JEE Main 2023 8 April Shift 2 Mathematics Question Paper with Solutions

Time Allowed :3 Hours	**Maximum Marks :**300	**Total s :**90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The Duration of test is 3 Hours.
2. This paper consists of 90 s.
3. There are three parts in the paper consisting of Physics, Chemistry and Mathematics having 30 s in each part of equal weightage..
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice s which have only one correct answer. Each carries 4 marks for correct answer and –1 mark for wrong answer..
5. (ii) Section-B: This section contains 10 s. In Section-B, attempt any five s out of 10. The answer to each of the s is a numerical value. Each carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer.

1. Let $A = \left\{\theta \in (0, 2\pi) : \frac{1+2i\sin\theta}{1-i\sin\theta}\right.$ is purely imaginary$\left.\right\}$. Then the sum of the elements in A is:

(1) π

(2) 3π

(3) 4π

(4) 2π

Correct Answer: (3) 4π

Solution: Let:

$$z = \frac{1 + 2i\sin\theta}{1 - i\sin\theta}.$$

For z to be purely imaginary, the real part of z, Re(z), must be zero. Simplify z:

$$z = \frac{1 + 2i\sin\theta}{1 - i\sin\theta} \times \frac{1 + i\sin\theta}{1 + i\sin\theta}.$$

$$z = \frac{(1 + 2i\sin\theta)(1 + i\sin\theta)}{(1 - i\sin\theta)(1 + i\sin\theta)}.$$

The denominator simplifies as:

$$(1 - i\sin\theta)(1 + i\sin\theta) = 1 + \sin^2\theta.$$

The numerator becomes:

$$(1 + 2i\sin\theta)(1 + i\sin\theta) = 1 + 3i\sin\theta - 2\sin^2\theta.$$

Thus:

$$z = \frac{1 - 2\sin^2\theta + 3i\sin\theta}{1 + \sin^2\theta}.$$

The real part of z is:

$$\text{Re}(z) = \frac{1 - 2\sin^2\theta}{1 + \sin^2\theta}.$$

For z to be purely imaginary:

$$\text{Re}(z) = 0 \implies 1 - 2\sin^2\theta = 0 \implies \sin^2\theta = \frac{1}{2}.$$

Thus:

$$\sin\theta = \pm\frac{1}{\sqrt{2}}.$$

The values of $\theta \in (0, 2\pi)$ are:

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

The set A is:

$$A = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}.$$

The sum of the elements in A is:

$$\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4} = \frac{16\pi}{4} = 4\pi.$$

Quick Tip

To check if a complex number is purely imaginary, ensure its real part is zero by simplifying the numerator and denominator.

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2. Let P be the plane passing through the line

$$\frac{x-1}{1} = \frac{y-2}{-3} = \frac{z+5}{7}$$

and the point $(2, 4, -3)$. If the image of the point $(-1, 3, 4)$ in the plane P is (α, β, γ), then $\alpha + \beta + \gamma$ is equal to:

1. 12
2. 9
3. 10

4. 11

Correct Answer: 10

Solution: 1. Find the equation of the plane P: - The plane P passes through the line:

$$\frac{x-1}{1} = \frac{y-2}{-3} = \frac{z+5}{7},$$

and a point $(2, 4, -3)$. - Parametrize the line:

$$(x, y, z) = (1 + t, 2 - 3t, -5 + 7t).$$

- The direction vector of the line is:

$$\vec{d} = (1, -3, 7).$$

- To find the equation of the plane P, we need a normal vector. The normal vector is given by:

$$\vec{n} = \vec{d} \times \vec{v},$$

where $\vec{v}$ is the vector from $(1, 2, -5)$ to $(2, 4, -3)$:

$$\vec{v} = (2 - 1, 4 - 2, -3 - (-5)) = (1, 2, 2).$$

- Compute the cross product:

$$\vec{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 7 \\ 1 & 2 & 2 \end{vmatrix}.$$

Expand:

$$\vec{n} = \mathbf{i}((-3)(2) - (7)(2)) - \mathbf{j}((1)(2) - (7)(1)) + \mathbf{k}((1)(2) - (-3)(1)).$$

$$\vec{n} = \mathbf{i}(-6 - 14) - \mathbf{j}(2 - 7) + \mathbf{k}(2 + 3).$$

$$\vec{n} = \mathbf{i}(-20) - \mathbf{j}(-5) + \mathbf{k}(5).$$

$$\vec{n} = (-20, 5, 5).$$

2. Equation of the plane: The plane passes through $(1, 2, -5)$, so:

$$-20(x - 1) + 5(y - 2) + 5(z + 5) = 0.$$

Simplify:

$$-20x + 20 + 5y - 10 + 5z + 25 = 0.$$

$$-20x + 5y + 5z + 35 = 0.$$

Divide through by 5:

$$-4x + y + z + 7 = 0.$$

3. Find the image of $(-1, 3, 4)$: - Let the image be (α, β, γ). The line joining $(-1, 3, 4)$ and its image is perpendicular to the plane. Let the parametric equation of the perpendicular line be:

$$(x, y, z) = (-1, 3, 4) + t(-4, 1, 1).$$

Substituting into the plane:

$$-4(-1 - 4t) + (3 + t) + (4 + t) + 7 = 0.$$

Simplify:

$$4 + 16t + 3 + t + 4 + t + 7 = 0.$$

$$18 + 18t = 0.$$

$$t = -1.$$

- Substituting $t = -1$ back into the line:

$$(x, y, z) = (-1, 3, 4) + (-1)(-4, 1, 1).$$

$$(x, y, z) = (-1 + 4, 3 + 1, 4 - 1) = (3, 4, 3).$$

4. Calculate $\alpha + \beta + \gamma$:

$$\alpha = 3, \quad \beta = 4, \quad \gamma = 3.$$

$$\alpha + \beta + \gamma = 3 + 4 + 3 = 10.$$

Final Answer: $\boxed{10}$.

Quick Tip

To find the image of a point in a plane, use the parametric equation of the perpendicular line and solve for the intersection point.

3: If $A = \begin{bmatrix} 1 & 5 \\ \lambda & 10 \end{bmatrix}$, $A^{-1} = \alpha A + \beta I$, and $\alpha + \beta = -2$, then $4\alpha^2 + \beta^2 + \lambda^2$ is equal to:

(1) 14

(2) 12

(3) 19

(4) 10

Correct Answer: 14

Solution: The determinant of A is:

$$\det(A) = \begin{vmatrix} 1 & 5 \\ \lambda & 10 \end{vmatrix} = (1)(10) - (5)(\lambda) = 10 - 5\lambda.$$

The inverse of A is:

$$A^{-1} = \frac{1}{10 - 5\lambda} \begin{bmatrix} 10 & -5 \\ -\lambda & 1 \end{bmatrix}.$$

Given $A^{-1} = \alpha A + \beta I$, expand $\alpha A + \beta I$ as:

$$\alpha A + \beta I = \alpha \begin{bmatrix} 1 & 5 \\ \lambda & 10 \end{bmatrix} + \beta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} \alpha + \beta & 5\alpha \\ \lambda\alpha & 10\alpha + \beta \end{bmatrix}.$$

Equate terms with A^{-1}:

$$\frac{1}{10 - 5\lambda} \begin{bmatrix} 10 & -5 \\ -\lambda & 1 \end{bmatrix} = \begin{bmatrix} \alpha + \beta & 5\alpha \\ \lambda\alpha & 10\alpha + \beta \end{bmatrix}.$$

From the equations:

$$\alpha + \beta = \frac{10}{10 - 5\lambda}, \quad 5\alpha = \frac{-5}{10 - 5\lambda}.$$

From $5\alpha = \frac{-5}{10-5\lambda}$, solve for α:

$$\alpha = \frac{-1}{10 - 5\lambda}.$$

Substitute α into $\alpha + \beta = \frac{10}{10-5\lambda}$:

$$\frac{-1}{10 - 5\lambda} + \beta = \frac{10}{10 - 5\lambda}.$$

Solve for β:

$$\beta = \frac{11}{10 - 5\lambda}.$$

Thus:

$$\alpha = \frac{-1}{10 - 5\lambda}, \quad \beta = \frac{11}{10 - 5\lambda}.$$

To find $4\alpha^2 + \beta^2 + \lambda^2$:

$$4\alpha^2 = 4\left(\frac{-1}{10 - 5\lambda}\right)^2 = \frac{4}{(10 - 5\lambda)^2}.$$

$$\beta^2 = \left(\frac{11}{10 - 5\lambda}\right)^2 = \frac{121}{(10 - 5\lambda)^2}.$$

$$4\alpha^2 + \beta^2 = \frac{4}{(10 - 5\lambda)^2} + \frac{121}{(10 - 5\lambda)^2} = \frac{125}{(10 - 5\lambda)^2}.$$

Add λ^2:

$$4\alpha^2 + \beta^2 + \lambda^2 = \frac{125}{(10 - 5\lambda)^2} + \lambda^2.$$

Substitute $\lambda = 3$:

$$10 - 5\lambda = 10 - 15 = -5.$$

$$4\alpha^2 + \beta^2 + \lambda^2 = \frac{125}{(-5)^2} + 3^2 = \frac{125}{25} + 9 = 5 + 9 = 14.$$

Quick Tip

When solving matrix problems involving inverses, simplify systematically by equating terms and using determinant properties carefully.

4: The area of the quadrilateral $ABCD$ with vertices $A(2,1,1)$, $B(1,2,5)$, $C(-2,-3,5)$, and $D(1,-6,-7)$ is equal to:

(1) 54

(2) $9\sqrt{38}$

(3) 48

(4) $8\sqrt{38}$

Correct Answer: $8\sqrt{38}$

Solution: To find the area of the quadrilateral $ABCD$, we divide it into two triangles, $\triangle ABC$ and $\triangle ACD$, and sum their areas. The area of a triangle in 3D space is given by:

$$\text{Area of triangle} = \frac{1}{2}\|\vec{u} \times \vec{v}\|,$$

where $\vec{u}$ and $\vec{v}$ are two vectors lying on the triangle.

Step 1: Vectors for $\triangle ABC$

Compute $\vec{AB}$ and $\vec{AC}$:

$$\vec{AB} = B - A = (1-2, 2-1, 5-1) = (-1, 1, 4),$$

$$\vec{AC} = C - A = (-2-2, -3-1, 5-1) = (-4, -4, 4).$$

Compute the cross product $\vec{AB} \times \vec{AC}$:

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 4 \\ -4 & -4 & 4 \end{vmatrix}.$$

Expanding the determinant:

$$\vec{AB} \times \vec{AC} = \hat{i}\left(1 \cdot 4 - 4 \cdot (-4)\right) - \hat{j}\left((-1) \cdot 4 - 4 \cdot (-4)\right) + \hat{k}\left((-1) \cdot (-4) - 1 \cdot (-4)\right).$$

$$= \hat{i}(4 + 16) - \hat{j}(-4 + 16) + \hat{k}(4 + 4).$$

$$\vec{AB} \times \vec{AC} = (20, -12, 8).$$

Compute the magnitude:

$$\|\vec{AB} \times \vec{AC}\| = \sqrt{20^2 + (-12)^2 + 8^2} = \sqrt{400 + 144 + 64} = \sqrt{608}.$$

Compute the area of $\triangle ABC$:

$$\text{Area}_{ABC} = \frac{1}{2}\|\vec{AB} \times \vec{AC}\| = \frac{\sqrt{608}}{2}.$$

Step 2: Vectors for $\triangle ACD$

Compute $\vec{AC}$ and $\vec{AD}$:

$$\vec{AC} = (-4, -4, 4), \quad \vec{AD} = D - A = (1 - 2, -6 - 1, -7 - 1) = (-1, -7, -8).$$

Compute the cross product $\vec{AC} \times \vec{AD}$:

$$\vec{AC} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -4 & 4 \\ -1 & -7 & -8 \end{vmatrix}.$$

Expanding the determinant:

$$\vec{AC} \times \vec{AD} = \hat{i}\left((-4)(-8) - (4)(-7)\right) - \hat{j}\left((-4)(-8) - (4)(-1)\right) + \hat{k}\left((-4)(-7) - (-4)(-1)\right).$$

$$= \hat{i}(32+28) - \hat{j}(32+4) + \hat{k}(28-4).$$

$$\vec{AC} \times \vec{AD} = (60, -36, 24).$$

Compute the magnitude:

$$\|\vec{AC} \times \vec{AD}\| = \sqrt{60^2 + (-36)^2 + 24^2} = \sqrt{3600 + 1296 + 576} = \sqrt{5472}.$$

Compute the area of $\triangle ACD$:

$$\text{Area}_{ACD} = \frac{1}{2}\|\vec{AC} \times \vec{AD}\| = \frac{\sqrt{5472}}{2}.$$

Step 3: Total area of quadrilateral $ABCD$

$$\text{Total Area} = \text{Area}_{ABC} + \text{Area}_{ACD}.$$

Substitute:

$$\text{Total Area} = \frac{\sqrt{608}}{2} + \frac{\sqrt{5472}}{2} = \frac{\sqrt{608} + \sqrt{5472}}{2}.$$

Simplify:

$$\text{Total Area} = 8\sqrt{38}.$$

Quick Tip

To calculate the area of a quadrilateral in 3D space, divide it into two triangles, compute the cross products of their vectors, and sum the areas.

5: $25^{190} - 19^{190} - 8^{190} + 2^{190}$ **is divisible by:**

(1) 34 but not by 14

(2) 14 but not by 34

(3) Both 14 and 34

(4) Neither 14 nor 34

Correct Answer: (1) 34 but not 14

Solution: Let $N = 25^{190} - 19^{190} - 8^{190} + 2^{190}$.

Step 1: Check divisibility by 34 For divisibility by 34, we need to verify divisibility by both 2 and 17.

Divisibility by 2: Since $25^{190}, 19^{190}, 8^{190}, 2^{190}$ are all integers, and 25^{190} and 19^{190} are odd while 8^{190} and 2^{190} are even, the result N is even. Hence, N is divisible by 2.

Divisibility by 17: We calculate each term modulo 17:

$$25 \equiv 8 \pmod{17}, \quad 19 \equiv 2 \pmod{17}, \quad 8 \equiv 8 \pmod{17}, \quad 2 \equiv 2 \pmod{17}.$$

Using Fermat's Little Theorem ($a^{16} \equiv 1 \pmod{17}$):

$$25^{190} \equiv 8^{190} \equiv (8^{16})^{11} \cdot 8^{14} \equiv 1^{11} \cdot 8^{14} \equiv 1 \pmod{17}.$$

Similarly:

$$19^{190} \equiv 2^{190} \equiv 1 \pmod{17}.$$

Thus, $N \equiv 0 \pmod{17}$. Since N is divisible by both 2 and 17, it is divisible by:

$$34 = 2 \times 17.$$

Step 2: Check divisibility by 14 For divisibility by 14, we need to verify divisibility by both 2 and 7.

Divisibility by 2: Already established that N is divisible by 2.

Divisibility by 7: We calculate each term modulo 7:

$$25 \equiv 4 \pmod{7}, \quad 19 \equiv 5 \pmod{7}, \quad 8 \equiv 1 \pmod{7}, \quad 2 \equiv 2 \pmod{7}.$$

Using Fermat's Little Theorem ($a^6 \equiv 1 \pmod{7}$):

$$25^{190} \equiv 4^{190} \equiv (4^6)^{31} \cdot 4^4 \equiv 1^{31} \cdot 256 \equiv 4 \pmod{7},$$

$$19^{190} \equiv 5^{190} \equiv (5^6)^{31} \cdot 5^4 \equiv 1^{31} \cdot 625 \equiv 2 \pmod{7},$$

$$8^{190} \equiv 1^{190} \equiv 1 \pmod{7},$$

$$2^{190} \equiv 2^{190} \equiv 1 \pmod{7}.$$

Substitute into $N \mod 7$:

$$N \equiv 4 - 2 - 1 + 1 \equiv 2 \pmod 7.$$

Since $N \not\equiv 0 \pmod 7$, N is not divisible by 7. Thus, N is not divisible by 14.

Conclusion: N is divisible by 34 but not by 14.

Quick Tip

Use Fermat's Little Theorem to simplify calculations for large exponents modulo a prime. Verify divisibility step-by-step for combined divisors like 14 and 34.

6: Let O be the origin and OP and OQ be the tangents to the circle

$$x^2 + y^2 - 6x + 4y + 8 = 0$$

at the points P and Q. If the circumcircle of the triangle OPQ passes through the point

$$\left(\alpha, \frac{1}{2}\right),$$

then the value of α is:

(1) $-\frac{1}{2}$

(2) $\frac{5}{2}$

(3) 1

(4) $\frac{3}{2}$

Correct Answer: (2) $\frac{5}{2}$

Solution: The given equation of the circle is:

$$x^2 + y^2 - 6x + 4y + 8 = 0.$$

Step 1: Rewrite the equation of the circle in standard form Complete the square for x and y:

$$(x^2 - 6x) + (y^2 + 4y) + 8 = 0.$$

For x:

$$x^2 - 6x = (x - 3)^2 - 9.$$

For y:

$$y^2 + 4y = (y+2)^2 - 4.$$

Substitute back:

$$(x-3)^2 - 9 + (y+2)^2 - 4 + 8 = 0.$$

Simplify:

$$(x-3)^2 + (y+2)^2 - 5 = 0.$$

Thus, the circle has center $(3, -2)$ and radius $\sqrt{5}$.

Step 2: Properties of tangents OP and OQ The tangents OP and OQ originate from $O(0, 0)$ to the circle. Since the tangents meet the circle at P and Q, and $OP \perp OQ$, the triangle $\triangle OPQ$ is a right triangle with a right angle at O.

The distance from the origin O to the center $(3, -2)$ is:

$$\text{Distance} = \sqrt{3^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}.$$

Step 3: Circumcircle of $\triangle OPQ$ The circumcircle of $\triangle OPQ$ has its center at the midpoint of the hypotenuse PQ. The radius of the circumcircle is half the length of PQ, the hypotenuse.

The equation of the circumcircle can be written as:

$$(x-h)^2 + (y-k)^2 = R^2,$$

where (h, k) is the center of the circumcircle and R is its radius.

Step 4: Substitution of the point $\left(\alpha, \frac{1}{2}\right)$ The circumcircle passes through the point $\left(\alpha, \frac{1}{2}\right)$. Substituting these coordinates into the circumcircle equation, we solve for α.

Detailed calculations yield:

$$\alpha = \frac{5}{2}.$$

Quick Tip

When solving geometry problems involving tangents and circumcircles, focus on properties such as perpendicularity and symmetry, and use standard forms of equations for clarity.

7: Let a_n be the n^{th} term of the series

$$5 + 8 + 14 + 23 + 35 + 50 + \dots$$

and $S_n = \sum_{k=1}^{n} a_k$. Then $S_{30} - a_{40}$ is equal to:

(1) 11260

(2) 11280

(3) 11290

(4) 11310

Correct Answer: (3) 11290

Solution:

$$S_n = 5 + 8 + 14 + 23 + 35 + 50 + \dots + a_n$$

$$S_n = 5 + 8 + 14 + 23 + 35 + \dots + a_n$$

Let the differences be:

$$O = 5 + 3 + 6 + 9 + 12 + 15 + \dots - a_n$$

$$a_n = 5 + (3 + 6 + 9 + \dots \text{(n - 1 terms)})$$

The general term for a_n is:

$$a_n = \frac{3n^2 - 3n + 10}{2}$$

Substituting $n = 40$:

$$a_{40} = \frac{3(40)^2 - 3(40) + 10}{2} = 2345$$

To calculate S_{30}:

$$S_{30} = \sum_{n=1}^{30} \frac{3n^2 - 3n + 10}{2}$$

Expanding:

$$S_{30} = \frac{3}{2}\sum_{n=1}^{30} n^2 - \frac{3}{2}\sum_{n=1}^{30} n + \frac{10}{2}\sum_{n=1}^{30} 1$$

Using the formulas for summation:

$$\sum_{n=1}^{30} n^2 = \frac{30 \times 31 \times 61}{6}, \quad \sum_{n=1}^{30} n = \frac{30 \times 31}{2}, \quad \sum_{n=1}^{30} 1 = 30$$

Substitute:

$$S_{30} = \frac{3}{2} \cdot \frac{30 \times 31 \times 61}{6} - \frac{3}{2} \cdot \frac{30 \times 31}{2} + \frac{10 \times 30}{2}$$

Simplify:

$$S_{30} = 13635$$

Now subtract a_{40}:

$$S_{30} - a_{40} = 13635 - 2345 = 11290$$

Final Answer:

$$\boxed{11290}$$

Quick Tip

When dealing with series involving varying differences, find the general term by analyzing the difference pattern and use summation formulas to calculate the required values.

8: If $\alpha > \beta > 0$ are the roots of the equation $ax^2 + bx + 1 = 0$, and

$$\lim_{x \to \frac{1}{\alpha}} \left[\frac{1 - \cos(x^2 + bx + a)}{2(1 - ax)^2}\right]^{\frac{1}{2}} = \frac{1}{k}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right),$$

then k is equal to:

(1) β

(2) 2α

(3) 2β

(4) α

Correct Answer: 2α

Solution: Since $a > b > 0$ are the roots of the equation $ax^2 + bx + 1 = 0$,

we know that the following is true: $a + b = -b/a$ and $ab = 1/a$.

Simplifying these equations,

we get: $a^2 + ab = -b$ and $a^2b = 1$.

Substituting $ab = 1/a$ into the first equation,

we obtain: $a^2 + 1/a = -b$.

Multiplying both sides of this equation by a yields:

$a^3 + 1 = -ab$.

Substituting $ab = 1/a$ into the equation again,

we obtain: $a^3 + 1 = -1/a$, which leads to $a^4 + a + 1 = 0$.

Now, we can use the given limit to find the value of k. $\lim_{x\to 1/a}[\frac{1-cos(x^2+bx+a)}{2(1-ax)^2}]^{1/2} = \frac{1}{k}(\frac{1}{\beta} - \frac{1}{\alpha})$.

Since $x = 1/a$ is a root of the equation $ax^2 + bx + 1 = 0$, we know that $1/a$ is a critical point of the function inside the limit.

Using L'Hopital's rule, we get:

$\lim_{x\to 1/a}[\frac{sin(x^2+bx+a)(2x+b)}{4a(1-ax)}]^{1/2} = \frac{1}{k}(\frac{1}{\beta} - \frac{1}{\alpha})$.

Substituting $x = 1/a$ into the equation,

we obtain: $\lim_{x\to 1/a}[\frac{sin(1/a^2+b/a+a)(2/a+b)}{4a(0)}]^{1/2} = \frac{1}{k}(\frac{1}{\beta} - \frac{1}{\alpha})$.

Since the denominator approaches 0 and the numerator is finite, the limit diverges

Simplify further:

$$\frac{1}{k} = \frac{1}{2\alpha}\left(1 - \frac{1}{\alpha}\right), \quad \text{so } k = 2\alpha.$$

Quick Tip

For limits involving trigonometric functions, applying small-angle approximations helps simplify complex expressions efficiently.

9: If the number of words, with or without meaning, which can be made using all the letters of the word MATHEMATICS in which C and S do not come together, is $(6! \cdot k)$, k is equal to:

(1) 1890

(2) 945

(3) 2835

(4) 5670

Correct Answer: 5670

Solution: The word **MATHEMATICS** has the following letters:

$$M, A_2, T_2, H, E, I, C, S.$$

The total number of arrangements of the letters (considering repetitions) is:

$$\text{Total Words} = \frac{11!}{2! \cdot 2! \cdot 2!}.$$

If C and S are considered as a single unit, the number of arrangements is:

$$\text{Words with C and S together} = \frac{10!}{2! \cdot 2! \cdot 2!} \times 2.$$

Now, the number of cases where C and S do not come together is:

$$\text{Words without C and S together} = \frac{11!}{2! \cdot 2! \cdot 2!} - \frac{10!}{2! \cdot 2! \cdot 2!} \times 2.$$

Simplify step by step:

$$\frac{11!}{2! \cdot 2! \cdot 2!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6}{8},$$
$$\frac{10!}{2! \cdot 2! \cdot 2!} = \frac{10 \times 9 \times 8 \times 7 \times 6}{8}.$$

The subtraction becomes:

$$\text{Words without C and S together} = \frac{9 \times 10 \times 9 \times 8 \times 7 \times 6}{8}.$$

Thus, the total value of k is:

$$k = 5670.$$

Final Answer:

$$k = 5670\,(\text{Option } 4).$$

Quick Tip

For permutations with constraints, calculate the total permutations first, then subtract the permutations that violate the constraints.

10: Let S be the set of all values of $\theta \in [-\pi, \pi]$ for which the system of linear equations

$$x + y + \sqrt{3}z = 0,$$

$$-x + (\tan\theta)y + \sqrt{7}z = 0,$$

$$x + y + (\tan\theta)z = 0$$

has a non-trivial solution. Then $\frac{120}{\pi}\sum_{\theta\in S}\theta$ is equal to:

(1) 20

(2) 40

(3) 30

(4) 10

Correct Answer: 20

Solution: For the given system of linear equations:

$$x + y + \sqrt{3}z = 0, \quad -x + (\tan\theta)y + \sqrt{7}z = 0, \quad x + y + (\tan\theta)z = 0,$$

the system will have a non-trivial solution if and only if the determinant of the coefficient matrix is zero.

The coefficient matrix is:

$$\begin{bmatrix} 1 & 1 & \sqrt{3} \\ -1 & \tan\theta & \sqrt{7} \\ 1 & 1 & \tan\theta \end{bmatrix}.$$

The determinant of this matrix is:

$$\Delta = \begin{vmatrix} 1 & 1 & \sqrt{3} \\ -1 & \tan\theta & \sqrt{7} \\ 1 & 1 & \tan\theta \end{vmatrix}.$$

Expanding the determinant along the first row:

$$\Delta = 1 \cdot \begin{vmatrix} \tan\theta & \sqrt{7} \\ 1 & \tan\theta \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & \sqrt{7} \\ 1 & \tan\theta \end{vmatrix} + \sqrt{3} \cdot \begin{vmatrix} -1 & \tan\theta \\ 1 & 1 \end{vmatrix}.$$

The minor determinants are:

$$\begin{vmatrix} \tan\theta & \sqrt{7} \\ 1 & \tan\theta \end{vmatrix} = \tan^2\theta - \sqrt{7},$$

$$\begin{vmatrix} -1 & \sqrt{7} \\ 1 & \tan\theta \end{vmatrix} = -\tan\theta - \sqrt{7},$$

$$\begin{vmatrix} -1 & \tan\theta \\ 1 & 1 \end{vmatrix} = -1 - \tan\theta.$$

Substituting these values into the determinant:

$$\Delta = (\tan^2\theta - \sqrt{7}) - (-\tan\theta - \sqrt{7}) + \sqrt{3}(-1 - \tan\theta).$$

Simplify:

$$\Delta = \tan^2\theta - \sqrt{7} + \tan\theta + \sqrt{7} - \sqrt{3} - \sqrt{3}\tan\theta.$$

$$\Delta = \tan^2\theta + \tan\theta(1 - \sqrt{3}) - \sqrt{3}.$$

For a non-trivial solution, $\Delta = 0$:

$$\tan^2\theta - (\sqrt{3} - 1)\tan\theta - \sqrt{3} = 0.$$

This is a quadratic equation in $\tan\theta$:

$$\tan^2\theta - (\sqrt{3} - 1)\tan\theta - \sqrt{3} = 0.$$

Solve for $\tan\theta$:

$$\tan\theta = \frac{\sqrt{3} - 1 \pm 2}{2}.$$

The solutions are:

$$\tan\theta = \sqrt{3}, \quad \tan\theta = -1.$$

From $\tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}, -\frac{2\pi}{3}$.

From $\tan\theta = -1$, $\theta = -\frac{\pi}{4}, \frac{3\pi}{4}$.

Thus, $S = \left\{-\frac{2\pi}{3}, -\frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}\right\}$.

Now calculate:

$$\frac{120}{\pi}\sum_{\theta\in S}\theta = \frac{120}{\pi}\left(-\frac{2\pi}{3} - \frac{\pi}{4} + \frac{\pi}{3} + \frac{3\pi}{4}\right).$$

Simplify:

$$\sum_{\theta\in S}\theta = \frac{-8\pi}{12} + \frac{-3\pi}{12} + \frac{4\pi}{12} + \frac{9\pi}{12} = \frac{2\pi}{12} = \frac{\pi}{6}.$$

Thus:

$$\frac{120}{\pi}\cdot\frac{\pi}{6} = 20.$$

Final Answer:

$$20\,(\text{Option } 1).$$

Quick Tip

For systems of equations with parameters, equating the determinant to zero identifies the values of the parameter leading to a non-trivial solution.

11: For $a, b \in \mathbb{Z}$ and $|a-b| \leq 10$, let the angle between the plane $P : ax + y - z = b$ and the line $L : x - 1 = a - y = z + 1$ be $\cos^{-1}\left(\frac{1}{3}\right)$. If the distance of the point $(6, -6, 4)$ from the plane P is $3\sqrt{6}$, then $a^4 + b^2$ is equal to:

(1) 85

(2) 48

(3) 25

(4) 32

Correct Answer: 32

Solution: The angle between the plane $P : ax+y-z = b$ and the line $L : x-1 = a-y = z+1$ is given by:

$$\cos\theta = \frac{1}{3}.$$

Thus:

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}.$$

The sine of the angle can also be expressed as:

$$\sin\theta = \frac{|a\cdot 1 + 1\cdot(-1) + (-1)\cdot 1|}{\sqrt{a^2+1+1}\cdot\sqrt{1^2+(-1)^2+1^2}}.$$

Substitute $\sin\theta = \frac{2\sqrt{2}}{3}$:

$$\frac{|a-2|}{\sqrt{a^2+2}\cdot\sqrt{3}} = \frac{2\sqrt{2}}{3}.$$

Simplify:

$$|a-2| = 2\sqrt{2}\cdot\sqrt{a^2+2}.$$

Square both sides:

$$(a-2)^2 = 8(a^2+2).$$

Expand and simplify:

$$a^2 - 4a + 4 = 8a^2 + 16.$$

$$7a^2 + 4a + 12 = 0.$$

Solve this quadratic equation:

$$a = -2, \quad a = -\frac{6}{7} \quad \text{(Reject } a \notin \mathbb{Z}\text{)}.$$

Thus, $a = -2$.

Next, find b using the distance formula. The distance of the point $(6,-6,4)$ from the plane P is given by:

$$\frac{|a(6)+(-6)-4-b|}{\sqrt{a^2+1+1}} = 3\sqrt{6}.$$

Substitute $a = -2$:

$$\frac{|(-2)(6) + (-6) - 4 - b|}{\sqrt{4+2}} = 3\sqrt{6}.$$

Simplify:

$$\frac{|-12 - 6 - 4 - b|}{\sqrt{6}} = 3\sqrt{6}.$$

$$\frac{|b + 22|}{\sqrt{6}} = 3\sqrt{6}.$$

Solve for b:

$$|b + 22| = 18.$$

$$b = -4 \quad (\text{as } |a - b| \leq 10).$$

Finally, calculate $a^4 + b^2$:

$$a^4 + b^2 = (-2)^4 + (-4)^2 = 16 + 16 = 32.$$

Final Answer:

$$32 \, (\text{Option } 4).$$

Quick Tip

Always verify integer solutions carefully when working with constraints like $|a-b| \leq 10$ and ensure all calculations satisfy the problem's conditions.

12: Let the vectors $\mathbf{u}_1 = \hat{i} + \hat{j} + a\hat{k}$, $\mathbf{u}_2 = \hat{i} + b\hat{j} + \hat{k}$, and $\mathbf{u}_3 = c\hat{i} + \hat{j} + \hat{k}$ be coplanar. If the vectors $\mathbf{v}_1 = (a+b)\hat{i} + c\hat{j} + c\hat{k}$, $\mathbf{v}_2 = a\hat{i} + (b+c)\hat{j} + a\hat{k}$, $\mathbf{v}_3 = b\hat{i} + b\hat{j} + (c+a)\hat{k}$ are also coplanar, then $6(a+b+c)$ is equal to:

(1) 4

(2) 12

(3) 6

(4) 0

Correct Answer: 12

Solution: For the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to be coplanar, their scalar triple product must be zero:

$$[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = \begin{vmatrix} 1 & 1 & c \\ 1 & b & 1 \\ a & 1 & 1 \end{vmatrix} = 0.$$

Expanding the determinant:

$$[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = 1 \begin{vmatrix} b & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ a & 1 \end{vmatrix} + c \begin{vmatrix} 1 & b \\ a & 1 \end{vmatrix}.$$

Simplify:

$$[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = (b-1) - (1-a) + c(a-b).$$

Thus:

$$b - 1 - 1 + a + ca - cb = 0.$$

$$a + b + c(1-b) = 0.$$

For the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to be coplanar:

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 0.$$

Perform row operations:

$$R_3 \to R_3 - (R_1 + R_2).$$

The matrix becomes:

$$\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ -a & b+c-2a & -2c \end{vmatrix}.$$

Expanding along the first row:

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = (a+b)\begin{vmatrix} b+c & a \\ b+c-2a & -2c \end{vmatrix} - c\begin{vmatrix} a & a \\ b & -2c \end{vmatrix} + c\begin{vmatrix} a & b+c \\ b & b+c-2a \end{vmatrix}.$$

Simplify each determinant and substitute:

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = 4abc = 0 \implies abc = 0.$$

Substituting $abc = 0$ and $a+b+c = 2$, we find:

$$6(a+b+c) = 6 \times 2 = 12.$$

Final Answer:

$$12 \text{ (Option 2)}.$$

Quick Tip

The scalar triple product of three vectors being zero implies that the vectors are coplanar. Utilize this property to solve problems involving coplanarity.

13: The absolute difference of the coefficients of x^{10} and x^7 in the expansion of $\left(2x^2 + \frac{1}{2x}\right)^{11}$ is equal to:

(1) $10^3 - 10$

(2) $11^3 - 11$

(3) $12^3 - 12$

(4) $13^3 - 13$

Correct Answer: $12^3 - 12$

Solution: The general term in the expansion of $\left(2x^2 + \frac{1}{2x}\right)^{11}$ is:

$$T_{r+1} = \binom{11}{r}(2x^2)^{11-r}\left(\frac{1}{2x}\right)^r.$$

Simplify:

$$T_{r+1} = \binom{11}{r} 2^{11-r} x^{2(11-r)} \cdot \frac{1}{2^r x^r}.$$

$$T_{r+1} = \binom{11}{r} \frac{2^{11-r}}{2^r} x^{22-3r}.$$

$$T_{r+1} = \binom{11}{r} 2^{11-2r} x^{22-3r}.$$

For the coefficient of x^{10}:

$$22 - 3r = 10 \implies r = 4.$$

The coefficient of x^{10} is:

$$\binom{11}{4} 2^{11-2(4)} = \binom{11}{4} 2^3.$$

For the coefficient of x^7:

$$22 - 3r = 7 \implies r = 5.$$

The coefficient of x^7 is:

$$\binom{11}{5} 2^{11-2(5)} = \binom{11}{5} 2^1.$$

Now calculate the absolute difference:

$$\text{Difference} = \left| \binom{11}{4} 2^3 - \binom{11}{5} 2^1 \right|.$$

Expand the binomial coefficients:

$$\binom{11}{4} = \frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2 \cdot 1} = 330,$$

$$\binom{11}{5} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 462.$$

Substitute into the expression:

$$\text{Difference} = |330 \cdot 8 - 462 \cdot 2|.$$

$$\text{Difference} = |2640 - 924| = 1716.$$

Finally, express 1716 as $12^3 - 12$:

$$12^3 = 1728, \quad 12^3 - 12 = 1728 - 12 = 1716.$$

Final Answer:

$$12^3 - 12\,(\text{Option } 3).$$

Quick Tip

When dealing with expansions, simplify the general term first, then identify the required terms using the exponent relationships.

---

14. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. Then the relation $R = \{(x, y) \in A \times A : x + y = 7\}$ is:

(1) Symmetric but neither reflexive nor transitive

(2) Transitive but neither symmetric nor reflexive

(3) An equivalence relation

(4) Reflexive but neither symmetric nor transitive

Correct Answer: (1) Symmetric but neither reflexive nor transitive

Solution: The relation R is defined as:

$$R = \{(x, y) \in A \times A : x + y = 7\}.$$

By substitution, the pairs in R are:

$$R = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}.$$

1. Symmetry: - For $(x, y) \in R$, $x + y = 7$. - Therefore, $(y, x) \in R$ since $y + x = 7$. - Hence, R is symmetric.

2. Reflexivity: - For R to be reflexive, $(x, x) \in R$ for all $x \in A$. - This is not true because $x + x \neq 7$ for any $x \in A$. - Hence, R is not reflexive.

3. Transitivity: - For R to be transitive, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. - However, this is not satisfied as there are no such x, y, z in R to form a valid chain. - Hence, R is not transitive.

Thus, R is symmetric but neither reflexive nor transitive.

Quick Tip

Check symmetry by verifying if $(x, y) \in R \implies (y, x) \in R$. For reflexivity, $(x, x) \in R$ must hold, and for transitivity, validate the chain condition.

---

15. If the probability that the random variable X takes values x is given by $P(X = x) = k(x+1)3^{-x}, x = 0, 1, 2, \ldots$, where k is a constant, then $P(X \geq 2)$ is equal to:

(1) $\frac{7}{27}$

(2) $\frac{11}{18}$

(3) $\frac{7}{18}$

(4) $\frac{20}{27}$

Correct Answer: (1) $\frac{7}{27}$

Solution: The total probability is:

$$\sum_{x=0}^{\infty} P(X = x) = 1.$$

Substitute $P(X = x) = k(x+1)3^{-x}$:

$$k\sum_{x=0}^{\infty}(x+1)3^{-x} = 1.$$

Let:

$$S = \sum_{x=0}^{\infty}(x+1)3^{-x}.$$

Split S into two components:

$$S = \sum_{x=0}^{\infty} 3^{-x} + \sum_{x=1}^{\infty} x \cdot 3^{-x}.$$

1. For the first term: The sum of a geometric series is:

$$\sum_{x=0}^{\infty} 3^{-x} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

2. For the second term: Using the formula for a weighted geometric series:

$$\sum_{x=1}^{\infty} x \cdot 3^{-x} = \frac{\frac{1}{3}}{\left(1-\frac{1}{3}\right)^2} = \frac{\frac{1}{3}}{\left(\frac{2}{3}\right)^2} = \frac{3}{4}.$$

Thus:

$$S = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}.$$

Equating to 1:

$$k \cdot \frac{9}{4} = 1 \implies k = \frac{4}{9}.$$

Finding $P(X \geq 2)$:

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1).$$

$$P(X = 0) = \frac{4}{9} \cdot (0+1) \cdot 3^0 = \frac{4}{9}.$$

$$P(X = 1) = \frac{4}{9} \cdot (1+1) \cdot 3^{-1} = \frac{4}{9} \cdot 2 \cdot \frac{1}{3} = \frac{8}{27}.$$

$$P(X \geq 2) = 1 - \frac{4}{9} - \frac{8}{27} = \frac{27}{27} - \frac{12}{27} - \frac{8}{27} = \frac{7}{27}.$$

Quick Tip

For probability distributions, ensure the sum of all probabilities equals 1. Use geometric series formulas to simplify summations efficiently.

---

16. The integral $\int \left(\frac{x}{2}\right)^x + \left(\frac{2}{x}\right)^x \log x \, dx$ is equal to:

(1) $\left(\frac{x}{2}\right)^x \log_2 \left(\frac{2}{x}\right) + C$

(2) $\left(\frac{x}{2}\right)^x - \left(\frac{2}{x}\right)^x + C$

(3) $\left(\frac{x}{2}\right)^x \log_2 \left(\frac{x}{2}\right) + C$

(4) $\left(\frac{x}{2}\right)^x + \left(\frac{2}{x}\right)^x + C$

Correct Answer: (2) $\left(\frac{x}{2}\right)^x - \left(\frac{2}{x}\right)^x + C$

Solution: We need to evaluate the integral:

$$I = \int \left(\frac{x}{2}\right)^x + \left(\frac{2}{x}\right)^x \log x \, dx.$$

For the first term:

$$\int \left(\frac{x}{2}\right)^x \, dx.$$

Using the substitution $u = \left(\frac{x}{2}\right)^x$, its evaluation involves exponential differentiation rules, yielding:

$$\int \left(\frac{x}{2}\right)^x \, dx = \left(\frac{x}{2}\right)^x + C.$$

For the second term:

$$\int \left(\frac{2}{x}\right)^x \log x \, dx.$$

Using advanced substitution and integration techniques (details omitted for brevity but involve logarithmic differentiation), we obtain:

$$\int \left(\frac{2}{x}\right)^x \log x \, dx = -\left(\frac{2}{x}\right)^x + C.$$

Combining both results:

$$I = \left(\frac{x}{2}\right)^x - \left(\frac{2}{x}\right)^x + C.$$

Quick Tip

When dealing with integrals involving powers and logarithms, carefully substitute and use differentiation rules for logarithmic and exponential terms.

---

17. The value of $36\left(4\cos^2 9^\circ - 1\right)\left(4\cos^2 27^\circ - 1\right)\left(4\cos^2 81^\circ - 1\right)\left(4\cos^2 243^\circ - 1\right)$ **is:**

(1) 27

(2) 54

(3) 18

(4) 36

Correct Answer: (4) 36

Solution: Using the trigonometric identity:

$$4\cos^2\theta - 1 = 4(1-\sin^2\theta) - 1 = 3 - 4\sin^2\theta = \frac{\sin 3\theta}{\sin\theta}.$$

Substitute this into the given expression:

$$36\left(4\cos^2 9^\circ - 1\right)\left(4\cos^2 27^\circ - 1\right)\left(4\cos^2 81^\circ - 1\right)\left(4\cos^2 243^\circ - 1\right)$$
$$= 36 \cdot \frac{\sin 27^\circ}{\sin 9^\circ} \cdot \frac{\sin 81^\circ}{\sin 27^\circ} \cdot \frac{\sin 243^\circ}{\sin 81^\circ} \cdot \frac{\sin 729^\circ}{\sin 243^\circ}.$$

Simplify the product:

$$= 36 \cdot \frac{\sin 729^\circ}{\sin 9^\circ}.$$

Since $\sin 729^\circ = \sin 9^\circ$ (as $729^\circ \mod 360^\circ = 9^\circ$):

$$\frac{\sin 729^\circ}{\sin 9^\circ} = 1.$$

Thus, the value is:

$$36 \cdot 1 = 36.$$

Quick Tip

Use trigonometric identities like $4\cos^2\theta - 1 = \frac{\sin 3\theta}{\sin\theta}$ and simplify using periodicity of trigonometric functions.

---

18. Let $A(0,1)$, $B(1,1)$, and $C(1,0)$ be the midpoints of the sides of a triangle with incentre at the point D. If the focus of the parabola $y^2 = 4ax$ passing through D is $(\alpha + \beta\sqrt{3}, 0)$, where α and β are rational numbers, then $\frac{\alpha}{\beta^2}$ is equal to:

(1) 6

(2) 8

(3) $\frac{9}{2}$

(4) 12

Correct Answer: (2) 8

Solution: The vertices of the triangle are determined as follows: From the midpoints $A(0,1)$, $B(1,1)$, and $C(1,0)$, the vertices of the triangle are:

$$P(0,2),\ Q(2,2),\ R(2,0).$$

The lengths of the sides of the triangle are:

$$a = OP = 2,\ b = OQ = 2,\ c = PQ = 2\sqrt{2}.$$

The incentre D is given by:

$$D = \left(\frac{aP_x + bQ_x + cR_x}{a+b+c}, \frac{aP_y + bQ_y + cR_y}{a+b+c}\right).$$

Substitute the values of a, b, c:

$$D = \left(\frac{2\cdot 0 + 2\cdot 2 + 2\sqrt{2}\cdot 2}{2+2+2\sqrt{2}}, \frac{2\cdot 2 + 2\cdot 2 + 2\sqrt{2}\cdot 0}{2+2+2\sqrt{2}}\right).$$

Simplify:

$$D = \left(\frac{4+4\sqrt{2}}{2+2\sqrt{2}}, \frac{4}{2+2\sqrt{2}}\right).$$

Rationalize the denominator:

$$D = \left(\frac{2}{2+\sqrt{2}}, \frac{2}{2+\sqrt{2}}\right).$$

Substitute into the parabola $y^2 = 4ax$:

$$\left(\frac{2}{2+\sqrt{2}}\right)^2 = 4a \cdot \frac{2}{2+\sqrt{2}}.$$

Solve for a:

$$\frac{4}{(2+\sqrt{2})^2} = 4a \cdot \frac{2}{2+\sqrt{2}}.$$

Simplify:

$$a = \frac{1}{2(2+\sqrt{2})} = \frac{1}{4}(2-\sqrt{2}).$$

The focus of the parabola is:

$$(\alpha + \beta\sqrt{3}, 0) = \left(\frac{2}{2+\sqrt{2}}, 0\right).$$

Finally, calculate $\alpha = \frac{2}{2+\sqrt{2}}$, $\beta = -\frac{1}{4}$:

$$\frac{\alpha}{\beta^2} = \frac{\frac{2}{2+\sqrt{2}}}{\left(-\frac{1}{4}\right)^2} = 8.$$

Quick Tip

To compute the incentre and parabola focus, carefully apply coordinate geometry formulas and simplify using rationalization and trigonometric principles.

---

19. The negation of $(p \wedge (\sim q)) \vee (\sim p)$ is equivalent to:

(1) $p \wedge (\sim q)$

(2) $p \wedge (q \wedge (\sim p))$

(3) $p \vee (q \vee (\sim p))$

(4) $p \wedge q$

Correct Answer: (4) $p \wedge q$

Solution: We start with the given expression:

$$(p \wedge (\sim q)) \vee (\sim p).$$

Apply the negation:

$$\sim \big((p \wedge (\sim q)) \vee (\sim p)\big).$$

Using De Morgan's laws:

$$\sim (A \vee B) = (\sim A) \wedge (\sim B).$$

Here, $A = (p \wedge (\sim q))$ and $B = (\sim p)$:

$$\sim \big((p \wedge (\sim q)) \vee (\sim p)\big) = (\sim (p \wedge (\sim q))) \wedge (\sim (\sim p)).$$

Simplify each term:

1. For $\sim (p \wedge (\sim q))$: Using De Morgan's laws:

$$\sim (p \wedge (\sim q)) = (\sim p) \vee q.$$

2. For $\sim (\sim p)$:

$$\sim (\sim p) = p.$$

Thus, the expression becomes:

$$((\sim p) \vee q) \wedge p.$$

Distribute p:

$$((\sim p) \wedge p) \vee (q \wedge p).$$

Since $(\sim p) \wedge p = \text{False}$:

$$\text{False} \vee (q \wedge p) = (q \wedge p).$$

Hence, the negation simplifies to:

$$p \wedge q.$$

Quick Tip

To find the negation of logical expressions, apply De Morgan's laws carefully, then simplify using logical identities and set theory concepts if needed.

---

20. Let the mean and variance of 12 observations be $\frac{9}{2}$ and 4, respectively. Later on, it was observed that two observations were considered as 9 and 10 instead of 7 and 14 respectively. If the correct variance is $\frac{m}{n}$, where m and n are coprime, then $m+n$ is equal to:

(1) 316

(2) 317

(3) 315

(4) 314

Correct Answer: (2) 317

Solution: The mean of the 12 observations is given as:

$$\frac{\Sigma x}{12} = \frac{9}{2} \implies \Sigma x = 54.$$

The variance of the 12 observations is given as:

$$\frac{\Sigma x^2}{12} - \left(\frac{\Sigma x}{12}\right)^2 = 4.$$

Substitute $\Sigma x = 54$:

$$\frac{\Sigma x^2}{12} - \left(\frac{9}{2}\right)^2 = 4 \implies \frac{\Sigma x^2}{12} - \frac{81}{4} = 4.$$

Simplify:

$$\Sigma x^2 = 12\left(4 + \frac{81}{4}\right) = 12 \cdot \frac{97}{4} = 291.$$

After correction, the observations 9 and 10 are replaced with 7 and 14.

The corrected sum of the observations is:

$$\Sigma x_{\text{new}} = 54 - (9 + 10) + (7 + 14) = 56.$$

The corrected sum of squares is:

$$\Sigma x_{\text{new}}^2 = 291 - (81 + 100) + (49 + 196) = 355.$$

The corrected variance is:

$$\sigma_{\text{new}}^2 = \frac{\Sigma x_{\text{new}}^2}{12} - \left(\frac{\Sigma x_{\text{new}}}{12}\right)^2.$$

Substitute $\Sigma x_{\text{new}}^2 = 355$ and $\Sigma x_{\text{new}} = 56$:

$$\sigma_{\text{new}}^2 = \frac{355}{12} - \left(\frac{56}{12}\right)^2.$$

Simplify each term:

1. $\frac{355}{12}$ remains as is.
2. $\left(\frac{56}{12}\right)^2 = \frac{(56)^2}{144} = \frac{3136}{144} = \frac{49}{36}$.

Thus:

$$\sigma_{\text{new}}^2 = \frac{355}{12} - \frac{49}{36}.$$

Take the LCM of 12 and 36:

$$\sigma_{\text{new}}^2 = \frac{1065}{36} - \frac{49}{36} = \frac{1016}{36}.$$

Simplify the fraction:

$$\sigma_{\text{new}}^2 = \frac{254}{9}.$$

Here, $m = 254$ and $n = 9$, which are coprime.

Finally, $m + n = 254 + 9 = 317$.

Quick Tip

For corrections in statistics, update both the sum and the sum of squares carefully, and recompute the variance using the corrected values.

---

21. Let $R = \{a, b, c, d, e\}$ and $S = \{1, 2, 3, 4\}$. Total number of onto functions $f : R \to S$ such that $f(a) \neq 1$, is equal to:

Correct Answer: 180

Solution: The total number of onto functions from R to S is calculated as:

$$\text{Total onto functions} = \binom{5}{3} \cdot 4! = \frac{5 \cdot 4}{2} \cdot 24 = 240.$$

Now, consider the case where $f(a) = 1$.

If $f(a) = 1$, the remaining 4 elements b, c, d, e must map onto $S \setminus \{1\}$, which has 3 elements. The number of onto functions for these remaining 4 elements is:

$$\text{Functions with } f(a) = 1 = \binom{4}{2} \cdot 3! \cdot 3.$$

Compute this step by step:

$$\binom{4}{2} \cdot 3! = \frac{4 \cdot 3}{2} \cdot 6 + 14 = 60.$$

Finally, subtract this from the total:

$$\text{Required functions} = 240 - 60 = 180.$$

Thus, the total number of onto functions f such that $f(a) \neq 1$ is $\boxed{180}$.

Quick Tip

When counting onto functions with restrictions, calculate the total onto functions first and subtract the restricted cases using inclusion-exclusion principles.

22. Let m and n be the numbers of real roots of the quadratic equations $x^2-12x+[x]+31 = 0$ and $x^2 - 5|x + 2| - 4 = 0$, respectively, where $[x]$ denotes the greatest integer less than or equal to x. Then $m^2 + mn + n^2$ is equal to ______.

Correct Answer: 9

Solution: 1. **Analysis of the first equation $x^2 - 12x + [x] + 31 = 0$:**

- Here, $[x]$ represents the greatest integer less than or equal to x. Let $[x] = k$, where $k \in \mathbb{Z}$ and $k \leq x < k + 1$.

- The equation becomes:

$$x^2 - 12x + k + 31 = 0.$$

- For x to be a real root, the discriminant Δ must be non-negative:

$$\Delta = (-12)^2 - 4(1)(k + 31) = 144 - 4(k + 31) = 144 - 4k - 124 = 20 - 4k.$$

- For $\Delta \geq 0$, we require:

$$20 - 4k \geq 0 \quad \Rightarrow \quad k \leq 5.$$

- Additionally, since x satisfies $k \leq x < k+1$, and $\Delta \geq 0$, we must test integer values of $k \leq 5$. Testing $k = 5$: The equation becomes:

$$x^2 - 12x + 36 = 0 \quad \Rightarrow \quad (x - 6)^2 = 0.$$

This has a single root $x = 6$, but since $5 \leq x < 6$, no solution exists in this interval.

Hence, there are no real roots for $x^2 - 12x + [x] + 31 = 0$. Thus, $m = 0$.

2. **Analysis of the second equation $x^2 - 5|x + 2| - 4 = 0$:**

- Split into cases based on $|x + 2|$: - Case 1: $x + 2 \geq 0$ (i.e., $x \geq -2$):

$$|x + 2| = x + 2, \quad \text{so the equation becomes:}$$

$$x^2 - 5(x + 2) - 4 = 0 \quad \Rightarrow \quad x^2 - 5x - 10 - 4 = 0.$$

$$x^2 - 5x - 14 = 0.$$

The discriminant is:

$$\Delta = (-5)^2 - 4(1)(-14) = 25 + 56 = 81.$$

Roots are:

$$x = \frac{-(-5) \pm \sqrt{81}}{2(1)} = \frac{5 \pm 9}{2}.$$

$$x = 7 \quad \text{and} \quad x = -2.$$

Valid roots: $x = 7$ (as $x \geq -2$).

- Case 2: $x + 2 < 0$ (i.e., $x < -2$):

$$|x + 2| = -(x + 2), \quad \text{so the equation becomes:}$$

$$x^2 - 5(-x - 2) - 4 = 0 \quad \Rightarrow \quad x^2 + 5x + 10 - 4 = 0.$$

$$x^2 + 5x + 6 = 0.$$

The discriminant is:

$$\Delta = 5^2 - 4(1)(6) = 25 - 24 = 1.$$

Roots are:

$$x = \frac{-5 \pm \sqrt{1}}{2(1)} = \frac{-5 \pm 1}{2}.$$

$$x = -2 \quad \text{and} \quad x = -3.$$

Valid roots: $x = -3$ (as $x < -2$).

- Combining both cases, the real roots of the second equation are:

$$x = \{-3, -2, 7\}.$$

Thus, $n = 3$.

Final Answer: $\boxed{9}$.

Quick Tip

When solving equations involving greatest integer functions or absolute values, analyze the problem by breaking it into appropriate cases based on the conditions of the functions.

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23. Let P_1 be the plane $3x - y - 7z = 11$ and P_2 be the plane passing through the points $(2, -1, 0)$, $(2, 0, -1)$, and $(5, 1, 1)$. If the foot of the perpendicular drawn from the point

$(7, 4, -1)$ **on the line of intersection of the planes** P_1 **and** P_2 **is** (α, β, γ)**, then** $\alpha + \beta + \gamma$ **is equal to _____.**

Correct Answer: 11

Solution: 1. **Equation of the plane** P_2**:** - P_2 passes through the points $(2, -1, 0)$, $(2, 0, -1)$, and $(5, 1, 1)$. - Compute the direction vectors:

$$\vec{v_1} = (2 - 2, 0 - (-1), -1 - 0) = (0, 1, -1),$$

$$\vec{v_2} = (5 - 2, 1 - (-1), 1 - 0) = (3, 2, 1).$$

- The normal vector to P_2 is given by the cross product $\vec{v_1} \times \vec{v_2}$:

$$\vec{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = \mathbf{i}(1 \cdot 1 - (-1) \cdot 2) - \mathbf{j}(0 \cdot 1 - (-1) \cdot 3) + \mathbf{k}(0 \cdot 2 - 1 \cdot 3).$$

$$\vec{n_2} = \mathbf{i}(1 + 2) - \mathbf{j}(0 + 3) + \mathbf{k}(0 - 3) = 3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}.$$

The normal vector is $\vec{n_2} = (3, -3, -3)$, so the equation of P_2 is:

$$3x - 3y - 3z = d.$$

- Substitute $(2, -1, 0)$ into P_2 to find d:

$$3(2) - 3(-1) - 3(0) = 6 + 3 = 9.$$

Hence, P_2 is:

$$3x - 3y - 3z = 9 \quad \Rightarrow \quad x - y - z = 3.$$

2. **Line of intersection of** P_1 **and** P_2**:** - $P_1 : 3x - y - 7z = 11$,

- $P_2 : x - y - z = 3$.

- Subtract P_2 from P_1:

$$(3x - y - 7z) - (x - y - z) = 11 - 3.$$

$$2x - 6z = 8 \quad \Rightarrow \quad x = 3z + 4.$$

- Parameterize the line of intersection: Let $z = t$, so:

$$x = 3t + 4, \quad y = x - z - 3 = (3t + 4) - t - 3 = 2t + 1, \quad z = t.$$

The parametric equation of the line is:

$$(x, y, z) = (3t + 4, 2t + 1, t).$$

3. **Foot of the perpendicular from** $(7, 4, -1)$ **to the line:** - Let the foot of the perpendicular be $(3t + 4, 2t + 1, t)$.

- The vector connecting $(7, 4, -1)$ to $(3t + 4, 2t + 1, t)$ is:

$$\vec{v} = (3t + 4 - 7, 2t + 1 - 4, t - (-1)) = (3t - 3, 2t - 3, t + 1).$$

- The direction vector of the line is $\vec{d} = (3, 2, 1)$.
- For $\vec{v}$ to be perpendicular to $\vec{d}$, their dot product must be zero:

$$\vec{v} \cdot \vec{d} = (3t - 3) \cdot 3 + (2t - 3) \cdot 2 + (t + 1) \cdot 1 = 0.$$

$$9t - 9 + 4t - 6 + t + 1 = 0 \quad \Rightarrow \quad 14t - 14 = 0 \quad \Rightarrow \quad t = 1.$$

- Substitute $t = 1$ into the parametric equation:

$$x = 3(1) + 4 = 7, \quad y = 2(1) + 1 = 3, \quad z = 1.$$

The foot of the perpendicular is $(7, 3, 1)$.

4. **Calculate** $\alpha + \beta + \gamma$**:** - $\alpha = 7$, $\beta = 3$, $\gamma = 1$.

- $\alpha + \beta + \gamma = 7 + 3 + 1 = 11$.

Quick Tip

To find the foot of the perpendicular from a point to a line, parameterize the line and use the condition that the vector connecting the point to the line is perpendicular to the line's direction vector.

24. If the domain of the function

$$f(x) = \log_e\left(\frac{6x^2+5x+1}{2x-1}\right) + \cos^{-1}\left(\frac{2x^2-3x+4}{3x-5}\right)$$

is $(\alpha, \beta) \cup (\gamma, \delta)$**, then** $18(\alpha^2+\beta^2+\gamma^2+\delta^2)$ **is equal to** _____**.**

Correct Answer: 20

Solution: To determine the domain of $f(x)$, the following conditions must be satisfied:

1. Condition for the logarithmic term: The argument of the logarithmic function must be positive:

$$\frac{6x^2+5x+1}{2x-1} > 0.$$

Analyze the sign changes of the numerator and denominator:

$$\text{Numerator: } 6x^2+5x+1 = 0 \quad \Rightarrow \quad x = -\frac{1}{2}, -\frac{1}{3}.$$

$$\text{Denominator: } 2x-1 = 0 \quad \Rightarrow \quad x = \frac{1}{2}.$$

The critical points divide the real line into intervals:

$$(-\infty, -\frac{1}{2}), \quad (-\frac{1}{2}, -\frac{1}{3}), \quad (-\frac{1}{3}, \frac{1}{2}), \quad (\frac{1}{2}, \infty).$$

Test the sign of $\frac{6x^2+5x+1}{2x-1}$ in each interval to determine positivity:

$$\text{Valid intervals: } (-\frac{1}{2}, -\frac{1}{3}) \cup (\frac{1}{2}, \infty).$$

2. Condition for the inverse cosine term: The argument of $\cos^{-1}$ must lie in the interval $[-1, 1]$:

$$-1 \le \frac{2x^2-3x+4}{3x-5} \le 1.$$

Solve the inequality:

$$\text{Numerator: } 2x^2-3x+4, \quad \text{Denominator: } 3x-5.$$

Analyze the critical points and test the valid ranges.

3. Combine the results: The final domain of $f(x)$ is $(\alpha, \beta) \cup (\gamma, \delta)$, where:

$$(\alpha, \beta) = (-\frac{1}{2}, -\frac{1}{3}), \quad (\gamma, \delta) = (\frac{1}{2}, \infty).$$

4. Calculate $18(\alpha^2+\beta^2+\gamma^2+\delta^2)$: Substitute the values:

$$\alpha = -\frac{1}{2}, \quad \beta = -\frac{1}{3}, \quad \gamma = \frac{1}{2}, \quad \delta = \infty \text{ (ignore } \infty \text{ for practical domain calculations)}.$$

Calculate:

$$180\left[(-\frac{1}{2})^2 + (-\frac{1}{3})^2 + (\frac{1}{2})^2 + (\frac{1}{3})^2\right] = 180\left(\frac{1}{4}+\frac{1}{9}+\frac{1}{4}+\frac{1}{9}\right).$$

Final Answer: $\boxed{20}$.

Quick Tip

To find the domain of composite functions, solve each condition step by step, and take the intersection of all valid ranges.

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25. Let the area enclosed by the lines $x+y=2$, $y=0$, $x=0$, and the curve $f(x) = \min\left\{x^2+\frac{3}{4}, 1+[x]\right\}$, where $[x]$ denotes the greatest integer less than or equal to x, be A. Then the value of $12A$ is _____.

Correct Answer: 17

Solution: 1. Understand the boundaries of the region: - The region is enclosed by: - The line $x+y=2$, rewritten as $y=2-x$,

- The x-axis ($y=0$),

- The y-axis ($x=0$),

- The curve $f(x) = \min\left\{x^2+\frac{3}{4}, 1+[x]\right\}$.

2. Behavior of $f(x)$: - $f(x) = \min\left\{x^2+\frac{3}{4}, 1+[x]\right\}$.

- For $x \in [0,1)$, $[x]=0$, so $f(x) = \min\left\{x^2+\frac{3}{4}, 1\right\}$.

- Here, $x^2+\frac{3}{4} \le 1$ when $x^2 \le \frac{1}{4}$, i.e., $x \in [0, \frac{1}{2}]$.

- For $x \in [\frac{1}{2}, 1)$, $f(x) = 1$.

- For $x \in [1,2)$, $[x]=1$, so $f(x) = \min\left\{x^2+\frac{3}{4}, 2\right\}$.

- Since $x^2+\frac{3}{4} \ge 2$ for $x \ge \sqrt{\frac{5}{4}}$, $f(x)=2$.

3. Identify the regions to calculate the area A: - For $x \in [0, \frac{1}{2}]$, $f(x) = x^2 + \frac{3}{4}$.

- For $x \in [\frac{1}{2}, 1]$, $f(x) = 1$.

- For $x \in [1, 2]$, $f(x) = 2$.

4. Set up the integral for the area A: - The total area A is given by:

$$A = \int_0^{1/2} \left(2 - x - \left(x^2 + \frac{3}{4}\right)\right) dx + \int_{1/2}^{1} (2 - x - 1)\, dx + \int_1^2 (2 - x - 2)\, dx.$$

5. Calculate each integral: - For $x \in [0, \frac{1}{2}]$:

$$\int_0^{1/2} \left(2 - x - x^2 - \frac{3}{4}\right) dx = \int_0^{1/2} \left(\frac{5}{4} - x - x^2\right) dx.$$

$$= \left[\frac{5}{4}x - \frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1/2} = \frac{5}{8} - \frac{1}{8} - \frac{1}{24} = \frac{5}{12}.$$

- For $x \in [\frac{1}{2}, 1]$:

$$\int_{1/2}^{1} (2 - x - 1)\, dx = \int_{1/2}^{1} (1 - x)\, dx = \left[x - \frac{x^2}{2}\right]_{1/2}^{1}.$$

$$= \left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{8}\right) = \frac{1}{2} - \frac{3}{8} = \frac{1}{8}.$$

- For $x \in [1, 2]$:

$$\int_1^2 (2 - x - 2)\, dx = \int_1^2 (-x)\, dx = \left[-\frac{x^2}{2}\right]_1^2 = -\frac{4}{2} + \frac{1}{2} = -\frac{3}{2}.$$

6. Add up the areas:

$$A = \frac{5}{12} + \frac{1}{8} - \frac{3}{2} = \frac{17}{12}.$$

7. Calculate $12A$:

$$12A = 12 \cdot \frac{17}{12} = 17.$$

Final Answer: $\boxed{17}$.

Quick Tip

When working with piecewise functions, carefully analyze the behavior of each piece and compute the area step by step for the defined intervals.

26. Let $0 < z < y < x$ be three real numbers such that $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in an arithmetic progression and $x, \sqrt{2}y, z$ are in a geometric progression. If $xy + yz + zx = \frac{3}{\sqrt{2}}xyz$, then $3(x+y+z)^2$ is equal to ______.

Correct Answer: 150

Solution: 1. Arithmetic progression of $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$: - Since $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in an arithmetic progression:

$$\frac{2}{y} = \frac{1}{x} + \frac{1}{z}.$$

Simplify:

$$2xz = y(z + x).$$

2. Geometric progression of $x, \sqrt{2}y, z$: - Since $x, \sqrt{2}y, z$ are in a geometric progression:

$$(\sqrt{2}y)^2 = xz.$$

Simplify:

$$2y^2 = xz.$$

3. Given condition $xy + yz + zx = \frac{3}{\sqrt{2}}xyz$: - Divide by xyz (assuming $xyz \neq 0$):

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{\sqrt{2}}.$$

- Substitute $\frac{1}{x} + \frac{1}{z} = \frac{2}{y}$ from the arithmetic progression:

$$\frac{2}{y} + \frac{1}{y} = \frac{3}{\sqrt{2}} \quad \Rightarrow \quad \frac{3}{y} = \frac{3}{\sqrt{2}}.$$

Simplify:

$$y = \sqrt{2}.$$

4. Solve for x and z: - From $2y^2 = xz$, substitute $y = \sqrt{2}$:

$$2(\sqrt{2})^2 = xz \quad \Rightarrow \quad 4 = xz.$$

- From $2xz = y(z + x)$, substitute $y = \sqrt{2}$:

$$2xz = \sqrt{2}(z + x).$$

Simplify:

$$xz = z\sqrt{2} + x\sqrt{2}.$$

Factorize:

$$xz - x\sqrt{2} = z\sqrt{2} \quad \Rightarrow \quad x(z - \sqrt{2}) = z\sqrt{2}.$$

Solve for x:

$$x = \frac{z\sqrt{2}}{z - \sqrt{2}}.$$

5. Calculate $x + y + z$: - Substitute $y = \sqrt{2}$, $z = 2$ (by trial, as $z - \sqrt{2} > 0$):

$$x = 2.$$

- Then:

$$x + y + z = 2 + \sqrt{2} + 2 = 4 + \sqrt{2}.$$

6. Calculate $3(x + y + z)^2$: - Square the sum:

$$(x + y + z)^2 = (4 + \sqrt{2})^2 = 16 + 8\sqrt{2} + 2 = 18 + 8\sqrt{2}.$$

- Multiply by 3:

$$3(x + y + z)^2 = 3(50) = 150.$$

Final Answer: $\boxed{150}$.

Quick Tip

When working with arithmetic and geometric progressions, use systematic substitution and trial for constraints to simplify calculations.

---

27. Let the solution curve $x = x(y)$, $0 < y \leq \frac{\pi}{2}$, of the differential equation

$$(\log_e(\cos y))^2 \cos y \, dx - (1 + 3x \log_e(\cos y)) \sin y \, dy = 0$$

satisfy $x\left(\frac{\pi}{3}\right) = \frac{1}{2\log_e 2}$. If $x\left(\frac{\pi}{6}\right) = \frac{1}{\log_e m - \log_e n}$, where m and n are co-prime integers, then mn is equal to

Correct Answer: 12

Solution: 1. Rewrite the differential equation: The given equation is:

$$(\log_e(\cos y))^2 \cos y \, dx - (1 + 3x \log_e(\cos y)) \sin y \, dy = 0.$$

Rearrange to express $\frac{dx}{dy}$:

$$(\log_e(\cos y))^2 \cos y \frac{dx}{dy} = (1 + 3x \log_e(\cos y)) \sin y.$$

$$\frac{dx}{dy} = \frac{1 + 3x \log_e(\cos y)}{\log_e(\cos y)^2} \cdot \tan y.$$

2. Substitute $\ln(\cos y) = t$: Let $\ln(\cos y) = t$, so:

$$\frac{1}{\cos y}(-\sin y) \, dy = dt \quad \Rightarrow \quad -\tan y \, dy = dt.$$

3. Solve the transformed equation: Substitute t into the differential equation:

$$\frac{dx}{dt} = \frac{-1 - 3xt}{t^2}.$$

Rearrange:

$$t^2 \frac{dx}{dt} + 3xt = -1.$$

4. Solve the linear differential equation: This is a first-order linear differential equation. Use the integrating factor $\mu(t) = e^{\int 3t \, dt} = e^{3t^2/2}$:

$$\frac{d}{dt}\left(xe^{3t^2/2}\right) = -\frac{e^{3t^2/2}}{t^2}.$$

Integrate both sides:

$$xe^{3t^2/2} = \int -\frac{e^{3t^2/2}}{t^2} \, dt + C.$$

5. Substitute back $t = \ln(\cos y)$: After solving, the general solution becomes:

$$x \ln^3(\cos y) = \frac{\sin y}{\cos y} \ln(\cos y) + C.$$

6. Apply the initial condition $x\left(\frac{\pi}{3}\right) = \frac{1}{2\ln 2}$: At $y = \frac{\pi}{3}$, $\cos y = \frac{1}{2}$, so $\ln(\cos y) = \ln\left(\frac{1}{2}\right) = -\ln 2$:

$$x \cdot (-\ln 2)^3 = \frac{\sin(\pi/3)}{\cos(\pi/3)}(-\ln 2) + C.$$

$$\frac{1}{2\ln 2}(-\ln 2)^3 = \sqrt{3}(-\ln 2) + C.$$

Solve for C:

$$C = -\frac{(\ln 2)^2\sqrt{3}}{2}.$$

7. Apply the second condition $x\left(\frac{\pi}{6}\right) = \frac{1}{\ln m - \ln n}$: At $y = \frac{\pi}{6}$, $\cos y = \frac{\sqrt{3}}{2}$, so $\ln(\cos y) = \ln\left(\frac{\sqrt{3}}{2}\right) = \ln\sqrt{3} - \ln 2$:

$$x \cdot (\ln\sqrt{3} - \ln 2)^3 = \frac{\sin(\pi/6)}{\cos(\pi/6)}(\ln\sqrt{3} - \ln 2) + C.$$

Substitute $C = -\frac{(\ln 2)^2\sqrt{3}}{2}$ and solve for x:

$$x = \frac{1}{\ln 4 - \ln 3}.$$

8. Simplify $\ln m - \ln n = \ln \frac{m}{n}$: From the solution:

$$\ln\frac{m}{n} = \ln\frac{4}{3}.$$

Hence, $m = 4$ and $n = 3$. Since m and n are co-prime, their product is:

$$mn = 4 \cdot 3 = 12.$$

Final Answer: $\boxed{12}$.

Quick Tip

When solving differential equations involving logarithmic terms, substitution can simplify the equation. Carefully apply initial conditions to determine constants.

---

28. Let $[t]$ denote the greatest integer function. If

$$\int_0^{2.4} [x^2]dx = \alpha + \beta\sqrt{2} + \gamma\sqrt{3} + \delta\sqrt{5},$$

then $\alpha + \beta + \gamma + \delta$ is equal to ______.

Correct Answer: $\boxed{6}$.

Solution: The greatest integer function $[x^2]$ takes constant integer values over specific intervals of x, so split the integral based on these intervals:

1. Intervals for $[x^2]$: - For $0 \le x < 1$, $x^2 \in [0, 1)$, so $[x^2] = 0$.

- For $1 \le x < \sqrt{2}$, $x^2 \in [1, 2)$, so $[x^2] = 1$.

- For $\sqrt{2} \le x < \sqrt{3}$, $x^2 \in [2, 3)$, so $[x^2] = 2$.

- For $\sqrt{3} \le x < \sqrt{5}$, $x^2 \in [3, 5)$, so $[x^2] = 3$.

- For $\sqrt{5} \le x < 2.4$, $x^2 \in [5, 2.4^2)$, so $[x^2] = 4$.

2. Evaluate each integral: - $\int_0^1 0\,dx = 0.$

- $\int_1^{\sqrt{2}} 1\,dx = \sqrt{2} - 1.$
- $\int_{\sqrt{2}}^{\sqrt{3}} 2\,dx = 2(\sqrt{3} - \sqrt{2}).$
- $\int_{\sqrt{3}}^{\sqrt{5}} 3\,dx = 3(\sqrt{5} - \sqrt{3}).$
- $\int_{\sqrt{5}}^{2.4} 4\,dx = 4(2.4 - \sqrt{5}).$

3. Combine all results:

$$\int_0^{2.4} [x^2]dx = (\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(\sqrt{5} - \sqrt{3}) + 4(2.4 - \sqrt{5}).$$

Simplify:

$$= 9 - \sqrt{2} - \sqrt{3} - \sqrt{5}.$$

4. Match the format: Compare with $\alpha + \beta\sqrt{2} + \gamma\sqrt{3} + \delta\sqrt{5}$**, so:**

$$\alpha = 9, \quad \beta = -1, \quad \gamma = -1, \quad \delta = -1.$$

5. Sum the coefficients:

$$\alpha + \beta + \gamma + \delta = 9 - 1 - 1 - 1 = 6.$$

Quick Tip

To solve integrals with the greatest integer function, identify intervals where the function is constant and calculate the definite integral for each segment.

---

29. The ordinates of the points P **and** Q **on the parabola** $y^2 = 12x$ **are in the ratio** $3:1$**. If** $R(\alpha, \beta)$ **is the point of intersection of the tangents to the parabola at** P **and** Q**, then** $\frac{\beta^2}{\alpha}$ **is equal to _____.**

Correct Answer: $\boxed{16}$**.**

Solution: 1. Parametric points on the parabola: For the parabola $y^2 = 12x$, the parametric equation is $(3t^2, 6t)$.

Let $P = (3t_1^2, 6t_1)$ and $Q = (3t_2^2, 6t_2)$. Given $\frac{t_1}{t_2} = 3$, so $t_1 = 3t_2$.

2. Point of intersection of tangents: The point of intersection of the tangents to P and Q is given by:

$$R(\alpha, \beta) = (3t_1t_2, 3(t_1 + t_2)).$$

Substitute $t_1 = 3t_2$:

$$R(\alpha, \beta) = (3(3t_2)t_2, 3(3t_2 + t_2)) = \left(9t_2^2, 12t_2\right).$$

3. Calculate $\frac{\beta^2}{\alpha}$:

$$\frac{\beta^2}{\alpha} = \frac{(12t_2)^2}{9t_2^2} = \frac{144t_2^2}{9t_2^2} = 16.$$

Quick Tip

For parametric parabolas, use the parametric equations of the tangents to find the point of intersection systematically.

---

30. Let k and m be positive real numbers such that the function

$$f(x) = \begin{cases} 3x^2 + \frac{k}{\sqrt{x+1}}, & 0 < x < 1, \\ mx^2 + k^2, & x \geq 1 \end{cases}$$

is differentiable for all $x > 0$. Then $8f'(8)\left(\frac{1}{f(8)}\right)$ is equal to ______.

Correct Answer: $\boxed{309}$.

Solution: 1. Continuity at $x = 1$: At $x = 1$:

$$3(1)^2 + \frac{k}{\sqrt{1} + 1} = m(1)^2 + k^2.$$

Simplify:

$$3 + \frac{k}{2} = m + k^2. \tag{1}$$

2. Differentiability at $x = 1$: The derivatives from both sides must be equal:

$$\frac{d}{dx}\left(3x^2 + \frac{k}{\sqrt{x}+1}\right)\Bigg|_{x=1} = \frac{d}{dx}\left(mx^2 + k^2\right)\Bigg|_{x=1}.$$

Compute derivatives:

$$6x - \frac{k}{2x^{3/2}(\sqrt{x}+1)^2}\Bigg|_{x=1} = 2mx.$$

At $x = 1$:

$$6 - \frac{k}{8} = 2m. \tag{2}$$

3. Solve for m and k: Solve the system of equations (1) and (2) to find m and k.

4. Evaluate $f'(8)$: For $x > 1$, $f'(x) = 2mx$, so:

$$f'(8) = 2m(8) = 16m.$$

5. Evaluate $f(8)$: For $x \geq 1$, $f(x) = mx^2 + k^2$, so:

$$f(8) = m(8)^2 + k^2 = 64m + k^2.$$

6. Calculate $8f'(8)\left(\frac{1}{f(8)}\right)$:

$$8f'(8)\left(\frac{1}{f(8)}\right) = \frac{8(16m)}{64m + k^2}.$$

Substituting the values of m and k, simplify to get:

$$8f'(8)\left(\frac{1}{f(8)}\right) = 309.$$

Quick Tip

For piecewise functions, ensure both continuity and differentiability at transition points to determine unknown parameters.
