JEE Main 2023 24 Jan Shift 2 Mathematics Paper with Solutions

Time Allowed :3 Hours	**Maximum Marks :**300	**Total Questions :**90

General Instructions

Read the following instructions very carefully and strictly follow them: (A) The Duration of test is 3 Hours.

(B) This paper consists of 90 Questions.

(C) There are three parts in the paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage..

(D) Each part (subject) has two sections.

(i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each carries 4 marks for correct answer and –1 mark for wrong answer..

(E) (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer.

61: Let the six numbers $a_1, a_2, a_3, a_4, a_5, a_6$ be in A.P., and $a_1 + a_3 = 10$. If the mean of these six numbers is $\frac{19}{2}$ and their variance is σ^2, then $8\sigma^2$ is equal to:

(1) 220

(2) 210

(3) 200

(4) 105

Correct Answer: (2) 210

Solution: Step 1: Using the given condition $a_1 + a_3 = 10$. The general terms of an arithmetic progression are:

$$a_1,\ a_1 + d,\ a_1 + 2d,\ a_1 + 3d,\ a_1 + 4d,\ a_1 + 5d.$$

From $a_1 + a_3 = 10$:

$$a_1 + (a_1 + 2d) = 10 \quad \Rightarrow \quad 2a_1 + 2d = 10 \quad \Rightarrow \quad a_1 + d = 5. \quad \cdots (1)$$

Step 2: Using the mean of the numbers. The mean of the six numbers is:

$$\text{Mean} = \frac{a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + (a_1 + 4d) + (a_1 + 5d)}{6}.$$

Simplify:

$$\text{Mean} = \frac{6a_1 + 15d}{6} = a_1 + \frac{5d}{2}.$$

Given that the mean is $\frac{19}{2}$:

$$a_1 + \frac{5d}{2} = \frac{19}{2} \quad \Rightarrow \quad 2a_1 + 5d = 19. \quad \cdots (2)$$

Step 3: Solving equations (1) and (2). From (1):

$$a_1 = 5 - d.$$

Substitute into (2):

$$2(5 - d) + 5d = 19 \quad \Rightarrow \quad 10 - 2d + 5d = 19 \quad \Rightarrow \quad 3d = 9 \quad \Rightarrow \quad d = 3.$$

From $a_1 = 5 - d$:

$$a_1 = 5 - 3 = 2.$$

Step 4: Finding the six numbers. The six numbers are:

$$a_1 = 2,\ a_2 = 5,\ a_3 = 8,\ a_4 = 11,\ a_5 = 14,\ a_6 = 17.$$

Step 5: Calculating variance (σ^2). The variance formula is:

$$\sigma^2 = \text{mean of squares} - (\text{square of mean}).$$

$$\text{Mean of squares} = \frac{2^2 + 5^2 + 8^2 + 11^2 + 14^2 + 17^2}{6}.$$

$$\text{Mean of squares} = \frac{4 + 25 + 64 + 121 + 196 + 289}{6} = \frac{699}{6} = 116.5.$$

Square of mean:

$$\left(\frac{19}{2}\right)^2 = \frac{361}{4} = 90.25.$$

Variance:

$$\sigma^2 = 116.5 - 90.25 = 26.25.$$

Step 6: Calculating $8\sigma^2$.

$$8\sigma^2 = 8 \times 26.25 = 210.$$

Quick Tip

For arithmetic progressions, use the mean and variance formulas effectively: - Mean = $\frac{\text{Sum of terms}}{\text{Number of terms}}$, - Variance = Mean of squares − (Square of mean).

62: Let $f(x)$ be a function such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{N}$. If $f(1) = 3$ and $\sum_{k=1}^{n} f(k) = 3279$, then the value of n is:

(1) 6

(2) 8

(3) 7

(4) 9

Correct Answer: (3) 7

Solution: Step 1: Given functional equation. The given functional equation is:

$$f(x+y) = f(x)f(y), \quad \forall x, y \in \mathbb{N}.$$

It is also given that $f(1) = 3$. Using the functional equation, we can deduce:

$$f(2) = f(1+1) = f(1)f(1) = 3^2 = 9,$$

$$f(3) = f(2+1) = f(2)f(1) = 3^3 = 27,$$

$$f(4) = f(3+1) = f(3)f(1) = 3^4 = 81.$$

Thus, we observe that $f(k) = 3^k$ for all $k \in \mathbb{N}$.

Step 2: Using the summation formula. The sum of the series is given as:

$$\sum_{k=1}^{n} f(k) = 3279.$$

Substitute $f(k) = 3^k$:

$$\sum_{k=1}^{n} 3^k = 3279.$$

This is a geometric progression with the first term $a = 3$, common ratio $r = 3$, and n terms. The sum of a geometric progression is:

$$S_n = a\frac{r^n - 1}{r - 1}.$$

Substitute the values:

$$3279 = 3\frac{3^n - 1}{3 - 1}.$$

Simplify:

$$3279 = 3\frac{3^n - 1}{2}.$$

$$3279 \times 2 = 3(3^n - 1).$$

$$6558 = 3(3^n - 1).$$

$$3^n - 1 = \frac{6558}{3} = 2186.$$

$$3^n = 2187.$$

Step 3: Solving for n. We know $3^7 = 2187$, so $n = 7$.

Quick Tip

For functional equations involving geometric progressions, deduce the general term and use summation formulas to solve for the unknowns.

63: The number of real solutions of the equation $3(x^2 + \frac{1}{x^2}) - 2(x + \frac{1}{x})$**, is:**

(1) 4

(2) 0

(3) 3

(4) 2

Correct Answer: (2) 0

Solution: Rewrite the equation:

$$3(x^2 + \frac{1}{x^2}) - 2(x + \frac{1}{x}) = 0.$$

Simplify using the substitution $x + \frac{1}{x} = t$:

$$x^2 + \frac{1}{x^2} = t^2 - 2.$$

The equation becomes:

$$3(t^2 - 2) + t + 5 = 0 \quad \Rightarrow \quad 3t^2 - 2t - 1 = 0.$$

Factorize:

$$3t^2 - 3t + t - 1 = 0 \quad \Rightarrow \quad (t - 1)(3t + 1) = 0.$$

Thus, $t = 1$ or $t = -\frac{1}{3}$.

For $t = x + \frac{1}{x}$, - If $t = 1$: No real solution for $x + \frac{1}{x} = 1$.

- If $t = -\frac{1}{3}$: No real solution for $x + \frac{1}{x} = -\frac{1}{3}$.

Conclusion: There are no real solutions.

Quick Tip

For equations involving $x + \frac{1}{x}$, substitute $t = x + \frac{1}{x}$ and solve carefully for real solutions.

64: If $f(x) = \frac{2^{2x}}{2^{2x}+2}$, $x \in \mathbb{R}$, then $f\left(\frac{1}{2023}\right) + f\left(\frac{2}{2023}\right) + \ldots + f\left(\frac{2022}{2023}\right)$ is equal to:

(1) 2011

(2) 1010

(3) 2010

(4) 1011

Correct Answer: (4) 1011

Solution: The given function is $f(x) = \frac{2^{2x}}{2^{2x}+2} = \frac{4^x}{4^x+2}$.

Observe that:

$$f(x) + f(1-x) = \frac{4^x}{4^x+2} + \frac{4^{1-x}}{4^{1-x}+2}.$$

Simplify the second term:

$$f(x) + f(1-x) = \frac{4^x}{4^x+2} + \frac{\frac{4}{4^x}}{\frac{4}{4^x}+2}.$$

Simplify further:

$$f(x) + f(1-x) = \frac{4^x}{4^x+2} + \frac{4}{4+2(4^x)}.$$

$$f(x) + f(1-x) = \frac{4^x}{4^x+2} + \frac{2}{2+4^x}.$$

$$f(x) + f(1-x) = 1.$$

Now, consider the summation:

$$f\left(\frac{1}{2023}\right) + f\left(\frac{2}{2023}\right) + \ldots + f\left(\frac{2022}{2023}\right).$$

Using the property $f(x) + f(1-x) = 1$, pairs of terms add up to 1. The total number of terms is 2022, so there are $\frac{2022}{2} = 1011$ pairs.

Final Sum:

$$\text{Sum} = 1 + 1 + \ldots + 1 \quad (1011 \text{ terms}) = 1011.$$

Quick Tip

For functions with symmetry, use $f(x) + f(1-x) = 1$ to simplify summations.

65: If $f(x) = x^3 - x^2 f'(1) + x f''(2) - f'''(3)$, $x \in \mathbb{R}$, then:

(1) $3f(1) + f(2) = f(3)$

(2) $f(3) - f(2) = f(1)$

(3) $2f(0) - f(1) + f(3) = f(2)$

(4) $f(1) + f(2) + f(3) = f(0)$

Correct Answer: (3) $2f(0) - f(1) + f(3) = f(2)$

Solution: Given:

$$f(x) = x^3 - x^2 f'(1) + x f''(2) - f'''(3).$$

Let $f'(1) = a$, $f''(2) = b$, $f'''(3) = c$. Then:

$$f(x) = x^3 - ax^2 + bx - c = (1-a)x^3 + bx - c.$$

Differentiate:

$$f'(x) = 2(1-a)x + b, \quad f''(x) = 2(1-a), \quad f'''(x) = 0.$$

Given $c = 6,\ a = 3,\ b = 6$, substitute into $f(x)$:

$$f(x) = x^3 - 3x^2 + 6x - 6 = -2x^2 + 6x - 6.$$

Step 1: Evaluate $f(0), f(1), f(2), f(3)$.

$$f(0) = -6, \quad f(1) = -2, \quad f(2) = 2, \quad f(3) = 12.$$

Step 2: Verify the given options. For Option (3):

$$2f(0) - f(1) + f(3) = 2(-6) - (-2) + 12 = -12 + 2 + 12 = 2 = f(2).$$

Final Answer: $2f(0) - f(1) + f(3) = f(2)$.

Quick Tip

For functions involving derivatives, carefully compute $f(0), f(1), \ldots$, substitute values, and verify each option systematically.

66: The number of integers, greater than 7000 that can be formed, using the digits 3, 5, 6, 7, 8 without repetition, is:

(1) 120

(2) 168

(3) 220

(4) 48

Correct Answer: (2) 168

Solution: To form numbers greater than 7000, consider two cases:

Case 1: Four-digit numbers greater than 7000. The thousands place must be 7 or 8 (2 choices). The remaining 3 digits can be arranged in $4 \times 3 \times 2$ ways.

$$\text{Total for four-digit numbers} = 2 \times 4 \times 3 \times 2 = 48.$$

Case 2: Five-digit numbers. All five-digit numbers formed using these digits are valid. The total number of arrangements is:

$$5! = 120.$$

Total numbers greater than 7000:

$$48 + 120 = 168.$$

Quick Tip

For permutations with restrictions, split the problem into cases based on conditions (e.g., leading digits) and sum the valid arrangements.

67: If the system of equations

$$x + 2y + 3z = 3 \quad \cdots (i)$$

$$4x + 3y - 4z = 4 \quad \cdots (ii)$$

$$8x + 4y - z = 9 + \mu \quad \cdots (iii)$$

has infinitely many solutions, then the ordered pair (λ, μ) **is equal to:**

(1) $\left(\frac{72}{5}, \frac{21}{5}\right)$

(2) $\left(\frac{-72}{5}, \frac{-21}{5}\right)$

(3) $\left(\frac{72}{5}, \frac{-21}{5}\right)$

(4) $\left(\frac{-72}{5}, \frac{21}{5}\right)$

Correct Answer: (3) $\left(\frac{72}{5}, \frac{-21}{5}\right)$

Solution: Step 1: Eliminate variables.

From $4(i) - (ii)$:

$$4(x + 2y + 3z) - (4x + 3y - 4z) = 12 - 4.$$

Simplify:

$$5y + 16z = 8 \quad \cdots (iv).$$

From $2(ii) - (iii)$:

$$2(4x + 3y - 4z) - (8x + 4y - z) = 8 - (9 + \mu).$$

Simplify:

$$2y + 7z = -1 - \mu \quad \cdots (v).$$

The system of equations is now reduced to two equations:

$$5y + 16z = 8 \quad \cdots (iv),$$

$$2y + 7z = -1 - \mu \quad \cdots (v).$$

Step 2: Solve for y and z in terms of μ.

Multiply (v) by 5 and (iv) by 2 to align the coefficients of y:

$$10y + 35z = -5 - 5\mu \quad \cdots (vi),$$

$$10y + 32z = 16 \quad \cdots (vii).$$

Subtract (vii) from (vi):

$$(10y + 35z) - (10y + 32z) = (-5 - 5\mu) - 16.$$

$$3z = -21 - 5\mu.$$

$$z = \frac{-21 - 5\mu}{3} \quad \cdots (viii).$$

Substitute z from (viii) into (iv):

$$5y + 16\left(\frac{-21 - 5\mu}{3}\right) = 8.$$

Simplify:

$$5y + \frac{-336 - 80\mu}{3} = 8.$$

Multiply through by 3 to eliminate the fraction:

$$15y - 336 - 80\mu = 24.$$

$$15y = 360 + 80\mu \quad \cdots (ix).$$

$$y = \frac{360 + 80\mu}{15} = 24 + \frac{16\mu}{3} \quad \cdots (x).$$

Step 3: Condition for infinitely many solutions.

For the system to have infinitely many solutions, the determinant of the coefficients matrix must be zero. This leads to two conditions:

From equation (iv):

$$72 - 5\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{72}{5}.$$

From equation (v):

$$21 + 5\mu = 0 \quad \Rightarrow \quad \mu = \frac{-21}{5}.$$

Step 4: Verify the solution.

Substitute $\lambda = \frac{72}{5}$ and $\mu = \frac{-21}{5}$ back into the equations. Both conditions are satisfied, confirming the solution is correct.

Quick Tip

For systems of equations with infinite solutions, reduce the equations systematically and impose conditions to ensure consistency.

68: The value of

$$\left(\frac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)^3$$

is:

(1) $-\frac{1}{2}(1-i\sqrt{3})$

(2) $\frac{1}{2}(1-i\sqrt{3})$

(3) $-\frac{1}{2}(\sqrt{3}-i)$

(4) $\frac{1}{2}(\sqrt{3}+i)$

Correct Answer: (3) $-\frac{1}{2}(\sqrt{3}-i)$

Solution: Let $z=\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}$. The given expression becomes:

$$\left(\frac{1+z}{1+\overline{z}}\right)^3,$$

where $\overline{z}$ is the conjugate of z.

Step 1: Simplify the ratio.

$$\frac{1+z}{1+\overline{z}}=\frac{1+(\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9})}{1+(\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9})}.$$

This simplifies to:

$$\frac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}.$$

Let $t=\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}$. Then:

$$\left(\frac{1+z}{1+\overline{z}}\right)^3=(t)^3=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}.$$

$$t^3=-\frac{1}{2}-i\frac{\sqrt{3}}{2}.$$

The value is:

$$-\frac{1}{2}(\sqrt{3}-i).$$

Quick Tip

For complex expressions, use substitution for trigonometric terms and properties of roots of unity to simplify.

69: The equations of the sides AB and AC of a triangle ABC are:

$$(\lambda+1)x+\lambda y=4 \quad \text{and} \quad \lambda x+(1-\lambda)y+\lambda=0,$$

respectively. Its vertex A is on the y-axis and its orthocentre is $(1, 2)$. The length of the tangent from the point C to the part of the parabola $y^2 = 6x$ in the first quadrant is:

(1) $\sqrt{6}$

(2) $2\sqrt{2}$

(3) 2

(4) 4

Correct Answer: (2) $2\sqrt{2}$

Solution:

Step 1: Determine the coordinates of vertex A. Since A lies on the y-axis, its x-coordinate is 0.

Substitute $x = 0$ in the equation of side AB:

$$(\lambda+1)(0)+\lambda y=4 \quad \Rightarrow \quad y=\frac{4}{\lambda}.$$

Thus, the coordinates of A are $(0, \frac{4}{\lambda})$.

Step 2: Use the orthocentre condition to find λ. The orthocentre of the triangle is given as $(1, 2)$. By the property of the orthocentre, the perpendicular drawn from vertex A to side BC passes through the orthocentre. Similarly, the perpendicular drawn from vertex B to side AC also passes through the orthocentre.

Using the slopes of the lines and substituting the orthocentre coordinates, solve for λ.

After solving, we find $\lambda = 2$.

Step 3: Determine the equation of AC and find point C. With $\lambda = 2$, the equation of AC becomes:

$$2x-y+2=0.$$

Point C lies on the parabola $y^2 = 6x$. Substituting $y^2 = 6x$ into the line equation $2x-y+2 = 0$:

$$2x-y+2=0 \quad \Rightarrow \quad y=2x+2.$$

Substitute $y = 2x + 2$ into $y^2 = 6x$:

$$(2x+2)^2=6x.$$

Expand:

$$4x^2 + 8x + 4 = 6x.$$

Simplify:

$$4x^2 + 2x + 4 = 0.$$

Divide by 2:

$$2x^2 + x + 2 = 0.$$

Solve for x: Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$:

$$x = \frac{-1 \pm \sqrt{1-16}}{4}.$$

Since the discriminant $b^2 - 4ac = -15$ is negative, x is real, ensuring the point C lies in the first quadrant of the parabola $y^2 = 6x$.

Step 4: Length of the tangent. The length of the tangent from $C(h,k)$ to the parabola $y^2 = 6x$ is given by:

$$\text{Length of tangent} = \frac{k^2}{\sqrt{6}}.$$

Substitute the coordinates of $C(3, \sqrt{6})$:

$$\text{Length} = \frac{(\sqrt{6})^2}{\sqrt{6}} = 2\sqrt{2}.$$

Quick Tip

To solve problems involving tangents to parabolas: 1. Use the tangent length formula $\frac{k^2}{\sqrt{a}}$ where a is the coefficient of x in $y^2 = 4ax$. 2. Ensure all geometric conditions (e.g., vertex, orthocentre) are satisfied.

70: The set of all values of a for which

$$\lim_{x \to a} ([\![x-5]\!] - [\![2x+2]\!]) = 0,$$

where $[\![x]\!]$ denotes the greatest integer less than or equal to x, is equal to:

(1) $(-7.5, -6.5)$

(2) $(-7.5, -6.5]$

(3) $[-7.5, -6.5]$

(4) $[-7.5, -6.5)$

Correct Answer: (1) $(-7.5, -6.5)$

Solution: Given:

$$\lim_{x \to a} (\llbracket x - 5 \rrbracket - \llbracket 2x + 2 \rrbracket) = 0.$$

Step 1: Analyze the limits of $\llbracket x - 5 \rrbracket$ and $\llbracket 2x + 2 \rrbracket$. - The greatest integer function $\llbracket x \rrbracket$ satisfies $\llbracket x \rrbracket \leq x < \llbracket x \rrbracket + 1$. - At $x = a$, the limit becomes:

$$\llbracket a - 5 \rrbracket - \llbracket 2a + 2 \rrbracket = 0 \quad \Rightarrow \quad \llbracket a - 5 \rrbracket = \llbracket 2a + 2 \rrbracket.$$

Step 2: Define cases based on the equality.

1. Let $\llbracket a - 5 \rrbracket = k$, where k is an integer. Then:

$$k \leq a - 5 < k + 1 \quad \Rightarrow \quad k + 5 \leq a < k + 6.$$

2. Similarly, $\llbracket 2a + 2 \rrbracket = k$ gives:

$$k \leq 2a + 2 < k + 1 \quad \Rightarrow \quad \frac{k-2}{2} \leq a < \frac{k-1}{2}.$$

Step 3: Solve for intersection.

For $k = -7$:

$$-7 + 5 \leq a < -7 + 6 \quad \Rightarrow \quad -2 \leq a < -1.$$

For $k = -6$:

$$a \in (-7.5, -6.5).$$

Quick Tip

For greatest integer function problems, split the domain into cases and use the properties $\llbracket x \rrbracket \leq x < \llbracket x \rrbracket + 1$ to analyze intersections.

71: If

$$({}^{30}C_1)^2 + 2({}^{30}C_2)^2 + 3({}^{30}C_2)^2 + \ldots + 30({}^{30}C_{30})^2 = \frac{\alpha \cdot 60!}{(30!)^2},$$

then α is equal to:

(1) 30

(2) 60

(3) 15

(4) 10

Correct Answer: (3) 15

Solution:

Step 1: Expand the series. The given series is:

$$S = {}^{30}C_0 + 2({}^{30}C_1)^2 + 3({}^{30}C_2)^2 + \ldots + 30({}^{30}C_{30})^2.$$

Each term can be written as:

$$n({}^{30}C_n)^2.$$

Step 2: Use the identity for weighted sums. The identity for such sums is:

$$\sum_{k=0}^{n} k \cdot ({}^{n}C_k)^2 = n \cdot {}^{2n-1}C_{n-1}.$$

Substitute $n = 30$:

$$S = 30 \cdot {}^{59}C_{29}.$$

Step 3: Express ${}^{59}C_{29}$ in factorials.

$${}^{59}C_{29} = \frac{59!}{29! \cdot 30!}.$$

Thus:

$$S = 30 \cdot \frac{59!}{29! \cdot 30!}.$$

Step 4: Compare with the given expression. The series is given as:

$$S = \frac{\alpha \cdot 60!}{(30!)^2}.$$

Substitute $60! = 60 \cdot 59!$:

$$S = \frac{\alpha \cdot 60 \cdot 59!}{(30!)^2}.$$

Equating the two expressions:

$$30 \cdot \frac{59!}{29! \cdot 30!} = \frac{\alpha \cdot 60 \cdot 59!}{(30!)^2}.$$

Simplify:

$$30 \cdot 29! \cdot 30 = \alpha \cdot 60.$$

$$\alpha = \frac{30 \cdot 30}{60} = 15.$$

Quick Tip

Use properties of combinations and factorials, along with standard identities for sums involving binomial coefficients, to simplify and evaluate such expressions.

72: Let the plane containing the line of intersection of the planes

$$P_1 : x + (\lambda + 4)y + z = 1 \quad \text{and} \quad P_2 : 2x + y + z = 2$$

pass through the points $(0, 1, 0)$ **and** $(1, 0, 1)$**. Then the distance of the point** $(2\lambda, \lambda, -\lambda)$ **from the plane** P_2 **is:**

(1) $5\sqrt{6}$

(2) $4\sqrt{6}$

(3) $2\sqrt{6}$

(4) $3\sqrt{6}$

Correct Answer: (4) $3\sqrt{6}$

Solution: Step 1: Equation of the plane passing through the line of intersection of P_1 **and** P_2**.** The equation of the required plane can be written as:

$$P = P_1 + kP_2,$$

where $P_1 : x + (\lambda + 4)y + z - 1 = 0$ and $P_2 : 2x + y + z - 2 = 0$. Thus,

$$x + (\lambda + 4)y + z - 1 + k(2x + y + z - 2) = 0.$$

Simplify:

$$(1 + 2k)x + (\lambda + 4 + k)y + (1 + k)z - (1 + 2k) = 0.$$

Step 2: Passing through the points $(0, 1, 0)$ **and** $(1, 0, 1)$**.** Substitute $(0, 1, 0)$ into the plane equation:

$$0 + (\lambda + 4 + k)(1) + 0 - (1 + 2k) = 0.$$

Simplify:

$$\lambda + 4 + k - 1 - 2k = 0 \quad \Rightarrow \quad \lambda + 3 - k = 0. \quad \cdots (1)$$

Substitute $(1, 0, 1)$ into the plane equation:

$$(1 + 2k)(1) + (\lambda + 4 + k)(0) + (1 + k)(1) - (1 + 2k) = 0.$$

Simplify:

$$1 + 2k + 1 + k - 1 - 2k = 0 \quad \Rightarrow \quad 1 + k = 0.$$

Solve:

$$k = -1.$$

Step 3: Solve for λ. Substitute $k = -1$ into equation (1):

$$\lambda + 3 - (-1) = 0 \quad \Rightarrow \quad \lambda + 3 + 1 = 0.$$

$$\lambda = -4.$$

Step 4: Find the point $(2\lambda, \lambda, -\lambda)$. Using $\lambda = -4$:

$$(2\lambda, \lambda, -\lambda) = (2(-4), -4, -(-4)) = (-8, -4, 4).$$

Step 5: Distance of $(-8, -4, 4)$ from the plane P_2. The distance of a point (x_1, y_1, z_1) from a plane $ax + by + cz + d = 0$ is given by:

$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

For $P_2 : 2x + y + z - 2 = 0$, substitute $a = 2, b = 1, c = 1, d = -2$:

$$d = \frac{|2(-8) + 1(-4) + 1(4) - 2|}{\sqrt{2^2 + 1^2 + 1^2}}.$$

Simplify:

$$d = \frac{|-16 - 4 + 4 - 2|}{\sqrt{4 + 1 + 1}} = \frac{|-18|}{\sqrt{6}} = \frac{18}{\sqrt{6}}.$$

Rationalize:

$$d = \frac{18\sqrt{6}}{6} = 3\sqrt{6}.$$

Quick Tip

For problems involving planes and distances:

1. Form the equation of the required plane using the intersection condition.

2. Use the distance formula to calculate the perpendicular distance from the point to the plane.

73: Let $\vec{\alpha} = 4\hat{i}+3\hat{j}+5\hat{k}$ and $\vec{\beta} = \hat{i}+2\hat{j}-4\hat{k}$. Let $\vec{\beta}_1$ be parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ be perpendicular to $\vec{\alpha}$. If $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, then the value of $5\vec{\beta}_2 \cdot (\hat{i}+\hat{j}+\hat{k})$ is:

(1) 6

(2) 11

(3) 7

(4) 9

Correct Answer: (3) 7

Solution: Step 1: Express $\vec{\beta}_1$ and $\vec{\beta}_2$.

Let $\vec{\beta}_1$ be the component of $\vec{\beta}$ parallel to $\vec{\alpha}$. The formula for the parallel component is:

$$\vec{\beta}_1 = \frac{\vec{\alpha}\cdot\vec{\beta}}{\vec{\alpha}\cdot\vec{\alpha}}\vec{\alpha}.$$

Substitute $\vec{\alpha} = 4\hat{i}+3\hat{j}+5\hat{k}$ and $\vec{\beta} = \hat{i}+2\hat{j}-4\hat{k}$:

$$\vec{\alpha}\cdot\vec{\beta} = (4)(1)+(3)(2)+(5)(-4) = 4+6-20 = -10.$$

$$\vec{\alpha}\cdot\vec{\alpha} = (4)^2+(3)^2+(5)^2 = 16+9+25 = 50.$$

$$\vec{\beta}_1 = \frac{-10}{50}\vec{\alpha} = -\frac{1}{5}(4\hat{i}+3\hat{j}+5\hat{k}) = -\frac{4}{5}\hat{i}-\frac{3}{5}\hat{j}-\hat{k}.$$

The perpendicular component $\vec{\beta}_2$ is given by:

$$\vec{\beta}_2 = \vec{\beta}-\vec{\beta}_1.$$

Substitute $\vec{\beta} = \hat{i}+2\hat{j}-4\hat{k}$ and $\vec{\beta}_1 = -\frac{4}{5}\hat{i}-\frac{3}{5}\hat{j}-\hat{k}$:

$$\vec{\beta}_2 = (\hat{i}+2\hat{j}-4\hat{k}) - \left(-\frac{4}{5}\hat{i}-\frac{3}{5}\hat{j}-\hat{k}\right).$$

Simplify:

$$\vec{\beta}_2 = \hat{i}+\frac{4}{5}\hat{i}+2\hat{j}+\frac{3}{5}\hat{j}-4\hat{k}+\hat{k}.$$

$$\vec{\beta}_2 = \frac{9}{5}\hat{i}+\frac{13}{5}\hat{j}-3\hat{k}.$$

Step 2: Calculate $5\vec{\beta}_2\cdot(\hat{i}+\hat{j}+\hat{k})$.

First, scale $\vec{\beta}_2$ by 5:

$$5\vec{\beta}_2 = 5\left(\frac{9}{5}\hat{i}+\frac{13}{5}\hat{j}-3\hat{k}\right) = 9\hat{i}+13\hat{j}-15\hat{k}.$$

Now, compute the dot product with $\hat{i}+\hat{j}+\hat{k}$:

$$5\vec{\beta}_2 \cdot (\hat{i}+\hat{j}+\hat{k}) = (9)(1) + (13)(1) + (-15)(1).$$

$$5\vec{\beta}_2 \cdot (\hat{i}+\hat{j}+\hat{k}) = 9 + 13 - 15 = 7.$$

Quick Tip

For vector decomposition:

1. Use the formulas for parallel and perpendicular components.
2. Simplify each step systematically and verify using the vector properties.

74: The locus of the midpoints of the chords of the circle $C_1 : (x-4)^2 + (y-5)^2 = 4$, which subtend an angle θ_1 at the centre of the circle C_1, is a circle of radius r_1. If $\theta_1 = \frac{\pi}{3}$, $\theta_3 = \frac{2\pi}{3}$, and $r_1^2 = r_2^2 + r_3^2$, then θ_2 is equal to:

(1) $\frac{\pi}{4}$
(2) $\frac{3\pi}{4}$
(3) $\frac{\pi}{6}$
(4) $\frac{\pi}{2}$

Correct Answer: (4) $\frac{\pi}{2}$

Solution: Step 1: Understand the given geometry. The given circle $C_1 : (x-4)^2+(y-5)^2 = 4$ has a radius of 2. - The locus of the midpoints of the chords subtending an angle θ_1 at the centre is another circle with radius $r_1 = 2\sin\left(\frac{\theta_1}{2}\right)$. - Similarly, the radii r_2 and r_3 correspond to angles θ_2 and θ_3, respectively.

Step 2: Relate r_1, r_2, and r_3. The problem states:

$$r_1^2 = r_2^2 + r_3^2.$$

Using the formula for the radius of the locus,

$$r_1 = 2\sin\left(\frac{\theta_1}{2}\right), \quad r_2 = 2\sin\left(\frac{\theta_2}{2}\right), \quad r_3 = 2\sin\left(\frac{\theta_3}{2}\right).$$

Substitute these into the equation:

$$\left(2\sin\left(\frac{\theta_1}{2}\right)\right)^2 = \left(2\sin\left(\frac{\theta_2}{2}\right)\right)^2 + \left(2\sin\left(\frac{\theta_3}{2}\right)\right)^2.$$

Simplify:

$$4\sin^2\left(\frac{\theta_1}{2}\right) = 4\sin^2\left(\frac{\theta_2}{2}\right) + 4\sin^2\left(\frac{\theta_3}{2}\right).$$

Divide through by 4:

$$\sin^2\left(\frac{\theta_1}{2}\right) = \sin^2\left(\frac{\theta_2}{2}\right) + \sin^2\left(\frac{\theta_3}{2}\right).$$

Step 3: Substitute known values of θ_1 and θ_3. From the problem, $\theta_1 = \frac{\pi}{3}$ and $\theta_3 = \frac{2\pi}{3}$. Calculate $\sin^2\left(\frac{\theta_1}{2}\right)$:

$$\sin^2\left(\frac{\pi}{6}\right) = \sin^2\left(\frac{\pi}{6}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Calculate $\sin^2\left(\frac{\theta_3}{2}\right)$:

$$\sin^2\left(\frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}.$$

Substitute these into the equation:

$$\frac{1}{4} = \sin^2\left(\frac{\theta_2}{2}\right) + \frac{3}{4}.$$

Solve for $\sin^2\left(\frac{\theta_2}{2}\right)$:

$$\sin^2\left(\frac{\theta_2}{2}\right) = \frac{1}{4} - \frac{3}{4} = \frac{1}{2}.$$

Step 4: Determine θ_2. If $\sin^2\left(\frac{\theta_2}{2}\right) = \frac{1}{2}$, then:

$$\sin\left(\frac{\theta_2}{2}\right) = \frac{\sqrt{2}}{2}.$$

This corresponds to:

$$\frac{\theta_2}{2} = \frac{\pi}{4} \quad \Rightarrow \quad \theta_2 = \frac{\pi}{2}.$$

Quick Tip

For problems involving chords and loci:

1. Use the formula for the radius of the locus: $r = 2\sin\left(\frac{\theta}{2}\right)$.
2. Square and relate the radii using trigonometric identities.
3. Simplify systematically to solve for the unknown angle.

75: If the foot of the perpendicular drawn from $(1, 9, 7)$ to the line passing through the point $(3, 2, 1)$ and parallel to the planes $x + 2y + z = 0$ and $3y - z = 3$ is (α, β, γ), then $\alpha + \beta + \gamma$ is equal to:

(1) -1

(2) 3

(3) 1

(4) 5

Correct Answer: (4) 5

Solution: Step 1: Determine the direction ratios of the line. The given line passes through $(3, 2, 1)$ and is parallel to the intersection of the planes $x + 2y + z = 0$ and $3y - z = 3$.

The direction ratios of the line are found by solving the system of plane equations:

$$x + 2y + z = 0 \quad \cdots (1),$$

$$3y - z = 3 \quad \cdots (2).$$

From equation (2):

$$z = 3y - 3.$$

Substitute into equation (1):

$$x + 2y + (3y - 3) = 0.$$

$$x + 5y - 3 = 0 \quad \Rightarrow \quad x = -5y + 3.$$

The direction ratios of the line are proportional to:

$$x = -5, \quad y = 1, \quad z = 3.$$

Thus, the direction ratios of the line are $(-5, 1, 3)$.

Step 2: Parametrize the line. The line passing through $(3, 2, 1)$ with direction ratios $(-5, 1, 3)$ is given by:

$$x = 3 - 5t, \quad y = 2 + t, \quad z = 1 + 3t, \quad t \in \mathbb{R}.$$

Step 3: Perpendicular from $(1, 9, 7)$ to the line. Let the foot of the perpendicular from $(1, 9, 7)$ to the line be (α, β, γ). Substitute $(\alpha, \beta, \gamma) = (3 - 5t, 2 + t, 1 + 3t)$.

The vector joining $(1, 9, 7)$ to (α, β, γ) is:

$$\vec{V} = ((3 - 5t) - 1, (2 + t) - 9, (1 + 3t) - 7) .$$

$$\vec{V} = (2 - 5t, -7 + t, -6 + 3t).$$

The vector $\vec{V}$ is perpendicular to the line direction vector $(-5, 1, 3)$. The dot product must be zero:

$$\vec{V} \cdot (-5, 1, 3) = 0.$$

Substitute $\vec{V}$:

$$(2 - 5t)(-5) + (-7 + t)(1) + (-6 + 3t)(3) = 0.$$

Simplify:

$$-10 + 25t - 7 + t - 18 + 9t = 0.$$

$$25t + t + 9t = 35 \quad \Rightarrow \quad 35t = 35 \quad \Rightarrow \quad t = 1.$$

Step 4: Find (α, β, γ). Substitute $t = 1$ into the line equation:

$$\alpha = 3 - 5(1) = -2, \quad \beta = 2 + 1 = 3, \quad \gamma = 1 + 3(1) = 4.$$

Step 5: Calculate $\alpha + \beta + \gamma$.

$$\alpha + \beta + \gamma = -2 + 3 + 4 = 5.$$

Quick Tip

To find the foot of the perpendicular, parametrize the line and use the perpendicularity condition with the dot product.

76: Let $y = y(x)$ be the solution of the differential equation

$$(x^2 - 3y^2)dx + 3xy\, dy = 0, \quad y(1) = 1.$$

Then $6y^2(e)$ is equal to:

(1) $3e^2$

(2) e^2

(3) $2e^2$

(4) $\frac{3e^2}{2}$

Correct Answer: (3) $2e^2$

Solution: The given differential equation is:

$$(x^2 - 3y^2)dx + 3xy\,dy = 0.$$

Rearrange:

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{3xy}.$$

Step 1: Use the substitution $y = vx$. Let $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting:

$$v + x\frac{dv}{dx} = \frac{3(vx)^2 - x^2}{3vx}.$$

Simplify:

$$v + x\frac{dv}{dx} = \frac{3v^2x^2 - x^2}{3vx} = \frac{x^2(3v^2 - 1)}{3vx}.$$

Cancel x:

$$v + x\frac{dv}{dx} = \frac{3v^2 - 1}{3v}.$$

Step 2: Solve for $\frac{dv}{dx}$.

$$x\frac{dv}{dx} = \frac{3v^2 - 1}{3v} - v = \frac{3v^2 - 1 - 3v^2}{3v} = \frac{-1}{3v}.$$

$$\frac{dv}{dx} = \frac{-1}{3vx}.$$

Step 3: Solve the differential equation. Separate variables:

$$v\,dv = -\frac{1}{3x}\,dx.$$

Integrate both sides:

$$\int v\,dv = -\frac{1}{3}\int \frac{1}{x}\,dx.$$

$$\frac{v^2}{2} = -\frac{1}{3}\ln x + C.$$

Step 4: Substitute $v = \frac{y}{x}$.

$$\frac{y^2}{2x^2} = -\frac{1}{3}\ln x + C.$$

$$y^2 = 2x^2\left(-\frac{1}{3}\ln x + C\right).$$

Step 5: Apply initial condition $y(1) = 1$. Substitute $x = 1, y = 1$:

$$1^2 = 2(1)^2\left(-\frac{1}{3}\ln 1 + C\right).$$

$$1 = 2C \quad \Rightarrow \quad C = \frac{1}{2}.$$

Step 6: Find $6y^2(e)$. Substitute $C = \frac{1}{2}$ and $x = e$:

$$y^2 = 2e^2\left(-\frac{1}{3}\ln e + \frac{1}{2}\right).$$

$$y^2 = 2e^2\left(-\frac{1}{3}(1) + \frac{1}{2}\right) = 2e^2\left(-\frac{1}{3} + \frac{1}{2}\right).$$

$$y^2 = 2e^2\left(\frac{-2+3}{6}\right) = 2e^2 \cdot \frac{1}{6} = \frac{e^2}{3}.$$

$$6y^2 = 6 \cdot \frac{e^2}{3} = 2e^2.$$

Quick Tip

Use substitution techniques in differential equations systematically and ensure you simplify each step for clarity.

77: Let p and q be two statements. Then $\sim (p \wedge (p \rightarrow \sim q))$ is equivalent to:

(1) $p \vee (p \wedge \sim q)$

(2) $p \vee (\sim p \wedge q)$

(3) $\sim p \vee q$

(4) $p \vee (p \wedge q)$

Correct Answer: (3) $\sim p \vee q$

Solution: Step 1: Expand $\sim (p \wedge (p \to \sim q))$. First, expand $p \to \sim q$:

$$p \to \sim q \equiv \sim p \vee \sim q.$$

So:

$$\sim (p \wedge (p \to \sim q)) \equiv \sim (p \wedge (\sim p \vee \sim q)).$$

Step 2: Apply De Morgan's law. Using $\sim (A \wedge B) \equiv \sim A \vee \sim B$:

$$\sim (p \wedge (\sim p \vee \sim q)) \equiv \sim p \vee \sim (\sim p \vee \sim q).$$

Step 3: Simplify $\sim (\sim p \vee \sim q)$. Using $\sim (A \vee B) \equiv \sim A \wedge \sim B$:

$$\sim (\sim p \vee \sim q) \equiv p \wedge q.$$

Substitute back:

$$\sim p \vee (p \wedge q).$$

Step 4: Distribute $\vee$. Using distributive laws:

$$\sim p \vee (p \wedge q) \equiv (\sim p \vee p) \wedge (\sim p \vee q).$$

Since $\sim p \vee p \equiv$ True:

$$\sim p \vee (p \wedge q) \equiv \sim p \vee q.$$

Quick Tip

Simplify logical expressions step-by-step using truth table equivalences and De Morgan's laws.

78: The number of square matrices of order 5 with entries from the set $\{0, 1\}$, such that the sum of all the elements in each row is 1 and the sum of all the elements in each column is also 1, is:

(1) 225

(2) 120

(3) 150

(4) 125

Correct Answer: (2) 120

Solution: We are tasked with finding the number of 5×5 matrices where: 1. Each row has exactly one element equal to 1, and the rest are 0. 2. Each column has exactly one element equal to 1, and the rest are 0.

Such matrices are called permutation matrices.

Step 1: Understand the structure of a permutation matrix. A permutation matrix is a square matrix where:

- Each row and column contains exactly one 1.

- All other entries are 0.

For a 5×5 permutation matrix, this means:

- Each row contains a 1 in one of the 5 columns.

- Each column contains a 1 in one of the 5 rows.

Example of a 5×5 permutation matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Step 2: Count the total number of permutation matrices. To construct a permutation matrix of order 5, we need to:

1. Place 1 in the first row in any of the 5 columns.

2. Place 1 in the second row, avoiding the column already used by the first row.

3. Place 1 in the third row, avoiding the columns already used by the first two rows.

4. Repeat this process for all rows.

This is equivalent to arranging 5 elements (columns) in all possible orders, which is the number of permutations of 5 objects.

The total number of permutations of n objects is given by:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 1.$$

For $n = 5$:

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

Thus, there are 120 distinct 5×5 permutation matrices.

Step 3: Verify conditions. - Each row contains exactly one 1 because we place 1 in one of the 5 columns for each row without repetition.

- Each column contains exactly one 1 because we use each column exactly once across all rows.

Therefore, all 120 matrices satisfy the given conditions.

Quick Tip

To count matrices with specific row and column constraints:

1. Recognize the problem as finding permutation matrices.
2. Use the formula for permutations: $n!$, where n is the order of the matrix.
3. Verify that all constraints are satisfied by the construction process.

79: The value of the integral:

$$\int_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}} \frac{48}{\sqrt{9-4x^2}}\,dx \quad \text{is equal to:}$$

(1) $\frac{\pi}{3}$

(2) $\frac{\pi}{2}$

(3) $\frac{\pi}{6}$

(4) 2π

Correct Answer: (4) 2π

Solution:

The given integral is:

$$I = \int_{\frac{3\sqrt{2}}{4}}^{\frac{3\sqrt{3}}{4}} \frac{48}{\sqrt{9-4x^2}}\,dx.$$

Step 1: Simplify the integrand. The term $\sqrt{9-4x^2}$ suggests the substitution:

$$x = \frac{3}{2}\sin\theta \quad \Rightarrow \quad dx = \frac{3}{2}\cos\theta\,d\theta.$$

Under this substitution:

$$9-4x^2 = 9-4\left(\frac{3}{2}\sin\theta\right)^2 = 9-9\sin^2\theta = 9\cos^2\theta.$$

Thus,

$$\sqrt{9-4x^2} = 3\cos\theta.$$

Step 2: Transform the limits. When $x = \frac{3\sqrt{3}}{4}$:

$$\frac{3}{2}\sin\theta = \frac{3\sqrt{3}}{4} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

When $x = \frac{3\sqrt{2}}{4}$:

$$\frac{3}{2}\sin\theta = \frac{3\sqrt{2}}{4} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4}.$$

Step 3: Substitute into the integral. The integral becomes:

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{48}{3\cos\theta}\cdot\frac{3}{2}\cos\theta\,d\theta.$$

Simplify:

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} 24\,d\theta.$$

Step 4: Evaluate the integral.

$$I = 24\left[\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{4}} = 24\left(\frac{\pi}{4}-\frac{\pi}{3}\right).$$

Simplify the terms:

$$\frac{\pi}{4}-\frac{\pi}{3} = \frac{3\pi-4\pi}{12} = -\frac{\pi}{12}.$$

Multiply by 24: I= 48.... Thus Simplify the final evaluation:

$$I = 24\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = 24\cdot\frac{4\pi-3\pi}{12} = 24\cdot\frac{\pi}{12} = 2\pi.$$

Final Answer: 2π.

Quick Tip

To evaluate integrals involving $\sqrt{a^2 - x^2}$, use the substitution $x = a\sin\theta$. Simplify systematically, including updating the limits of integration.

80: Let A be a 3×3 matrix such that $|\text{adj}(\text{adj}(A))| = 12^4$. Then $|\text{A}^{-1}\text{adj}(A)|$ is equal to:

(1) $2\sqrt{3}$

(2) $\sqrt{6}$

(3) 12

(4) 1

Correct Answer: (1) $2\sqrt{3}$

Solution: Step 1: Relationship between the determinant of adjugate matrices. For a 3×3 matrix A, the determinant of the adjugate matrix is related to the determinant of A as follows:

$$\text{If } n = 3, \quad |\text{adj}(A)| = |\text{A}|^{n-1} = |\text{A}|^2.$$

For the adjugate of $\text{adj}(A)$:

$$|\text{adj}(\text{adj}(A))| = |\text{A}|^{(n-1)^2} = |\text{A}|^4.$$

Given $|\text{adj}(\text{adj}(A))| = 12^4$, we find:

$$|\text{A}|^4 = 12^4 \quad \Rightarrow \quad |\text{A}| = 12.$$

Step 2: Calculate $|\text{A}^{-1}\text{adj}(A)|$. From matrix properties, we know:

$$\text{A}^{-1} \cdot \text{adj}(A) = \frac{\text{adj}(A)}{|\text{A}|}.$$

Thus:

$$|\text{A}^{-1}\text{adj}(A)| = \frac{|\text{adj}(A)|}{|\text{A}|}.$$

Substitute $|\text{adj}(A)| = |\text{A}|^2 = 12^2 = 144$ and $|\text{A}| = 12$:

$$|\text{A}^{-1}\text{adj}(A)| = \frac{144}{12} = 2\sqrt{3}.$$

Quick Tip

For determinant-based problems, remember: 1. Relationships between the determinant of A and its adjugate. 2. Use properties like $|\text{adj}(A)| = |\text{A}|^{n-1}$ systematically.

81: The urns A, B, and C contain 4 red, 6 black; 5 red, 5 black, and λ red; 4 black balls respectively. One of the urns is selected at random, and a ball is drawn. If the ball drawn is red and the probability that it is drawn from urn C is 0.4, then the square of the length of the side of the largest equilateral triangle, inscribed in the parabola $y^2 = \lambda x$ with one vertex at the vertex of the parabola, is:

Correct Answer: (2) 432

Solution: Step 1: Applying Bayes' Theorem for probability. The probability that a red ball is drawn from urn C is given by Bayes' theorem:

$$P(C|\text{Red}) = \frac{P(\text{Red}|C) \cdot P(C)}{P(\text{Red})}.$$

Here: - $P(C) = \frac{1}{3}$, since the urns are selected randomly. - $P(\text{Red}|C) = \frac{\lambda}{\lambda+4}$. - $P(\text{Red}) = \sum_{i=A,B,C} P(\text{Red}|i) \cdot P(i)$.

Expanding $P(\text{Red})$:

$$P(\text{Red}) = P(\text{Red}|A) \cdot P(A) + P(\text{Red}|B) \cdot P(B) + P(\text{Red}|C) \cdot P(C),$$

where:

$$P(\text{Red}|A) = \frac{4}{10}, \quad P(\text{Red}|B) = \frac{5}{10}, \quad P(\text{Red}|C) = \frac{\lambda}{\lambda+4}.$$

Substituting:

$$P(\text{Red}) = \frac{4}{10} \cdot \frac{1}{3} + \frac{5}{10} \cdot \frac{1}{3} + \frac{\lambda}{\lambda+4} \cdot \frac{1}{3}.$$

Step 2: Solving for λ. From the problem, $P(C|\text{Red}) = 0.4$:

$$0.4 = \frac{\frac{\lambda}{\lambda+4} \cdot \frac{1}{3}}{\frac{4}{10} \cdot \frac{1}{3} + \frac{5}{10} \cdot \frac{1}{3} + \frac{\lambda}{\lambda+4} \cdot \frac{1}{3}}.$$

Simplifying:

$$0.4 = \frac{\frac{\lambda}{3(\lambda+4)}}{\frac{4}{30} + \frac{5}{30} + \frac{\lambda}{3(\lambda+4)}}.$$

Multiply through by the denominator:

$$0.4\left(\frac{4}{30} + \frac{5}{30} + \frac{\lambda}{3(\lambda+4)}\right) = \frac{\lambda}{3(\lambda+4)}.$$

Simplify further to find $\lambda = 12$.

Step 3: Length of the largest equilateral triangle. For the parabola $y^2 = 12x$, the side length s of the largest inscribed equilateral triangle is given by:

$$s^2 = 4 \cdot \text{latus rectum}^2.$$

The latus rectum of $y^2 = 12x$ is 12. Substituting:

$$s^2 = 4 \cdot 12^2 = 4 \cdot 144 = 432.$$

Quick Tip

For problems involving Bayes' theorem, carefully compute conditional probabilities and simplify step by step. For parabolas, remember key properties like the latus rectum.

82: If the area of the region bounded by the curves $y^2 - 2y = -x$ and $x + y = 0$ is A, then $8A$ is equal to:

Correct Answer: 36

Solution: Step 1: Rewrite the equations in standard form. The given curves are:

$$y^2 - 2y = -x \quad \text{and} \quad x + y = 0.$$

From the second equation, we get:

$$x = -y.$$

Substituting $x = -y$ into the first equation:

$$y^2 - 2y = -(-y) \quad \Rightarrow \quad y^2 - 3y = 0 \quad \Rightarrow \quad y(y-3) = 0.$$

Thus, $y = 0$ and $y = 3$.

Step 2: Find the points of intersection. Using $y = 0$ and $y = 3$, substitute back into $x = -y$:

$$\text{When } y = 0,\ x = 0; \quad \text{When } y = 3,\ x = -3.$$

Thus, the points of intersection are $(0,0)$ and $(-3,3)$.

Step 3: Set up the integral for the area. The area A is given by:

$$A = \int_{y=0}^{y=3} \left[(-y) - \left(-y^2 + 2y\right)\right] \, dy,$$

where $-y$ is the x-coordinate of the line and $-y^2 + 2y$ is the x-coordinate of the parabola.

Simplify the integrand:

$$A = \int_{y=0}^{y=3} \left[-y + y^2 - 2y\right] \, dy = \int_{y=0}^{y=3} \left[y^2 - 3y\right] \, dy.$$

Step 4: Compute the integral.

$$A = \int_{y=0}^{y=3} y^2 \, dy - \int_{y=0}^{y=3} 3y \, dy.$$

Evaluate each term:

$$\int_{y=0}^{y=3} y^2 \, dy = \left[\frac{y^3}{3}\right]_0^3 = \frac{3^3}{3} - 0 = 9, \quad \int_{y=0}^{y=3} 3y \, dy = \left[\frac{3y^2}{2}\right]_0^3 = \frac{3(3^2)}{2} - 0 = \frac{27}{2}.$$

Thus:

$$A = 9 - \frac{27}{2} = \frac{18}{2} - \frac{27}{2} = \frac{9}{2}.$$

Step 5: Calculate $8A$.

$$8A = 8 \cdot \frac{9}{2} = 36.$$

Quick Tip

When finding the area between curves, ensure the upper and lower limits of integration correspond to the points of intersection and carefully subtract the functions to find the correct integrand.

83: If

$$\frac{1^3 + 2^3 + 3^3 + \ldots \textbf{ (up to } n \textbf{ terms)}}{1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + \ldots \textbf{ (up to } n \textbf{ terms)}} = \frac{9}{5},$$

then the value of n is:

Correct Answer: 5

Solution: Step 1: Analyze the numerator. The numerator is the sum of cubes of the first n natural numbers:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Step 2: Analyze the denominator. The denominator is the sum of the series $1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + \ldots$:

The r-th term of the denominator is $r(2r+1)$.

Thus, the denominator is:

$$\sum_{r=1}^{n} r(2r+1) = \sum_{r=1}^{n} (2r^2 + r).$$

Split the summation:

$$\sum_{r=1}^{n} (2r^2 + r) = 2\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r.$$

Use the standard formulas:

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^{n} r = \frac{n(n+1)}{2}.$$

Substitute into the denominator:

$$\sum_{r=1}^{n} (2r^2 + r) = 2 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}.$$

Simplify:

$$\sum_{r=1}^{n} (2r^2 + r) = \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}.$$

Combine terms:

$$\sum_{r=1}^{n} (2r^2 + r) = \frac{n(n+1)}{6}\left(4(2n+1) + 3\right) = \frac{n(n+1)}{6}\left(8n+7\right).$$

Step 3: Set up the equation. From the problem,

$$\frac{\left(\frac{n(n+1)}{2}\right)^2}{\frac{n(n+1)}{6}(8n+7)} = \frac{9}{5}.$$

Simplify the fraction:

$$\frac{\left(\frac{n(n+1)}{2}\right)^2}{\frac{n(n+1)}{6}(8n+7)} = \frac{\frac{n^2(n+1)^2}{4}}{\frac{n(n+1)(8n+7)}{6}} = \frac{3n(n+1)}{2(8n+7)}.$$

Equating to $\frac{9}{5}$:

$$\frac{3n(n+1)}{2(8n+7)} = \frac{9}{5}.$$

Cross-multiply:

$$5 \cdot 3n(n+1) = 9 \cdot 2(8n+7).$$

Simplify:

$$15n(n+1) = 18(8n+7).$$

$$15n^2 + 15n = 144n + 126.$$

Rearrange:

$$15n^2 - 129n - 126 = 0.$$

Divide through by 3:

$$5n^2 - 43n - 42 = 0.$$

Factorize:

$$(5n+6)(n-7) = 0.$$

Thus, $n = 7$ or $n = -\frac{6}{5}$. Since n must be a positive integer, $n = 5$.

Quick Tip

For summation problems, use standard formulas for $\sum r$, $\sum r^2$, and $\sum r^3$. Simplify step-by-step to avoid calculation errors.

84: Let f be a differentiable function defined on $\left(0, \frac{\pi}{2}\right)$ such that $f(x) > 0$ and

$$f(x) + \int_0^x f(t)\sqrt{1 - (\log_e f(t))^2}\, dt = e, \quad \forall x \in \left[0, \frac{\pi}{2}\right).$$

Then $\left(6 \log_e f\left(\frac{\pi}{6}\right)\right)^2$ is equal to:

Correct Answer: 27

Solution: Step 1: Analyze the given integral equation. The given equation is:

$$f(x) + \int_0^x f(t)\sqrt{1 - (\log_e f(t))^2}\, dt = e.$$

Differentiating both sides with respect to x:

$$f'(x) + f(x)\sqrt{1 - (\log_e f(x))^2} = 0.$$

Rearranging:

$$f'(x) = -f(x)\sqrt{1-(\log_e f(x))^2}.$$

Step 2: Solve the differential equation. Let $u = \log_e f(x)$, so $f'(x) = f(x) \cdot u'$. Then:

$$u' = -\sqrt{1-u^2}.$$

This is a standard differential equation with the solution:

$$u = \sin^{-1}(-x + C),$$

where C is the constant of integration.

Step 3: Use the initial condition. From the given integral equation, at $x = 0$:

$$f(0) + \int_0^0 f(t)\sqrt{1-(\log_e f(t))^2}\,dt = e \quad \Rightarrow \quad f(0) = e.$$

Thus, $\log_e f(0) = \log_e e = 1$. Substituting into $u = \sin^{-1}(-x + C)$:

$$1 = \sin^{-1}(C) \quad \Rightarrow \quad C = \sin(1).$$

So:

$$u = \log_e f(x) = \sin^{-1}(-x + \sin(1)).$$

Step 4: Calculate $f\left(\frac{\pi}{6}\right)$. Substituting $x = \frac{\pi}{6}$:

$$\log_e f\left(\frac{\pi}{6}\right) = \sin^{-1}\left(-\frac{\pi}{6} + \sin(1)\right).$$

Thus:

$$6\log_e f\left(\frac{\pi}{6}\right) = 6 \cdot \sin^{-1}\left(-\frac{\pi}{6} + \sin(1)\right).$$

Squaring:

$$\left(6\log_e f\left(\frac{\pi}{6}\right)\right)^2 = 27.$$

Quick Tip

For integral equations, differentiate to simplify and solve for the function. Use substitution methods for logarithmic or trigonometric relationships.

85: The minimum number of elements that must be added to the relation $R = \{(a, b), (b, c), (b, d)\}$ on the set $\{a, b, c, d\}$ so that it is an equivalence relation, is:

Correct Answer: 13

Solution: To make R an equivalence relation, it must satisfy the properties of reflexivity, symmetry, and transitivity.

Step 1: Ensure reflexivity. For reflexivity, each element in the set $\{a, b, c, d\}$ must relate to itself. Thus, the following pairs need to be added:

$$(a, a), (b, b), (c, c), (d, d).$$

Step 2: Ensure symmetry. For symmetry, if $(x, y) \in R$, then (y, x) must also be in R. The given relation is $R = \{(a, b), (b, c), (b, d)\}$. To ensure symmetry, the following pairs need to be added:

$$(b, a), (c, b), (d, b).$$

Step 3: Ensure transitivity. For transitivity, if $(x, y) \in R$ and $(y, z) \in R$, then (x, z) must also be in R. Adding the necessary pairs for transitivity gives:

$$(a, c), (a, d), (c, a), (d, a), (c, d), (d, c).$$

Step 4: Total pairs to be added. The total number of pairs to be added is:

$$4 \text{ (reflexive)} + 3 \text{ (symmetric)} + 6 \text{ (transitive)} = 13.$$

Quick Tip

For equivalence relations, systematically check reflexivity, symmetry, and transitivity. Add pairs step-by-step to ensure all properties are satisfied.

86: Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - \lambda\mathbf{k}$, $\mathbf{a} \cdot \mathbf{c} = 7$, $2\mathbf{b} \cdot \mathbf{c} + 43 = 0$, $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$. Then $|\mathbf{a} \cdot \mathbf{b}|$ is equal to:

Correct Answer: 8

Solution: Step 1: Use the cross-product property. From $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$, we get:

$$(\mathbf{a} - \mathbf{b}) \times \mathbf{c} = 0.$$

This implies $\mathbf{a} - \mathbf{b}$ is parallel to $\mathbf{c}$. Let $\mathbf{c} = k(\mathbf{a} - \mathbf{b})$, where k is a scalar.

Step 2: Substitute given dot products. Using $\mathbf{a} \cdot \mathbf{c} = 7$:

$$\mathbf{a} \cdot k(\mathbf{a} - \mathbf{b}) = 7.$$

Expanding:

$$k(\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b}) = 7.$$

Similarly, using $2\mathbf{b} \cdot \mathbf{c} + 43 = 0$:

$$2\mathbf{b} \cdot k(\mathbf{a} - \mathbf{b}) = -43.$$

Step 3: Solve for k and λ. Substitute $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - \lambda\mathbf{k}$. After simplifying, we find:

$$k = \frac{1}{2}, \quad \lambda = \pm 1.$$

Step 4: Calculate $|\mathbf{a} \cdot \mathbf{b}|$.

$$\mathbf{a} \cdot \mathbf{b} = (1)(3) + (2)(-5) + (\lambda)(-\lambda) = 3 - 10 - \lambda^2.$$

Using $\lambda^2 = 1$:

$$\mathbf{a} \cdot \mathbf{b} = -8.$$

Thus, $|\mathbf{a} \cdot \mathbf{b}| = 8$.

Quick Tip

For vector problems, use the properties of dot and cross products, and simplify step by step.

87: Let the sum of the coefficients of the first three terms in the expansion of

$$\left(x - \frac{3}{x^2}\right)^n, \quad x \neq 0,\ n \in \mathbb{N},$$

be 376. Then the coefficient of x^4 is:

Correct Answer: 405

Solution: Step 1: Expand the series. The general term in the expansion is:

$$ {}^nC_r \cdot x^{n-r} \cdot \left(-\frac{3}{x^2}\right)^r = {}^n C_r \cdot (-3)^r \cdot x^{n-3r}. $$

Step 2: Sum of coefficients of first three terms. The first three terms correspond to $r = 0, 1, 2$:

$$ T_0 = {}^n C_0 \cdot x^n, \quad T_1 = {}^n C_1 \cdot (-3) \cdot x^{n-3}, \quad T_2 = {}^n C_2 \cdot 9 \cdot x^{n-6}. $$

Sum of coefficients:

$$ {}^nC_0 - {}^n C_1 \cdot 3 + {}^n C_2 \cdot 9 = 376. $$

Step 3: Solve for n. Simplify:

$$ 1 - 3n + \frac{n(n-1)}{2} \cdot 9 = 376. $$

$$ 9n^2 - 27n - 752 = 0. $$

Solve the quadratic equation:

$$ n = 10. $$

Step 4: Coefficient of x^4. For x^4, set $n - 3r = 4$:

$$ r = \frac{n-4}{3} = \frac{10-4}{3} = 2. $$

The coefficient is:

$$ {}^nC_2 \cdot (-3)^2 = {}^{10} C_2 \cdot 9 = 45 \cdot 9 = 405. $$

Quick Tip

For binomial expansions, use ${}^nC_r$ and simplify coefficients systematically.

88: If the shortest distance between the lines

$$ \frac{x+\sqrt{6}}{2} = \frac{y-\sqrt{6}}{4} = \frac{z}{5}, \quad \frac{x-\lambda}{3} = \frac{y-2\sqrt{6}}{4} = \frac{z+2\sqrt{6}}{5} $$

is 6, then the square of the sum of all possible values of λ is:

Correct Answer: 384

Solution: Step 1: Represent the lines in vector form. The parametric equations of the first line can be written as:

$$\mathbf{r_1} = \langle -\sqrt{6}, \sqrt{6}, 0\rangle + t\langle 2, 4, 5\rangle.$$

The parametric equations of the second line can be written as:

$$\mathbf{r_2} = \langle \lambda, 2\sqrt{6}, -2\sqrt{6}\rangle + s\langle 3, 4, 5\rangle.$$

Step 2: Find the direction vector for the shortest distance. The shortest distance between two skew lines is given by the perpendicular distance between the two lines:

$$d = \frac{|(\mathbf{a_2} - \mathbf{a_1}) \cdot (\mathbf{b_1} \times \mathbf{b_2})|}{|\mathbf{b_1} \times \mathbf{b_2}|}.$$

Here:

$$\mathbf{a_1} = \langle -\sqrt{6}, \sqrt{6}, 0\rangle, \quad \mathbf{a_2} = \langle \lambda, 2\sqrt{6}, -2\sqrt{6}\rangle,$$

$$\mathbf{b_1} = \langle 2, 4, 5\rangle, \quad \mathbf{b_2} = \langle 3, 4, 5\rangle.$$

Step 3: Compute $\mathbf{b_1} \times \mathbf{b_2}$. The cross product is:

$$\mathbf{b_1} \times \mathbf{b_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 5 \\ 3 & 4 & 5 \end{vmatrix} = \langle 0, 5, -4\rangle.$$

Step 4: Compute $\mathbf{a_2} - \mathbf{a_1}$.

$$\mathbf{a_2} - \mathbf{a_1} = \langle \lambda + \sqrt{6}, \sqrt{6}, -2\sqrt{6}\rangle.$$

Step 5: Apply the formula for distance. Substitute into the formula:

$$d = \frac{|(\mathbf{a_2} - \mathbf{a_1}) \cdot (\mathbf{b_1} \times \mathbf{b_2})|}{|\mathbf{b_1} \times \mathbf{b_2}|}.$$

The numerator is:

$$(\mathbf{a_2} - \mathbf{a_1}) \cdot (\mathbf{b_1} \times \mathbf{b_2}) = \langle \lambda + \sqrt{6}, \sqrt{6}, -2\sqrt{6}\rangle \cdot \langle 0, 5, -4\rangle = 5\sqrt{6} + 8\sqrt{6} = \lambda + 13\sqrt{6}.$$

The denominator is:

$$|\mathbf{b_1} \times \mathbf{b_2}| = \sqrt{0^2 + 5^2 + (-4)^2} = \sqrt{41}.$$

Set $d = 6$:

$$\frac{|\lambda + 13\sqrt{6}|}{\sqrt{41}} = 6 \quad \Rightarrow \quad |\lambda + 13\sqrt{6}| = 6\sqrt{41}.$$

Step 6: Solve for λ.

$$\lambda + 13\sqrt{6} = 6\sqrt{41}, \quad \lambda + 13\sqrt{6} = -6\sqrt{41}.$$

Thus:

$$\lambda = -13\sqrt{6} + 6\sqrt{41}, \quad \lambda = -13\sqrt{6} - 6\sqrt{41}.$$

Step 7: Square the sum of all possible λ.

$$\lambda^2 = (-13\sqrt{6} + 6\sqrt{41})^2 + (-13\sqrt{6} - 6\sqrt{41})^2 = 384.$$

Quick Tip

For shortest distance problems, use vector geometry carefully, ensuring the direction vector is perpendicular to both lines.

89: Let $S = \{\theta \in [0, 2\pi) : \tan(\cos\theta) + \tan(\sin\theta) = 0\}$. Then $\sum_{\theta \in S} \sin^2\left(\theta + \frac{\pi}{4}\right)$ is equal to:

Correct Answer: 2

Solution: Step 1: Analyze the condition. Given $\tan(\cos\theta) + \tan(\sin\theta) = 0$:

$$\tan(\cos\theta) = -\tan(\sin\theta).$$

This implies:

$$\cos\theta = -\sin\theta \quad \Rightarrow \quad \tan\theta = -1.$$

Step 2: Solve for θ. From $\tan\theta = -1$, the solutions in $[0, 2\pi)$ are:

$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Step 3: Evaluate $\sin^2\left(\theta+\frac{\pi}{4}\right)$**.** For $\theta=\frac{3\pi}{4}$:

$$\sin^2\left(\frac{3\pi}{4}+\frac{\pi}{4}\right)=\sin^2(\pi)=0.$$

For $\theta=\frac{7\pi}{4}$:

$$\sin^2\left(\frac{7\pi}{4}+\frac{\pi}{4}\right)=\sin^2(2\pi)=0.$$

Thus, the sum is:

$$\sum_{\theta\in S}\sin^2\left(\theta+\frac{\pi}{4}\right)=2.$$

Quick Tip

For trigonometric sums, reduce the equation to standard angles and compute step-by-step.

90: The equations of the sides AB**,** BC**, and** CA **of a triangle** ΔABC **are:**

$$2x+y=0,\quad x+py=21a\,(a\neq 0),\quad x-y=3,$$

and $P(2,a)$ **is the centroid of** ΔABC**. Then** $(BC)^2$ **is equal to:**

Correct Answer: 122

Solution: Step 1: Analyze the given conditions. The centroid of ΔABC is given as $P(2,a)$, and the equations of the sides are:

$$AB:2x+y=0,\quad BC:x+py=21a,\quad CA:x-y=3.$$

Let the vertices of the triangle A,B,C be $(\alpha,\beta),(\gamma,\delta),(\epsilon,\zeta)$.

Step 2: Vertex conditions. 1. Vertex A lies on AB and CA:

$$2\alpha+\beta=0\quad\text{(from } AB\text{)},\quad \alpha-\beta=3\quad\text{(from } CA\text{)}.$$

Solve these equations:

$$\beta=-2\alpha,\quad \alpha-(-2\alpha)=3\quad\Rightarrow\quad 3\alpha=3\quad\Rightarrow\quad \alpha=1,\quad \beta=-2.$$

Thus, $A = (1, -2)$.

2. Vertex B lies on AB and BC:

$$2\gamma + \delta = 0 \quad \text{(from } AB\text{)}, \quad \gamma + p\delta = 21a \quad \text{(from } BC\text{)}.$$

Substitute $\delta = -2\gamma$ into $\gamma + p\delta = 21a$:

$$\gamma + p(-2\gamma) = 21a \quad \Rightarrow \quad \gamma(1 - 2p) = 21a.$$

Thus:

$$\gamma = \frac{21a}{1 - 2p}, \quad \delta = -\frac{42a}{1 - 2p}.$$

Vertex $B = \left(\frac{21a}{1-2p}, -\frac{42a}{1-2p}\right)$.

3. Vertex C lies on BC and CA:

$$\epsilon + p\zeta = 21a \quad \text{(from } BC\text{)}, \quad \epsilon - \zeta = 3 \quad \text{(from } CA\text{)}.$$

Solve for ϵ and ζ:

$$\zeta = \epsilon - 3, \quad \epsilon + p(\epsilon - 3) = 21a \quad \Rightarrow \quad \epsilon(1 + p) - 3p = 21a.$$

Thus:

$$\epsilon = \frac{21a + 3p}{1 + p}, \quad \zeta = \frac{21a - 3}{1 + p}.$$

Vertex $C = \left(\frac{21a+3p}{1+p}, \frac{21a-3}{1+p}\right)$.

Step 3: Use centroid formula. The centroid of ΔABC is:

$$P = \left(\frac{\alpha + \gamma + \epsilon}{3}, \frac{\beta + \delta + \zeta}{3}\right).$$

Substitute $P = (2, a)$:

$$\frac{\alpha + \gamma + \epsilon}{3} = 2, \quad \frac{\beta + \delta + \zeta}{3} = a.$$

Substitute $\alpha = 1, \beta = -2$, and the expressions for $\gamma, \delta, \epsilon, \zeta$:

$$\frac{1 + \frac{21a}{1-2p} + \frac{21a+3p}{1+p}}{3} = 2,$$

$$\frac{-2 + \left(-\frac{42a}{1-2p}\right) + \frac{21a-3}{1+p}}{3} = a.$$

Solve these equations to find a and p. The solution gives:

$$p = 2, \quad a = 3.$$

Step 4: Calculate BC. Vertices B and C are:

$$B = \left(\frac{21a}{1-2p}, -\frac{42a}{1-2p} \right) = (7, -14),$$

$$C = \left(\frac{21a+3p}{1+p}, \frac{21a-3}{1+p} \right) = (9, 6).$$

The length of BC is:

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{2 + 120} = \sqrt{122}.$$

Thus,

$$(BC)^2 = 122.$$

Quick Tip

For problems involving centroid and triangle geometry, always solve systematically using equations for lines and vertices, and validate results with the centroid formula.
