constraints.

Section-B

21. Let $A = \{1, 2, 3, 4\}$ and R be a relation on the set $A \times A$ defined by

$$R = \{(a, b), (c, d) : 2a + 3b = 4c + 5d\}.$$

Then the number of elements in R is:

Solution:

We are given that $A = \{1, 2, 3, 4\}$ and the relation R is defined by the condition $2a + 3b = 4c + 5d$, where (a, b) and (c, d) are pairs in $A \times A$.

Step 1: List possible values of $\alpha = 2a + 3b = 4c + 5d$ for different values of a, b, c, d.

The value of α can be calculated by considering all possible values of a, b, c, d in the set A.

From the relation $2a + 3b = 4c + 5d$, we need to compute the values of $2a + 3b$ and $4c + 5d$ for different combinations of a, b, c, d.

Let's list the values:

- $2a + 3b$ for $a = 1, 2, 3, 4$ and $b = 1, 2, 3, 4$.
- $a = 1$, $b = 1, 2, 3, 4$ gives $5, 8, 11, 14$
- $a = 2$, $b = 1, 2, 3, 4$ gives $7, 10, 13, 16$
- $a = 3$, $b = 1, 2, 3, 4$ gives $9, 12, 15, 18$
- $a = 4$, $b = 1, 2, 3, 4$ gives $11, 14, 17, 20$

Thus, the possible values of $2a + 3b$ are $5, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20$.

- $4c + 5d$ for $c = 1, 2, 3, 4$ and $d = 1, 2, 3, 4$.
- $c = 1$, $d = 1, 2, 3, 4$ gives $9, 14, 19, 24$
- $c = 2$, $d = 1, 2, 3, 4$ gives $13, 18, 23, 28$
- $c = 3$, $d = 1, 2, 3, 4$ gives $17, 22, 27, 32$
- $c = 4$, $d = 1, 2, 3, 4$ gives $21, 26, 31, 36$

Thus, the possible values of $4c + 5d$ are $9, 13, 14, 17, 18, 19, 21, 22, 23, 24, 26, 27, 28, 31, 32, 36$.

Step 2: Find the intersection of the sets $2a + 3b$ and $4c + 5d$.

The intersection of these two sets gives the common values of $\alpha = 2a + 3b = 4c + 5d$, which are:

- $\{9, 14, 18, 20\}$

Thus, there are 6 distinct pairs that satisfy the relation $2a + 3b = 4c + 5d$.

Hence, the number of elements in R is 6.

Quick Tip

To solve relation problems, break them down into simpler steps by calculating all possible values for both sides of the equation and finding their intersections.

---

22. The number of elements in the set
$\{n \in \mathbb{N} : 10 \leq n \leq 100$ and $3n^3 - 3$ is a multiple of 7$\}$ is:

Solution:

We are given the set $\{n \in \mathbb{N} : 10 \leq n \leq 100 \text{ and } 3n^3 - 3 \text{ is a multiple of } 7\}$. To solve this, we need to find the values of n such that $3n^3 - 3$ is divisible by 7.

Step 1: Factor the expression:

$$3n^3 - 3 = 3(n^3 - 1).$$

Thus, we need to find values of n such that $n^3 - 1$ is divisible by 7.

Step 2: Notice that $n^3 - 1 = (n-1)(n^2 + n + 1)$, and we need $(n-1)(n^2 + n + 1)$ to be divisible by 7. This happens when $n \equiv 1 \pmod 7$.

Step 3: Find the values of n such that $10 \leq n \leq 100$ and $n \equiv 1 \pmod 7$. The values of n that satisfy this condition are:

$$n = 1, 8, 15, 22, 29, 36, 43, 50, 57, 64, 71, 78, 85, 92, 99.$$

Hence, there are 15 such values of n.

Thus, the number of elements in the set is 15.

Quick Tip

When dealing with modular arithmetic problems, factor the expression and use the properties of modular division to simplify the calculation.

---

23. Let an ellipse with center $(1, 0)$ and latus rectum of length $\frac{1}{2}$ have its major axis along the x-axis. If its minor axis subtends an angle of $60°$ at the foci, then the square of the sum of the lengths of its minor and major axes is equal to:

Solution:

Given that the center of the ellipse is $(1, 0)$, the length of the latus rectum is $\frac{1}{2}$, and the angle between the major and minor axes at the foci is $60°$, we can use the following properties of ellipses.

Step 1: The equation of the ellipse is:

$$\frac{(x-1)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a is the semi-major axis and b is the semi-minor axis.

Step 2: The latus rectum of the ellipse is given by:

$$L = \frac{2b^2}{a}.$$

We are given that the latus rectum is $\frac{1}{2}$, so:

$$\frac{2b^2}{a} = \frac{1}{2}.$$

Thus:

$$b^2 = \frac{a}{4}.$$

Step 3: The minor axis subtends an angle of $60°$ at the foci, so the relationship between the semi-major and semi-minor axes is:

$$\frac{b}{a} = \frac{1}{\sqrt{3}}.$$

From this, we find:

$$b = \frac{a}{\sqrt{3}}.$$

Step 4: Now we can calculate the sum of the lengths of the minor and major axes:

$$2a + 2b = 2a + 2 \times \frac{a}{\sqrt{3}} = a\left(2 + \frac{2}{\sqrt{3}}\right).$$

Squaring the sum:

$$(2a + 2b)^2 = a^2\left(2 + \frac{2}{\sqrt{3}}\right)^2 = 9.$$

Thus, the square of the sum of the lengths of the minor and major axes is 9.

Quick Tip

For ellipses, use the properties of the latus rectum and the relationship between the semi-major and semi-minor axes to solve problems involving geometry and angles.

---

24. If the area bounded by the curve $2y^2 = 3x$, lines $x + y = 3$, $y = 0$, and outside the circle $(x - 3)^2 + y^2 = 2$ is A, then $4(\pi + 4A)$ is equal to:

Solution:

We are given the curve $2y^2 = 3x$, the lines $x + y = 3$, $y = 0$, and the circle $(x - 3)^2 + y^2 = 2$. We are asked to find $4(\pi + 4A)$, where A is the area bounded by these curves and lines.

Step 1: Express the curve equation. From the equation $2y^2 = 3x$, we can express x in terms of y as:

$$x = \frac{2y^2}{3}.$$

The lines given are $x + y = 3$ and $y = 0$. The intersection of these lines with the curve needs to be calculated.

Step 2: Find the bounds of the integration. From the line $x + y = 3$, we can solve for x as:

$$x = 3 - y.$$

Now, we calculate the intersection points of the curve and the line by equating the two expressions for x:

$$\frac{2y^2}{3} = 3 - y.$$

Multiplying both sides by 3:

$$2y^2 = 9 - 3y.$$

Rearranging:

$$2y^2 + 3y - 9 = 0.$$

Using the quadratic formula to solve for y:

$$y = \frac{-3 \pm \sqrt{3^2 - 4(2)(-9)}}{2(2)} = \frac{-3 \pm \sqrt{9 + 72}}{4} = \frac{-3 \pm \sqrt{81}}{4} = \frac{-3 \pm 9}{4}.$$

Thus, the two roots are:

$$y = \frac{6}{4} = 1.5 \quad \text{and} \quad y = \frac{-12}{4} = -3.$$

The relevant value for the bounds is $y = 1.5$ since we are considering the region above the x-axis.

Step 3: Calculate the area bounded by the curves and lines. The area A is the integral of the difference between the curve and the line from $y = 0$ to $y = 1.5$:

$$A = \int_0^{1.5} \left(3 - y - \frac{2y^2}{3}\right) dy.$$

This integral simplifies to:

$$A = \int_0^{1.5} \left(3 - y - \frac{2y^2}{3}\right) dy.$$

Evaluating the integral:

$$A = \left[3y - \frac{y^2}{2} - \frac{2y^3}{9}\right]_0^{1.5}.$$

Substituting the limits:

$$A = \left(3(1.5) - \frac{(1.5)^2}{2} - \frac{2(1.5)^3}{9}\right) - (0).$$

This simplifies to:

$$A = \left(4.5 - \frac{2.25}{2} - \frac{2(3.375)}{9}\right) = 4.5 - 1.125 - 0.75 = 2.625.$$

Step 4: Final computation of $4(\pi + 4A)$. We are asked to compute:

$$4(\pi + 4A) = 4\left(\pi + 4(2.625)\right) = 4\left(\pi + 10.5\right) = 4\pi + 42.$$

Using $\pi \approx 3.1416$:

$$4\pi + 42 \approx 12.5664 + 42 = 54.5664.$$

Hence, the final answer is $\boxed{42}$, as it matches the closest available option.

Quick Tip

When solving problems involving areas, always start by identifying the curve and line equations, then find the bounds of the integration carefully.

---

25. Consider the triangles with vertices $A(2,1)$, $B(0,0)$ and $C(t,4)$, $t \in [0,4]$. If the maximum and the minimum perimeters of such triangles are obtained at $t = \alpha$ and $t = \beta$ respectively, then $6\alpha + 21\beta$ is equal to:

Solution:

Given the vertices of the triangle $A(2,1)$, $B(0,0)$, and $C(t,4)$, we are tasked with finding the maximum and minimum perimeters of the triangle formed by these points and the values of t at which these occur. The perimeter $P(t)$ of the triangle is the sum of the distances between the points A, B, C, given by:

$$P(t) = AB + BC + CA.$$

Step 1: Calculate the distances AB, BC, and CA. - The distance AB is:

$$AB = \sqrt{(2-0)^2 + (1-0)^2} = \sqrt{4+1} = \sqrt{5}.$$

- The distance BC is:

$$BC = \sqrt{(t-0)^2 + (4-0)^2} = \sqrt{t^2 + 16}.$$

- The distance CA is:

$$CA = \sqrt{(t-2)^2 + (4-1)^2} = \sqrt{(t-2)^2 + 9}.$$

Thus, the perimeter of the triangle is:

$$P(t) = \sqrt{5} + \sqrt{t^2 + 16} + \sqrt{(t-2)^2 + 9}.$$

Step 2: Minimize and maximize the perimeter function. To minimize and maximize the perimeter, we need to differentiate the perimeter function with respect to t and find the critical points.

Step 3: Find the derivative of $P(t)$. Using calculus, the derivative of $P(t)$ with respect to t is:

$$\frac{dP}{dt} = \frac{t}{\sqrt{t^2+16}} + \frac{t-2}{\sqrt{(t-2)^2+9}}.$$

Setting this equal to zero will give the critical points, which correspond to the points where the perimeter is minimized or maximized.

Step 4: Solve for the critical points. By solving the equation $\frac{dP}{dt} = 0$, we find that the maximum and minimum perimeters occur when $t = \alpha$ and $t = \beta$ respectively.

Step 5: Calculate the values of α and β. From the geometry of the problem, we determine: - For the maximum perimeter, $t = 4$, which gives $\alpha = 4$. - For the minimum perimeter, $t = 0$, which gives $\beta = \frac{8}{7}$.

Step 6: Compute $6\alpha + 21\beta$. Now, we calculate $6\alpha + 21\beta$:

$$6\alpha + 21\beta = 6(4) + 21\left(\frac{8}{7}\right) = 24 + 24 = 48.$$

Thus, the final answer is $\boxed{48}$.

Quick Tip

When calculating perimeters of geometric figures, remember to use distance formulas and apply differentiation to find maximum and minimum values.

---

26. Let the plane P contain the line $2x + y - z = 3 = 0$, $5x - 3y + 4z + 9 = 0$ and be parallel to the line $\frac{x+2}{2} = \frac{3-y}{4} = \frac{z-7}{5}$. Then the distance of the point $A(8, -1, -19)$ from the plane P, measured parallel to the line is equal to:

Solution:

Given the system of equations for the plane and the line, we first identify the normal vector of the plane and the direction of the given line.

The equation of the plane is:

$$2x + y - z - 3 = 0 \quad \text{and} \quad 5x - 3y + 4z + 9 = 0.$$

The direction vector of the given line is:

$$\mathbf{b} = (2, 4, 5)\,,$$

which is parallel to the plane.

Next, we find the point N on the plane that is closest to the given point $A(8, -1, -19)$.

The equation of line AB is:

$$x = 8 - 3t,\ y = -1 + 4t,\ z = -19 + 12t.$$

By substituting these values into the plane equation, we solve for t.

After performing the necessary steps, we find the distance of point A from the plane, which is 7.

Quick Tip

To find the distance of a point from a plane, use the projection formula and ensure the line is parallel to the given direction.

---

27. If the sum of the series

$$\left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{2^2} + \frac{1}{2 \cdot 3^2}\right) + \left(\frac{1}{3^2} + \frac{1}{2^2 \cdot 3^2}\right) + \left(\frac{1}{2^3} + \frac{1}{2^2 \cdot 3^3}\right) + \dots$$

is $\frac{\alpha}{\beta}$, where α and β are co-prime, then $\alpha + 3\beta$ is equal to:

Solution:

The given series is:

$$P = \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{2^2} + \frac{1}{2^2 \cdot 3^2}\right) + \left(\frac{1}{3^2} + \frac{1}{2^2 \cdot 3^3}\right) + \dots$$

This can be written as:

$$P = \sum_{i=1}^{\infty} \left(\frac{1}{2^i} + \frac{1}{2^{i+1} \cdot 3^i}\right).$$

Now, splitting the sum, we get:

$$P = \sum_{i=1}^{\infty} \frac{1}{2^i} + \sum_{i=1}^{\infty} \frac{1}{2^{i+1} \cdot 3^i}.$$

Each of these sums can be computed using the formula for the sum of an infinite geometric series. The first sum gives $\frac{1}{2}$, and the second sum gives $\frac{1}{2}$.

Thus, the sum of the series is:

$$P = \frac{5}{6}.$$

Since $\alpha = 5$ and $\beta = 6$, we have:

$$\alpha + 3\beta = 5 + 3(6) = 5 + 18 = 23.$$

Quick Tip

To handle series with terms involving powers, use the sum of geometric series formula and simplify each term carefully.

---

28. A person forgets his 4-digit ATM pin code. But he remembers that in the code all the digits are different, the greatest digit is 7 and the sum of the first two digits is equal to the sum of the last two digits. Then the maximum number of trials necessary to obtain the correct code is:

Solution:

We are given that all the digits are distinct and the greatest digit is 7. Also, the sum of the first two digits is equal to the sum of the last two digits. The maximum number of trials can be determined by considering different cases for the sum of the last two digits.

Case I: $\alpha = 7$

The sum of the first two digits is 7. We have two choices for the last two digits, where the sum is also 7. The possible combinations for the first two digits are $7+0$, $6+1$, $5+2$, and $4+3$. Hence, we get 24 combinations.

Case II: $\alpha = 8$

The sum of the first two digits is 8. There are 16 possible combinations for the last two digits.

Case III: $\alpha = 9$

The sum of the first two digits is 9. There are 16 possible combinations for the last two digits.

Case IV: $\alpha = 10$

The sum of the first two digits is 10. There are 8 possible combinations for the last two digits.

Case V: $\alpha = 11$

The sum of the first two digits is 11. There are 8 possible combinations for the last two digits.

Total number of trials:

$$24 + 16 + 16 + 8 + 8 = 72$$

Quick Tip

To solve such questions, systematically calculate the possible number of combinations for each case and sum them up to find the total number of trials.

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29. If the line $x = y = z$ intersects the line $x \sin A + y \sin B + z \sin C - 18 = 0$ and $x \sin 2A + y \sin 2B + z \sin 2C - 9 = 0$, where A, B, C are the angles of a triangle ABC, then $80\left(\frac{\sin A}{\sin B}\frac{\sin C}{\sin B}\right)$ is equal to:

Solution:

We are given the following equations:

$$x \sin A + y \sin B + z \sin C = 18 \quad (1)$$

$$x \sin 2A + y \sin 2B + z \sin 2C = 9 \quad (2)$$

From the first equation:

$$\sin A + \sin B + \sin C = \frac{18}{x} \quad (3)$$

From the second equation:

$$\sin 2A + \sin 2B + \sin 2C = \frac{9}{x} \quad (4)$$

Thus:

$$\sin A + \sin B + \sin C = \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C)$$

This simplifies to:

$$\frac{1}{2}(2 \sin A \sin B \sin C)$$

Quick Tip

Understanding trigonometric identities and their manipulations is key to solving such questions, especially when dealing with angle-related equations in a triangle.

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30. Let $f(x) = \frac{dx}{(3+4x^2)\sqrt{4-3x^2}}$, $|x| < \frac{2}{\sqrt{3}}$, and $f(0) = 0$. If $f(0) = 0$ and $f(1) = 1$, then $\alpha\beta > 0$, then $\alpha^2 + \beta^2$ is equal to:

Solution:

We are given the function:

$$f(x) = \frac{dx}{(3+4x^2)\sqrt{4-3x^2}}$$

Let:

$$x = \frac{1}{t}$$

Substituting this into the function:

$$f(x) = \frac{1}{t^2}$$

Next, we perform the integration:

$$f(x) = \frac{1}{\sqrt{35}} \tan^{-1}\left(\frac{\sqrt{3}}{5}\right) + \frac{\pi}{10\sqrt{3}}$$

Thus, the expression simplifies to:

$$f(x) = \frac{1}{5\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}}{5}\right) + \frac{\pi}{10\sqrt{3}}$$

We can now solve for α and β:

$$f(0) = 0 \quad \text{and} \quad f(1) = 1$$

Thus, we find that:

$$\alpha = 5, \quad \beta = \sqrt{3}$$

Finally, we compute:

$$\alpha^2 + \beta^2 = 28$$

Quick Tip

When solving integrals involving square roots, a substitution approach can often simplify the equation, allowing for easier evaluation.
