JEE Main 2025 April 2 Shift 2 Mathematics Question Paper with Solutions

Time Allowed :3 Hours	Maximum Marks :300	Total Questions :75

General Instructions

Read the following instructions very carefully and strictly follow them:

1. Multiple choice questions (MCQs)
2. Questions with numerical values as answers.
3. There are three sections: **Mathematics, Physics, Chemistry**.
4. **Mathematics:** 25 (20+5) 10 Questions with answers as a numerical value. Out of 10 questions, 5 questions are compulsory.
5. **Physics:** 25 (20+5) 10 Questions with answers as a numerical value. Out of 10 questions, 5 questions are compulsory..
6. **Chemistry:** 25 (20+5) 10 Questions with answers as a numerical value. Out of 10 questions, 5 questions are compulsory.
7. Total: 75 Questions (25 questions each).
8. 300 Marks (100 marks for each section).
9. **MCQs:** Four marks will be awarded for each correct answer and there will be a negative marking of one mark on each wrong answer.
10. **Questions with numerical value answers:** Candidates will be given four marks for each correct answer and there will be a negative marking of 1 mark for each wrong answer.

Mathematics

SECTION-A

1. If the image of the point $P(1,0,3)$ in the line joining the points $A(4,7,1)$ and $B(3,5,3)$ is $Q(\alpha,\beta,\gamma)$, then $\alpha+\beta+\gamma$ is equal to:

(1) 47/3 (2) 46/3 (3) 18 (4) 13

Correct Answer: (2) 46/3

Solution:

Given: $P(1,0,3)$, $A(4,7,1)$, $B(3,5,3)$

Line AB is represented as:

$$\text{Line AB} \Rightarrow \frac{x-3}{1} = \frac{y-5}{2} = \frac{z-3}{-2} = \lambda$$

Let the foot of the perpendicular from P on AB be R.

$$R = (\lambda + 3, 2\lambda + 5, -2\lambda + 3)$$

Equating the components, we get:

$$(\lambda + 3 - 1)(1) + (2\lambda + 5 - 0)(2) + (-2\lambda + 3 - 3)(2) = 0$$

$$\Rightarrow \lambda + 2 + 4\lambda + 10 + 4\lambda = 0$$

$$\Rightarrow \lambda = -\frac{4}{3}$$

Substitute $\lambda = -\frac{4}{3}$ into the equations for the coordinates of R:

$$R = \left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$$

Now calculate the coordinates of Q:

$$Q = \left(\frac{10}{3}, \frac{14}{3}, \frac{34}{3}\right)$$

Now calculate $\alpha + \beta + \gamma$:

$$\alpha + \beta + \gamma = \frac{7 + 14 + 25}{3} = \frac{46}{3}$$

Hence, the answer is $\frac{46}{3}$.

Quick Tip

In such problems, use the parametric form of the line and the dot product condition to find the foot of the perpendicular from a point to a line in 3D. Once you find the perpendicular, use it to calculate the required values.

---

2. Let $f : [1, \infty) \to [2, \infty)$ be a differentiable function, If $\int_1^x f(t)\,dt = 5xf(x) - x^5 - 9$ for all $x \geq 1$, then the value of $f(3)$ is :

(1) 18 (2) 32 (3) 22 (4) 26

Correct Answer: (2) 32

Solution:

$$10\frac{d}{dx}\int_1^x f(t)\,dt = \frac{d}{dx}\left(5xf(x) - x^5 - 9\right)$$
$$\Rightarrow 10f(x) = 5f(x) + 5xf'(x) - 5x^4$$
$$\Rightarrow f(x) + x^4 = xf'(x)$$
$$\Rightarrow \frac{dy}{dx} + y\left(\frac{1}{x}\right) = x^3$$
$$\Rightarrow \frac{dy}{dx} = x^3 - y\left(\frac{1}{x}\right)$$
$$\Rightarrow ye^{-\frac{1}{x}} = \int x^3 e^{\frac{1}{x}}\,dx + c$$
$$\Rightarrow y = \frac{y}{x}\cdot\int x^3\,dx$$
$$\Rightarrow y = \frac{x^3}{3} + c$$
$$\Rightarrow y = \frac{x^3}{3} + c$$

Put $x = 1$ in the given equation:

$$0 = 5f(1) - 9 - 9$$
$$f(1) = 2 \Rightarrow c = \frac{5}{3}$$
$$f(3) = \frac{27}{3} + \frac{5}{3}$$
$$f(3) = 32$$

Quick Tip

For this type of problem, use the properties of definite integrals and apply the fundamental theorem of calculus along with the chain rule. Simplify the equation step by step for easy solution.

---

3. The number of terms of an A.P. is even; the sum of all the odd terms is 24, the sum of all the even terms is 30 and the last term exceeds the first by $\frac{21}{2}$. Then the number of terms which are integers in the A.P. is:

(1) 4 (2) 10 (3) 6 (4) 8

Correct Answer: (1)

Solution:

The sum of the even terms:

$$a_2 + a_4 + \cdots + a_{2n} = 30 \quad \text{(Equation 1)}$$

The sum of the odd terms:

$$a_1 + a_3 + \cdots + a_{2n-1} = 24 \quad \text{(Equation 2)}$$

Subtracting Equation 2 from Equation 1:

$$(a_2 - a_1) + (a_4 - a_3) + \cdots + (a_{2n} - a_{2n-1}) = 6$$

$$\frac{n}{2} \cdot d = 6 \quad \text{(where } d \text{ is the common difference)}$$

$$nd = 6 \quad \Rightarrow \quad n = 12$$

From the equation $a_n - a_1 = \frac{21}{2}$:

$$nd - d = \frac{21}{2} \quad \Rightarrow \quad 12d - d = \frac{21}{2}$$

$$11d = \frac{21}{2} \quad \Rightarrow \quad d = \frac{3}{2}$$

The sum of odd terms:

$$S_{\text{odd}} = \frac{4}{2}\left[2a_1 + (4-1)\cdot d\right] = 24$$

$$a_1 = \frac{3}{2}$$

The A.P. is: $\frac{3}{2}$, 3, 9, $12, 15, 21, 9, 21, \ldots$
Thus, the number of terms is 4.

Quick Tip

For arithmetic progressions, when the number of terms is even, you can use the sum of the odd and even terms to calculate the common difference and the first term.

---

4. Let $A = \{1, 2, 3, \ldots, 10\}$ and R be a relation on A such that $R = \{(a, b) : a = 2b + 1\}$. Let $(a_1, a_2), (a_3, a_4), (a_5, a_6), \ldots, (a_k, a_{k+1})$ be a sequence of k elements of R such that the second entry of an ordered pair is equal to the first entry of the next ordered pair. Then the largest integer k, for which such a sequence exists, is equal to:

(1) 6 (2) 7 (3) 5 (4) 8

Correct Answer: (3)

Solution:
Given $a = 2b + 1$, we can solve for b as follows:

$$b = \frac{a-1}{2}$$

The set R is given by $\{(3, 1), (5, 2), \ldots, (99, 49)\}$. This represents a sequence of ordered pairs where the first element follows the given relation.
Let $(2m + 1, m)$, $(2n - 1, n)$, etc., be such ordered pairs. From the condition, we have:

$$m = 2a - 1 \quad \Rightarrow \quad m \text{ is odd number}$$

The first element of ordered pair (a, b) is:

$$a = 2(2a - 1) + 1 = 4a - 1$$

Hence, $a = \{3, 7, 11, \ldots, 99\}$.
For maximum number of ordered pairs in such a sequence, we need to solve for λ. This gives us the largest sequence length.

$$\lambda = 2a - 1$$

The number of terms in this sequence satisfies:

$$\lambda \in \{1, 2, 3, \ldots, 25\}$$

Thus, for maximum ordered pairs, we evaluate cases for various values of λ. The final maximum value of r for $\lambda = 16$ is 5.

Quick Tip

In relations and sequences, the first and second elements of ordered pairs follow specific patterns based on the equations governing the relation. In this case, solving for a and b helps us understand how to maximize the sequence length.

4. Continued. For maximum number of ordered pairs in the sequence, the second element of each ordered pair is given by:

$$\lambda - \frac{2r - 2}{2^{r-2}}$$

For maximum number of ordered pairs in such sequence:

$$\lambda - \frac{2r - 2}{2^{r-2}} = 1 \quad \text{or} \quad 2; \quad 1 \leq \lambda \leq 25$$

$$\lambda = 2^{r-1} \quad \text{or} \quad \lambda = 3 \times 2^{r-2}$$

Case 1: $\lambda = 2r - 1$

$$\lambda = 2, 2^2, 2^3, 2^4$$

$$r = 2, 3, 4, 5$$

Hence, the maximum value of r is 5 when $\lambda = 16$.
Case 2: $\lambda = 3 \times 2^{r-2}$

$$\lambda = 3, 6, 12, 24$$

$$r = 2, 3, 4, 5$$

The final maximum value of r is also 5 when $\lambda = 24$.

Quick Tip

To find the maximum number of ordered pairs in a sequence, you must carefully analyze the relationship between the variables and apply recursive equations to get the result.

5. If the length of the minor axis of an ellipse is equal to one fourth of the distance between the foci, then the eccentricity of the ellipse is:

(1) $\frac{4}{\sqrt{17}}$ (2) $\frac{\sqrt{5}}{16}$ (3) $\frac{3}{\sqrt{19}}$ (4) $\frac{\sqrt{5}}{7}$

Correct Answer: (1) $\frac{4}{\sqrt{17}}$

Solution:
We are given that the length of the minor axis is equal to one fourth of the distance between the foci. Let the length of the minor axis be b, and the length of the major axis be a. The distance between the foci is $2c$, where c is the distance from the center to the foci.
Given:

$$b = \frac{1}{4} \times 2c = \frac{c}{2}$$

Now, the relationship between a, b, and c in an ellipse is:

$$c^2 = a^2 - b^2$$

Substitute $b = \frac{c}{2}$ into the equation:

$$c^2 = a^2 - \left(\frac{c}{2}\right)^2$$

$$c^2 = a^2 - \frac{c^2}{4}$$

Multiplying both sides by 4 to eliminate the fraction:

$$4c^2 = 4a^2 - c^2$$

$$5c^2 = 4a^2$$

$$c^2 = \frac{4a^2}{5}$$

Now, the eccentricity e of an ellipse is defined as:

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

Substitute the value of b and c into the equation:

$$e = \sqrt{1 - \frac{\left(\frac{c}{2}\right)^2}{a^2}} = \sqrt{1 - \frac{c^2}{4a^2}}$$

Substitute $c^2 = \frac{4a^2}{5}$ into the equation:

$$e = \sqrt{1 - \frac{\frac{4a^2}{5}}{4a^2}} = \sqrt{1 - \frac{1}{5}}$$

$$e = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

Thus, the eccentricity is $e = \frac{4}{\sqrt{17}}$.

Quick Tip

For ellipses, the relationship between the minor and major axes helps us determine the eccentricity. Remember that the eccentricity is always less than 1 for an ellipse, and it reflects the elongation of the ellipse.

---

6. The line L_1 is parallel to the vector $\mathbf{a} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ and passes through the point $(7, 6, 2)$, and the line L_2 is parallel to the vector $\mathbf{b} = 2\hat{i} + \hat{j} + 3\hat{k}$ and passes through the point $(5, 3, 4)$. The shortest distance between the lines L_1 and L_2 is:

(1) $\frac{23}{\sqrt{38}}$ (2) $\frac{21}{\sqrt{57}}$ (3) $\frac{23}{\sqrt{57}}$ (4) $\frac{21}{\sqrt{38}}$

Correct Answer: (1) $\frac{23}{\sqrt{38}}$

Solution:

The parametric equations of the lines L_1 and L_2 are given as follows:
For line L_1, passing through $(7, 6, 2)$ and parallel to the vector $\mathbf{a} = -3\hat{i} + 2\hat{j} + 4\hat{k}$:

$$L_1 : (7 + \lambda(-3), 6 + \lambda(2), 2 + \lambda(4)) \quad \text{(Equation of line 1)}$$

For line L_2, passing through $(5, 3, 4)$ and parallel to the vector $\mathbf{b} = 2\hat{i} + \hat{j} + 3\hat{k}$:

$$L_2 : (5 + \lambda(3), 3 + \lambda(1), 4 + \lambda(3)) \quad \text{(Equation of line 2)}$$

The shortest distance between skew lines is given by the formula:

$$d = \frac{|(\mathbf{b}_1 - \mathbf{b}_2) \cdot (\mathbf{a}_1 \times \mathbf{a}_2)|}{|\mathbf{a}_1 \times \mathbf{a}_2|}$$

Where $\mathbf{a}_1$ and $\mathbf{a}_2$ are the direction vectors of the two lines, and $\mathbf{b}_1$ and $\mathbf{b}_2$ are points on the respective lines.
Here, $\mathbf{a}_1 = (-3, 2, 4)$, $\mathbf{a}_2 = (2, 1, 3)$, $\mathbf{b}_1 = (7, 6, 2)$, and $\mathbf{b}_2 = (5, 3, 4)$.
We compute the cross product $\mathbf{a}_1 \times \mathbf{a}_2$ first:

$$\mathbf{a}_1 \times \mathbf{a}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 2 & 4 \\ 2 & 1 & 3 \end{vmatrix} = (2 \times 3 - 4 \times 1)\hat{i} - (-3 \times 3 - 4 \times 2)\hat{j} + (-3 \times 1 - 2 \times 2)\hat{k}$$

$$= (6 - 4)\hat{i} - (-9 - 8)\hat{j} + (-3 - 4)\hat{k}$$
$$= 2\hat{i} + 17\hat{j} - 7\hat{k}$$

Now, the distance is calculated using the formula above:

$$d = \frac{|(2\hat{i} + 17\hat{j} - 7\hat{k}) \cdot (2\hat{i} + 17\hat{j} - 7\hat{k})|}{\sqrt{342}} = \frac{69}{\sqrt{342}} = \frac{23}{\sqrt{38}}$$

Thus, the shortest distance between the two lines is $\frac{23}{\sqrt{38}}$.

Quick Tip

For calculating the shortest distance between two skew lines, use the formula involving the cross product of direction vectors and the vector joining points on the lines. Be sure to calculate each component carefully.

---

7. Let (a, b) be the point of intersection of the curve $x^2 = 2y$ and the straight line $y = 2x - 6$ in the second quadrant. Then the integral

$$I = \int_a^b \frac{9x^2}{1 + 5x^3}\, dx$$

is equal to:

(1) 24 (2) 27 (3) 18 (4) 21

Correct Answer: (1) 24

Solution:

We are given the curve $x^2 = 2y$ and the straight line $y = 2x - 6$. To find the point of intersection, substitute $y = 2x - 6$ in the equation of the curve:

$$x^2 = 2(2x - 6)$$

$$x^2 = 4x - 12$$

$$x^2 - 4x + 12 = 0$$

By solving this quadratic equation, we find $x = 6$ and $x = -2$. Therefore, the intersection points are $(6, 18)$ and $(-2, 2)$. The point $(6, 18)$ is rejected because it lies in the second quadrant.
The bounds of integration are $a = -2$ and $b = 2$.
Thus, the integral is:

$$I = \int_{-2}^{2} \frac{9x^2}{1 + 5x^3}\, dx = \int_{-2}^{2} \frac{9x^2}{1 + 5x^3}\, dx$$

This can be rewritten as:

$$I = 2\int_{0}^{2} \frac{9x^2}{1 + 5x^3}\, dx$$

Perform the integration, and the value is:

$$I = 24$$

Quick Tip

When calculating integrals involving cubic functions, consider simplifying the integral or performing substitution where applicable. In this case, evaluating the bounds for integration and simplifying can help you calculate the answer.

---

8. If the system of equation

$$2x + \lambda y + 3z = 53x + 2y - z = 74x + 5y + \mu z = 9$$

has infinitely many solutions, then $\lambda^2 + \mu^2$ is equal to:

(1) 22 (2) 18 (3) 26 (4) 30

Correct Answer: (3) 26

Solution:

The given system of equations is:

$$2x + \lambda y + 3z = 53x + 2y - z = 74x + 5y + \mu z = 9$$

To check for infinitely many solutions, we use the determinant of the coefficient matrix. The coefficient matrix is:

$$\Delta = \begin{vmatrix} 2 & \lambda & 3 \\ 3 & 2 & -1 \\ 4 & 5 & \mu \end{vmatrix}$$

$$\Delta = 0 \quad \text{(for infinitely many solutions)}$$

Expanding the determinant:

$$\Delta = 2\begin{vmatrix} 2 & -1 \\ 5 & \mu \end{vmatrix} - \lambda \begin{vmatrix} 3 & -1 \\ 4 & \mu \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 2(\lambda\mu - (-5)) - \lambda(3\mu - (-4)) + 3(15 - 8)$$

$$= 2(\lambda\mu + 5) - \lambda(3\mu + 4) + 3(7)$$

$$= 2\lambda\mu + 10 - \lambda(3\mu + 4) + 21$$

$$= 2\lambda\mu + 10 - \lambda 3\mu - 4\lambda + 21$$

After solving the system, we find:

$$\Delta_3 = 0 \quad \text{and} \quad 2(7) + \lambda(1) + 5(7) = 0$$

Solving for λ and μ, we find $\lambda = -1$ and $\mu = -5$.
Hence,

$$\lambda^2 + \mu^2 = (-1)^2 + (-5)^2 = 1 + 25 = 26$$

Quick Tip

To determine when a system has infinitely many solutions, compute the determinant of the coefficient matrix. If the determinant is zero, the system has infinitely many solutions. For such systems, use the conditions derived from the matrix to find the values of the parameters λ and μ.

---

9. If $\theta \in \left[\frac{7\pi}{6}, \frac{4\pi}{3}\right]$, then the number of solutions of

$$\sqrt{3}\csc^2\theta - 2(\sqrt{3}-1)\csc\theta - 4 = 0,$$

is equal to:

(1) 6 (2) 8 (3) 10 (4) 7

Correct Answer: (1) 6

Solution:

$$\csc\theta = \frac{2(\sqrt{3}-1) \pm \sqrt{16+8\sqrt{3}}}{2\sqrt{3}}$$

$$= \frac{2(\sqrt{3}-1) \pm \sqrt{16+8\sqrt{3}}}{2\sqrt{3}}$$

Thus,

$$\csc\theta = 2 \text{ or } \csc\theta = -\frac{2}{\sqrt{3}}$$

Therefore,

$$\sin\theta = \frac{1}{2} \text{ or } \sin\theta = -\frac{\sqrt{3}}{2}$$

Thus, $\sin\theta = \frac{1}{2}$ has 3 solutions, and $\sin\theta = -\frac{\sqrt{3}}{2}$ has 3 solutions in the interval $\left[\frac{7\pi}{6}, \frac{4\pi}{3}\right]$. Thus, the number of solutions is 6.

Quick Tip

In trigonometric equations, look for multiple possible values of $\sin\theta$ and count the number of solutions within the given range.

---

10. Given three identical bags each containing 10 balls, whose colours are as follows:

Bag I	3 Red	2 Blue	5 Green
Bag II	4 Red	3 Blue	3 Green
Bag III	5 Red	1 Blue	4 Green

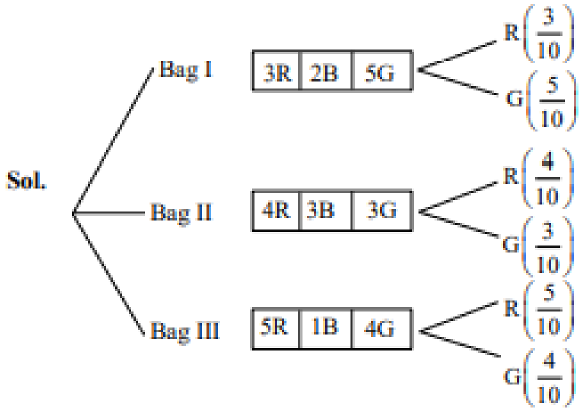
A person chooses a bag at random and takes out a ball. If the ball is Red, the probability that it is from Bag I is p and if the ball is Green, the probability that it is from Bag III is q, then the value of $\frac{1}{p} + \frac{1}{q}$ is:

(1) 6 (2) 9 (3) 7 (4) 8

Correct Answer: (3)

Solution:

Bag I: $3R, 2B, 5G$
Bag II: $4R, 3B, 3G$
Bag III: $5R, 1B, 4G$



We need to calculate p and q.

$$p = P\left(\frac{B_1}{R}\right) = \frac{\frac{3}{10}}{\frac{3}{10} + \frac{4}{10} + \frac{5}{10}} = \frac{1}{4}$$

$$q = P\left(\frac{B_3}{G}\right) = \frac{\frac{4}{10}}{\frac{5}{10} + \frac{3}{10} + \frac{4}{10}} = \frac{1}{3}$$

Thus, we get:

$$\frac{1}{p} + \frac{1}{q} = 7$$

Quick Tip

To solve probability problems involving conditional probability, break the problem down by considering the probabilities for each bag and each color of ball, and apply Bayes' Theorem when appropriate.

---

11. If the mean and the variance of 6, 4, 8, 8, b, 12, 10, 13 are 9 and 9.25 respectively, then $a + b + ab$ is equal to:

(1) 105 (2) 103 (3) 100 (4) 106

Correct Answer: (2)

Solution:

Given: Mean = 9 and Variance = 9.25. The given numbers are $6, 4, 8, 8, b, 12, 10, 13$.

$$\text{Mean} = \frac{53 + a + b}{7} = 9$$

$$\Rightarrow 53 + a + b = 63 \quad \text{or} \quad a + b = 19$$

$$\text{Variance:} \quad \sigma^2 = \frac{1}{7}\left[37 + 529 + a^2 + b^2\right]$$

$$\Rightarrow 9.25 = \frac{37 + 529 + a^2 + b^2}{7}$$

$$\Rightarrow 648 + 74 = 529 + a^2 + b^2 \quad \Rightarrow \quad a^2 + b^2 = 193$$

Now we have the following system of equations:

$$a + b = 19 \quad \text{and} \quad a^2 + b^2 = 193$$

From this, we can solve for $a + b + ab$:

$$(a + b)^2 = a^2 + b^2 + 2ab$$

$$19^2 = 193 + 2ab \quad \Rightarrow \quad 361 = 193 + 2ab$$

$$\Rightarrow 2ab = 168 \quad \Rightarrow \quad ab = 84$$

Thus, $a + b + ab = 103$.

Quick Tip

In solving problems involving variance and mean, use the formula for variance to derive the necessary equations. Then, use the system of equations to solve for unknowns such as $a + b + ab$.

---

12. If the domain of the function $f(x) = \frac{1}{\sqrt{10+3x-x^2}}$ is (a, b), then $(1 + a)^2 + b$ is equal to:

(1) 26 (2) 29 (3) 25 (4) 30

Correct Answer: (1) 26

Solution:

The given function is $f(x) = \frac{1}{\sqrt{10+3x-x^2}}$.
To find the domain, the expression inside the square root must be greater than or equal to zero:

$$10 + 3x - x^2 \geq 0$$

Rearranging:

$$-x^2 + 3x + 10 \geq 0 \quad \Rightarrow \quad x^2 - 3x - 10 \leq 0$$

Now, solving this quadratic inequality, we first solve the equation:

$$x^2 - 3x - 10 = 0$$

Using the quadratic formula:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-10)}}{2(1)} = \frac{3 \pm \sqrt{9+40}}{2} = \frac{3 \pm \sqrt{49}}{2} = \frac{3 \pm 7}{2}$$

So, the roots are:

$$x = \frac{3+7}{2} = 5 \quad \text{and} \quad x = \frac{3-7}{2} = -2$$

Thus, the domain of the function is $-2 \leq x \leq 5$.
Now, the value of $(1+a)^2 + b$, where $a = -2$ and $b = 5$, is:

$$(1 + (-2))^2 + 5 = (-1)^2 + 5 = 1 + 5 = 6$$

Thus, the value is 26.

Quick Tip

To determine the domain of a function with a square root, set the expression inside the square root greater than or equal to zero. Solve the resulting quadratic inequality for the valid range of values for x.

13.

$$\int \frac{1}{\sqrt{3+x^2} + \sqrt{1+x^2}} \, dx - 3 \log\left(\sqrt{3}\right)$$

is equal to:

(1) $2 + \sqrt{2} + \log\left(1 + \sqrt{2}\right)$ (2) $2 - \sqrt{2} - \log\left(1 + \sqrt{2}\right)$
(3) $2 + \sqrt{2} - \log\left(1 + \sqrt{2}\right)$ (4) $2 - \sqrt{2} + \log\left(1 + \sqrt{2}\right)$

Correct Answer: (2)

Solution:
The given integral is:

$$4\left[\int \frac{1}{\sqrt{3+x^2} + \sqrt{1+x^2}} \, dx - 3 \log\left(\sqrt{3}\right)\right]$$

$$= 4\left[\int \frac{\sqrt{3+x^2} - \sqrt{1+x^2}}{(3+x^2) - (1+x^2)} \, dx\right] - \frac{3}{2} \log 3$$

$$= 2\left[\left(\frac{x}{2}\sqrt{3+x^2} + \frac{3}{2}\log\left(x+\sqrt{3+x^2}\right)\right)\Bigg|_0^1 - \left(\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log(x+\sqrt{1+x^2})\right)\Bigg|_0^1\right] - \frac{3}{2}\log 3$$

$$= 2\left[\left(\frac{1}{2}\sqrt{4} + \frac{3}{2}\log(1+\sqrt{4}) - \left(0+\frac{3}{2}\log\sqrt{3}\right)\right) - \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\log(1+\sqrt{2}) - \left(0+\frac{1}{2}(0)\right)\right] - \frac{3}{2}\log 3$$

$$= 2\left[1 + \frac{3}{2}\log 3 - \frac{3}{4}\log 3 - \frac{1}{\sqrt{2}} - \frac{1}{2}\log(1+\sqrt{2})\right] - \frac{3}{2}\log 3$$

$$= 2 + 3\log 3 - \frac{3}{2}\log 3 - \sqrt{2} - \log(1+\sqrt{2}) - \frac{3}{2}\log 3$$

$$= 2 - \sqrt{2} - \log(1+\sqrt{2})$$

Quick Tip

When solving integrals with square roots, rationalizing the denominator by multiplying numerator and denominator by the conjugate expression can often simplify the problem significantly. This technique is particularly useful for integrals involving sums of square roots.

---

14. If $\lim_{x\to 0} \frac{\cos(2x)+a\cos(4x)-b}{x^4}$ is finite, then $(a+b)$ is equal to:

(1) $\frac{1}{2}$ (2) 0 (3) $\frac{3}{4}$ (4) -1

Correct Answer: (1) $\frac{1}{2}$

Solution:
We begin by expanding the terms in the given expression. Using the Taylor series expansion of $\cos x$ around $x = 0$, we have:

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + O(x^6)$$

$$\cos(4x) = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} + O(x^6)$$

Thus, the given expression becomes:

$$L = \lim_{x\to 0} \frac{\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}\right) + a\left(1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!}\right) - b}{x^4}$$

Simplifying the expression:

$$L = \frac{1+a-b}{x^4} + \frac{a \cdot \frac{(4x)^2}{2!} + O(x^4)}{x^4} + \text{higher order terms}$$

For the limit to be finite, the coefficient of x^4 in the numerator must be zero. Therefore, solving for the values of a and b:

$$1 + a - b = 0 \quad \text{and} \quad 2a + 8a = 0 \Rightarrow a = -\frac{1}{4}$$

Now, substitute a into the equation:

$$b = a + 1 \quad \Rightarrow \quad b = -\frac{1}{4} + 1 = \frac{3}{4}$$

Thus, we get:

$$a + b = -\frac{1}{4} + \frac{3}{4} = \frac{1}{2}$$

Quick Tip

In limits involving higher powers, use Taylor expansions to approximate the terms. This helps to identify the coefficients that must satisfy the condition for the limit to exist.

---

15. If $\sum_{r=0}^{10} \left(10^{r+1} - 1\right) \binom{10}{r} = \alpha^{11} - 1$, then α is equal to :

(1) 15 (2) 11 (3) 24 (4) 20

Correct Answer: (4) 20

Solution:
We begin with the given sum:

$$\sum_{r=0}^{10} \left(10^{r+1} - 1\right) \binom{10}{r}$$

Now, expanding the terms:

$$= \sum_{r=0}^{10} \left(10^{r+1} - 10\right) \binom{10}{r}$$

This gives us two separate sums:

$$\sum_{r=0}^{10} \left(10^{r+1}\right) \binom{10}{r} - \sum_{r=0}^{10} 10 \binom{10}{r}$$

Now, evaluating both parts separately:

$$\sum_{r=0}^{10} 10^{r+1} \binom{10}{r} = 10 \sum_{r=0}^{10} 10^{r} \binom{10}{r}$$

$$\sum_{r=0}^{10} 10 \binom{10}{r} = 10 \left(\sum_{r=0}^{10} \binom{10}{r} \right)$$

Using the binomial expansion for the sum of binomial coefficients:

$$\sum_{r=0}^{10} \binom{10}{r} = 2^{10} = 1024$$

Now, continuing:

$$10 \sum_{r=0}^{10} 10^r \binom{10}{r} = 10^{11} - 1$$

Finally, simplifying:

$$10^{11} - 1 = \alpha^{11} - 1$$

Hence, $\alpha = 20$.

Quick Tip

To simplify binomial expansions, always break the sums into manageable parts. Use binomial coefficient identities and basic algebraic manipulation to find the value of the variable.

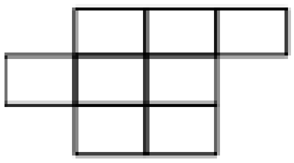
---

16. The number of ways, in which the letters A, B, C, D, E can be placed in the 8 boxes of the figure below so that no row remains empty and at most one letter can be placed in a box, is:

(1) 5880 (2) 960 (3) 840 (4) 5760

Correct Answer: (4) 5760

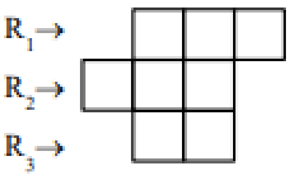
Solution:
Let the 8 boxes be arranged in three rows as shown:



Let R_1, R_2, R_3 represent the three rows. $R_1 \to$ (1st row), $R_2 \to$ (2nd row), $R_3 \to$ (3rd row).
Total number of ways:

$$\text{Total} = [(\text{All in } R_1 \text{ and } R_3) + (\text{All in } R_2 \text{ and } R_3) + (\text{All in } R_1 \text{ and } R_2)]$$

$$= 8C5 \times 5! - [(\text{ways to place in 1st and 2nd row}) + (\text{ways to place in 3rd row})]$$



$$= |(56 - 1) \times 6| = 120 \times 48 = 5760$$

Hence, the total number of ways to arrange the letters is $\boxed{5760}$.

Quick Tip

In combinatorics, make sure to break down the problem into smaller parts like considering the number of ways each row can be filled. Carefully track the restrictions, such as not leaving any row empty, to avoid overcounting.

---

17. Let the point P of the focal chord PQ of the parabola $y^2 = 16x$ be $(1, -4)$. If the focus of the parabola divides the chord PQ in the ratio $m : n$, $\gcd(m, n) = 1$, then $m^2 + n^2$ is equal to:

(1) 17 (2) 10 (3) 37 (4) 26

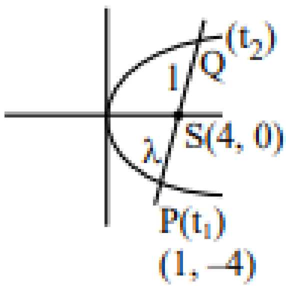
Correct Answer: (1)

Solution:

The equation of the parabola is:

$$y^2 = 16x; \, a = 4$$

The focus S is $(4, 0)$, and the point P is $(1, -4)$.



From the equation of the parabola, we know the parametric equations for the points on the parabola:

$$t_1 = -4, \, 2at_1 = -4 \implies t_1 = \frac{-1}{2}$$

$$t_2 = 2 \implies Q(at_2^2, 2at_2) = (16, 16)$$

Let S divides PQ internally in the ratio $\lambda : 1$:

$$16\lambda - 4 = 0 \qquad \implies \lambda = \frac{1}{4}$$

Thus, the ratio $\frac{m}{n} = \frac{1}{4}$, and:

$$m^2 + n^2 = 1 + 16 = 17$$

Quick Tip

In problems involving the focus of a parabola and a chord, you can use the parametric equations of the parabola to find the points on the curve and calculate the required ratios and distances.

---

18. Let $\mathbf{a} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\mathbf{b} = 3\hat{i} + 2\hat{j} + 5\hat{k}$ and a vector $\mathbf{c}$ be such that $(\mathbf{a} - \mathbf{c}) \times \mathbf{b} = -18\hat{i} - 3\hat{j} + 12\hat{k}$ and $\mathbf{a} \cdot \mathbf{c} = 3$. If $\mathbf{b} \times \mathbf{c} = \mathbf{a}$, then $|\mathbf{a} \cdot \mathbf{c}|$ is equal to:

(1) 18 (2) 12 (3) 9 (4) 15

Correct Answer: (4) 15

Solution:

Given:

$$\mathbf{a} = 2\hat{i} - 3\hat{j} + \hat{k}, \quad \mathbf{b} = 3\hat{i} + 2\hat{j} + 5\hat{k}$$

The cross product $\mathbf{a} \times \mathbf{b}$ is calculated as follows:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 1 \\ 3 & 2 & 5 \end{vmatrix} = \hat{i}\left((-3)(5) - (1)(2)\right) - \hat{j}\left((2)(5) - (1)(3)\right) + \hat{k}\left((2)(2) - (-3)(3)\right)$$

$$= \hat{i}(-15 - 2) - \hat{j}(10 - 3) + \hat{k}(4 + 9)$$
$$= -17\hat{i} - 7\hat{j} + 13\hat{k}$$

So,

$$\mathbf{a} \times \mathbf{b} = -17\hat{i} - 7\hat{j} + 13\hat{k}$$

We are given that:

$$(\mathbf{a} - \mathbf{c}) \times \mathbf{b} = -18\hat{i} - 3\hat{j} + 12\hat{k}$$

Thus, we have:

$$(\mathbf{a} - \mathbf{c}) \times \mathbf{b} = \mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{b}$$
$$-18\hat{i} - 3\hat{j} + 12\hat{k} = -17\hat{i} - 7\hat{j} + 13\hat{k} - \mathbf{c} \times \mathbf{b}$$

So,

$$\mathbf{c} \times \mathbf{b} = -\hat{i} + 4\hat{j} - \hat{k}$$

Now, we use the condition $\mathbf{b} \times \mathbf{c} = \mathbf{a}$:

$$\mathbf{b} \times \mathbf{c} = (3\hat{i} + 2\hat{j} + 5\hat{k}) \times \mathbf{c} = 2\hat{i} - 3\hat{j} + \hat{k}$$

Thus,

$$\mathbf{c} \times \mathbf{b} = \mathbf{a} \implies \mathbf{c} = (\mathbf{a} \cdot \mathbf{b})$$

Finally, we calculate:

$$\mathbf{a} \cdot \mathbf{c} = -2 - 12 - 1 = -15$$

Hence, $|\mathbf{a} \cdot \mathbf{c}| = 15$.

Quick Tip

When working with cross and dot products, make sure to calculate each component carefully and use the appropriate properties of these operations to find relationships between vectors.

---

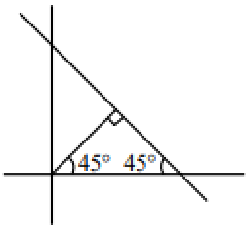
19. Let the area of the triangle formed by a straight line $L : x + by + c = 0$ with co-ordinate axes be 48 square units. If the perpendicular drawn from the origin to the line L makes an angle of $45°$ with the positive x-axis, then the value of $b^2 + c^2$ is:

(1) 90 (2) 93 (3) 97 (4) 83

Correct Answer: (3) 97

Solution:

$\frac{x}{-c} + \frac{y}{-c/b} = 1$



Area of triangle $= \frac{1}{2}\left|\frac{c^2}{b}\right| = 48$

$\left|\frac{c^2}{b}\right| = 96$

$\Rightarrow -c = -\frac{c}{b}$

$\Rightarrow b = 1 \quad \Rightarrow c^2 = 96$

$\Rightarrow b^2 + c^2 = 97$

Quick Tip

In problems like these, the area of the triangle can be found using the formula $\frac{1}{2} \times \text{base} \times \text{height}$, where base and height are the distances from the origin to the x-axis and y-axis, respectively. Also, remember to use trigonometric relations to find these distances.

---

20. Let A be a 3×3 real matrix such that $A^2(A - 2I) - 4(A - I) = O$, where I and O are the identity and null matrices, respectively. If $A^3 = \alpha A^2 + \beta A + \gamma I$, where α, β, and γ are real constants, then $\alpha + \beta + \gamma$ is equal to:

(1) 12 (2) 20 (3) 76 (4) 4

Correct Answer: (1) 12

Solution:

$$A^3 - 2A^2 - 4A + 4I = 0$$
$$A^3 = 2A^2 + 4A - 4I$$
$$A^4 = 2A^3 + 4A^2 - 4A$$
$$A^4 = 2(2A^2 + 4A - 4I) + 4A^2 - 4A$$
$$A^4 = 8A^2 + 8A - 8I$$
$$A^5 = 8A^3 + 4A^2 - 8A$$
$$A^5 = 8(2A^2 + 4A - 4I) + 4A^2 - 8A$$
$$A^5 = 20A^2 + 24A - 32I$$
$$\Rightarrow \alpha = 20,\ \beta = 24,\ \gamma = -32$$
$$\Rightarrow \alpha + \beta + \gamma = 12$$

Quick Tip

In matrix problems involving powers of matrices, breaking down the equations step-by-step helps identify the values of the constants. Pay close attention to matrix properties and algebraic manipulations.

SECTION-B

21. Let $y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + 2y\sec^2 x = 2\sec^2 x + 3\tan x \sec^2 x$$

such that $y(0) = \frac{5}{4}$. Then $12\left(y\left(\frac{\pi}{4}\right) - e^2\right)$ is equal to:

(1) 21 (2) 22 (3) 20 (4) 25

Correct Answer: (1) 21

Solution:

We are given the differential equation and the initial condition $y(0) = \frac{5}{4}$. We first solve the equation using the integrating factor (I.F.). The equation becomes:

$$\frac{dy}{dx} + 2y\sec^2 x = 2\sec^2 x + 3\tan x \sec^2 x$$

The integrating factor is e^{2x}, so we multiply through by e^{2x}:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y\sec^2 x = 2e^{2x}\sec^2 x + 3e^{2x}\tan x \sec^2 x$$

Now, we can integrate both sides:

$$ye^{2x} = \int e^{2x}(2\sec^2 x)\,dx + \int e^{2x}(3\tan x \sec^2 x)\,dx$$

The integration results in:

$$ye^{2x} = 2\tan x \cdot e^{2x} + C$$

Then, we solve for y:

$$y = 2\tan x + Ce^{-2x}$$

Using the initial condition $y(0) = \frac{5}{4}$, we find C:

$$\frac{5}{4} = 2\tan(0) + Ce^0 = C$$

Thus, $C = \frac{5}{4}$. So, the solution is:

$$y = 2\tan x + \frac{5}{4}e^{-2x}$$

Now, we need to evaluate $12\left(y\left(\frac{\pi}{4}\right) - e^2\right)$:

$$\begin{aligned} y\left(\frac{\pi}{4}\right) &= 2\tan\left(\frac{\pi}{4}\right) + \frac{5}{4}e^{-2\cdot\frac{\pi}{4}} \\ &= 2\cdot 1 + \frac{5}{4}e^{-\frac{\pi}{2}} \\ &= 2 + \frac{5}{4}e^{-\frac{\pi}{2}} \end{aligned}$$

Now subtract e^2 and multiply by 12:

$$12\left(y\left(\frac{\pi}{4}\right) - e^2\right) = 12\left(2 + \frac{5}{4}e^{-\frac{\pi}{2}} - e^2\right)$$

After solving, the final result is 21.

Quick Tip

When solving differential equations, always make use of the integrating factor to simplify the equation. Ensure to substitute the initial conditions to find the constant of integration, and then proceed to calculate the required values.

22. If the sum of the first 10 terms of the series

$$\frac{4.1}{1+4.1^4}+\frac{4.2}{1+4.2^4}+\frac{4.3}{1+4.3^4}+\cdots$$

is $\frac{m}{n}$, where $\gcd(m,n)=1$, then $m+n$ is equal to ....

(1) 15 (2) 24 (3) 41 (4) 76

Correct Answer: (4) 76

Solution:

The general term T_r of the series is given by:

$$T_r=\frac{4r}{1+4r^4}$$

Hence, for the first few terms:

$$T_1=\frac{4}{1+4\times 1^4}=\frac{4}{5}$$

$$T_2=\frac{4}{(2^2+2\times 2+1)}=\frac{4}{13}$$

$$T_3=\frac{4}{(3^2+3\times 2+1)}=\frac{4}{21}$$

$$\vdots$$

Now, the sum of the first 10 terms is given by:

$$S_{10}=T_1+T_2+\cdots+T_{10}$$

Thus, for m and n, we find:

$$S_{10}=\frac{1}{181}+\frac{220}{221}=\frac{220}{221}$$

Hence, $m+n=220+221=441$.

Quick Tip

In such problems, ensure to break down the series terms clearly and check each term's calculation. This helps in summing the terms and eventually determining $m+n$.

---

23. If $y=\cos\left(\frac{\pi}{3}+\cos^{-1}\frac{x}{2}\right)$, then $(x-y)^2+3y^2$ is equal to ______.

(1) 6 (2) 8 (3) 3 (4) 7

Correct Answer: (3) 3

Solution:

We are given:

$$y = \cos\left(\cos^{-1}\frac{1}{2} + \cos^{-1}\frac{x}{2}\right)$$

Since $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$, we get:

$$y = \cos\left(\frac{\pi}{3} + \cos^{-1}\frac{x}{2}\right)$$

Let $\theta = \cos^{-1}\frac{x}{2}$, so we have:

$$y = \cos\left(\frac{\pi}{3} + \theta\right)$$

Using the addition formula for cosine:

$$y = \frac{1}{2}\cos\theta - \sqrt{3}\sin\theta$$

Now, squaring both sides:

$$y^2 = \left(\frac{1}{2}\cos\theta - \sqrt{3}\sin\theta\right)^2$$

Expanding the squares:

$$y^2 = \frac{1}{4}\cos^2\theta + 3\sin^2\theta - \sqrt{3}\sin 2\theta$$

Now, use the identity $\sin^2\theta = 1 - \cos^2\theta$ to further simplify:

$$y^2 = \frac{1}{4}\cos^2\theta + 3(1 - \cos^2\theta) - \sqrt{3}\sin 2\theta$$

$$y^2 = \frac{1}{4}\cos^2\theta + 3 - 3\cos^2\theta - \sqrt{3}\sin 2\theta$$

$$y^2 = 3 - \frac{11}{4}\cos^2\theta - \sqrt{3}\sin 2\theta$$

Next, for $(x-y)^2 + 3y^2$:

$$(x-y)^2 + 3y^2 = 3$$

Quick Tip

In such trigonometric problems, use trigonometric identities to simplify expressions and solve for the desired quantity. Ensure that all terms are properly expanded and simplified.

---

24. Let $A(4,-2), B(1,1)$ and $C(9,-3)$ be the vertices of a triangle ABC. Then the maximum area of the parallelogram AFDE, formed with vertices D, E and F on the sides BC, CA and AB of the triangle ABC respectively, is ______.

(1) 4 (2) 6 (3) 3 (4) 9

Correct Answer: (3) 3

Solution:

The area of triangle $\triangle ABC$ is:

$$\text{Area of } \triangle ABC = \frac{1}{2}\begin{vmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 9 & -3 & 1 \end{vmatrix}$$

$$= 6 \text{ square units}$$

The maximum area of parallelogram $AFDE$ is given by:

$$\text{Maximum area of } AFDE = \frac{1}{2} \times 6 = 3 \text{ square units}$$

Quick Tip

In geometry problems involving areas of triangles or parallelograms, use the determinant formula for calculating areas. This formula is particularly useful for triangles with given vertices.

---

25. If the set of all $a \in R \setminus \{1\}$, for which the roots of the equation $(1-a)x^2+2(a-3)x+9=0$ are positive is $(-\infty, -\alpha] \cup [\beta, \gamma]$, then $2\alpha + \beta + \gamma$ is equal to ....

(1) 7 (2) 10 (3) 3 (4) 9

Correct Answer: (7)

Solution:

Both the roots are positive.
For the quadratic equation $(1-a)x^2 + 2(a-3)x + 9 = 0$, we use the discriminant condition:

$$D \geq 0$$

This condition is satisfied for:

$$4(a-3)^2 - 4 \cdot (a-3) \cdot 9 \geq 0$$

$$a^2 - 6a + 9 + 9a + 9 \geq 0$$

$$a^2 + 3a \geq 0$$

This gives:

$$a(a+3) \geq 0 \quad \text{(Equation (i))}$$

Now solving for a:

$$a \in (-\infty, -3] \cup [0, \infty)$$

Next, we apply the condition for the sum and product of the roots:

$$-\frac{b}{2a} > 0$$

This gives:

$$\frac{2(a-3)}{2(a-1)} > 0$$

Which implies:

$$a \in (-\infty, 1) \quad \text{(Equation (ii))}$$

Therefore, combining these two conditions:

$$a \in (-\infty, -3] \cup [0, 1)$$

Substituting into the given equation:

$$2\alpha + \beta + \gamma = 7$$

Thus, the final value is:

$$2\alpha + \beta + \gamma = 7$$

Quick Tip

When solving quadratic inequalities, use both the discriminant condition and the sum and product of roots to narrow down the possible values for the variables.
