Maharashtra Board Class 12 Mathematics & Statistics (40-J-862) 2024 Question Paper with Solutions

Time Allowed :3 Hour	**Maximum Marks :**80	**Total questions :**34

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The question paper is divided into FOUR sections.
2. Section A: Q. 1 contains Eight multiple choice type of questions, each carrying Two marks.
   Q. 2 contains Four very short answer type questions, each carrying One mark.
3. Section B: Q. 3 to Q. 14 contain Twelve short answer type questions, each carrying Two marks. (Attempt any Eight)
4. Section C: Q. 15 to Q. 26 contain Twelve short answer type questions, each carrying Three marks. (Attempt any Eight)
5. Section D: Q. 27 to Q. 34 contain Eight long answer type questions, each carrying Four marks. (Attempt any Five)
6. Use of log table is allowed. Use of calculator is not allowed.
7. Figures to the right indicate full marks.
8. Use of graph paper is not necessary. Only rough sketch of graph is expected.
9. For each multiple choice type of question, only the first attempt will be considered for evaluation.
10. Start answer to each section on a new page.

SECTION - A

1. Select and write the correct answer for the following multiple choice type of questions:

(i) The dual of statement $t \vee (p \vee q)$ is ___.

(a) $c \wedge (p \vee q)$

(b) $c \wedge (p \wedge q)$

(c) $t \wedge (p \wedge q)$

(d) $t \wedge (p \vee q)$

Correct Answer: (c) $t \wedge (p \wedge q)$

Solution:

Step 1: Understanding the Concept of Duality

Duality in logic involves the following transformations: - Replacing $\vee$ (OR) with $\wedge$ (AND), and vice versa. - Swapping 1 with 0, and vice versa.

Step 2: Applying Duality to the Given Expression

For the expression $t \vee (p \vee q)$, applying the duality principle gives:

$$t \vee (p \vee q) \quad \Rightarrow \quad t \wedge (p \wedge q)$$

Hence, the correct result is $t \wedge (p \wedge q)$.

Quick Tip

In duality, exchange $\vee$ with $\wedge$ and vice versa, ensuring the variables remain in the same order.

(ii) The principal solutions of the equation $\cos\theta = \frac{1}{2}$ are ___.

(a) $\frac{\pi}{6}, \frac{5\pi}{6}$

(b) $\frac{\pi}{3}, \frac{5\pi}{3}$

(c) $\frac{\pi}{6}, \frac{7\pi}{6}$

(d) $\frac{\pi}{3}, \frac{2\pi}{3}$

Correct Answer: (B) $\frac{\pi}{3}, \frac{5\pi}{3}$

Solution:

Step 1: Finding the General Solutions

The equation $\cos\theta = \frac{1}{2}$ holds at:

$$\theta = \pm\frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$$

Step 2: Principal Solutions

The general solutions in the principal range $[0, 2\pi]$ are:

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Quick Tip

For $\cos\theta = a$, the principal solutions are $\theta = \pm\cos^{-1}(a)$ within $[0, 2\pi]$.

(iii) If α, β, γ are direction angles of a line and $\alpha = 60^\circ, \beta = 45^\circ$, then γ is ___.

(a) 30° or 90°

(b) 45° or 60°

(c) 90° or 130°

(d) 60° or 120°

Correct Answer: (D) 60° or 120°

Solution:

Step 1: Applying the Direction Cosine Identity

The identity for the sum of the squares of the direction cosines states:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Step 2: Substituting the Given Values

Substitute the known values into the equation:

$$\cos^2(60^\circ) + \cos^2(45^\circ) + \cos^2\gamma = 1$$

$$\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right) + \cos^2\gamma = 1$$

$$\cos^2\gamma = \frac{1}{4}$$

$$\gamma = 60^\circ \text{ or } 120^\circ$$

Quick Tip

Remember, the sum of the squares of the cosines of direction angles is always equal to 1:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

(iv) The perpendicular distance of the plane $r \cdot (3\hat{i} + 4\hat{j} + 12\hat{k}) = 78$ from the origin is ___.

(a) 4

(b) 5

(c) 6

(d) 8

Correct Answer: (c) 6

Solution:

Step 1: Formula for Perpendicular Distance

The perpendicular distance d from the origin to the plane $Ax + By + Cz = D$ is calculated using the formula:

$$d = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}$$

Step 2: Substituting the Given Values

Given that:

$$A = 3, \quad B = 4, \quad C = 12, \quad D = 78$$

we substitute these values into the formula:

$$d = \frac{|78|}{\sqrt{3^2 + 4^2 + 12^2}}$$

$$= \frac{78}{\sqrt{169}} = \frac{78}{13} = 6$$

Quick Tip

The perpendicular distance of a plane $Ax + By + Cz = D$ from the origin is given by:

$$d = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}$$

(v) The slope of the tangent to the curve $x = \sin\theta$ and $y = \cos 2\theta$ at $\theta = \frac{\pi}{6}$ is ___.

(a) $-2\sqrt{3}$

(b) $\frac{-2}{\sqrt{3}}$

(c) -2

(d) $-\frac{1}{2}$

Correct Answer: (c) -2

Solution:

Step 1: Find $\frac{dy}{dx}$

$$\frac{dx}{d\theta} = \cos\theta, \quad \frac{dy}{d\theta} = -2\sin 2\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-2\sin 2\theta}{\cos\theta}$$

Step 2: Evaluate at $\theta = \frac{\pi}{6}$

$$\sin 2\theta = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos\theta = \frac{\sqrt{3}}{2}$$

$$\frac{dy}{dx} = \frac{-2 \times \frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = -2$$

Quick Tip

For parametric curves $x = f(\theta)$, $y = g(\theta)$, the slope is:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

(vi) If $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^4 x \, dx = k$, then k is ___.

(a) 1

(b) 2

(c) 4

(d) 0

Correct Answer: (d) 0

Solution:

Step 1: Analyzing the Function's Parity

The function inside the integral is:

$$f(x) = x^3 \sin^4 x$$

Since x^3 is odd and $\sin^4 x$ is even, their product $x^3 \sin^4 x$ is an odd function.

Step 2: Integrating an Odd Function

For any odd function $f(x)$, we know that:

$$\int_{-a}^{a} f(x)\,dx = 0$$

Given that our limits are symmetric about zero, we can directly conclude:

$$k = 0$$

Quick Tip

The integral of an odd function over symmetric limits always equals zero:

$$\int_{-a}^{a} f(x)\,dx = 0$$

(vii) The integrating factor of the linear differential equation

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

is ___.

(a) x

(b) $\frac{1}{x}$

(c) x^2

(d) $\frac{1}{x^2}$

Correct Answer: (c) x^2

Solution:

Step 1: Rewriting the Equation in Standard Form

The given equation:

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

can be rewritten as:

$$\frac{dy}{dx} + \frac{2}{x}y = x \log x$$

Step 2: Determining the Integrating Factor (IF)

The integrating factor is computed as:

$$IF = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

Quick Tip

For a linear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x),$$

the integrating factor is:

$$IF = e^{\int P(x)dx}$$

(viii) If the mean and variance of a binomial distribution are 18 and 12 respectively, then the value of n is ___.

(a) 36

(b) 54

(c) 18

(d) 27

Correct Answer: (B) 54

Solution:

Step 1: Review of Mean and Variance Formulas

For a binomial distribution with parameters n and p, the mean and variance are expressed as:

$$\mu = np, \quad \sigma^2 = np(1-p)$$

Step 2: Formulating the Equations

We are provided with the following equations:

$$np = 18, \quad np(1-p) = 12$$

By dividing the second equation by the first:

$$(1-p) = \frac{12}{18} = \frac{2}{3}$$

which simplifies to:

$$p = \frac{1}{3}$$

Step 3: Solving for n

Substitute $p = \frac{1}{3}$ into the first equation:

$$n \times \frac{1}{3} = 18$$

Solving for n:

$$n = 18 \times 3 = 54$$

Quick Tip

For a binomial distribution, the formulas for mean and variance are:

$$\mu = np, \quad \sigma^2 = np(1-p)$$

Use these to solve for n and p.

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2. Answer the following questions :

(1) Write the compound statement 'Nagpur is in Maharashtra and Chennai is in Tamilnadu' symbolically.

Solution:

Step 1: Define the Propositions

Let:

- p denote "Nagpur is in Maharashtra."

- q denote "Chennai is in Tamil Nadu."

Step 2: Express the Compound Statement

Since the statement uses the word "and," we apply the logical conjunction ($\wedge$):

$$p \wedge q$$

Quick Tip

In logic, the conjunction ($\wedge$) is used to represent "and," while the disjunction ($\vee$) is used to represent "or."

(2) If the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $p\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear, then find the value of p.

Solution:

Step 1: Condition for Vectors to be Collinear

Two vectors $\mathbf{A}$ and $\mathbf{B}$ are collinear if there exists a scalar k such that:

$$\mathbf{B} = k\mathbf{A}$$

Step 2: Writing the Vectors in Component Form

Given the vectors:

$$(2\hat{i} - 3\hat{j} + 4\hat{k}) \quad \text{and} \quad (p\hat{i} + 6\hat{j} - 8\hat{k})$$

Equating the corresponding components:

$$p = 2k, \quad 6 = -3k, \quad -8 = 4k$$

Step 3: Solving for the Scalar k

From the equation $6 = -3k$, solving for k:

$$k = -2$$

Step 4: Solving for p

Substitute $k = -2$ into the equation $p = 2k$:

$$p = 2(-2) = -4$$

Quick Tip

For two vectors to be collinear, their corresponding components must be proportional by the same scalar k.

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(3) Evaluate:

$$\int \frac{1}{x^2 + 25} dx.$$

Solution:

Step 1: Recognizing the Standard Integral Formula

We apply the standard integral formula:

$$\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C.$$

Step 2: Substituting $a = 5$

Given that $a^2 = 25$, we find $a = 5$. Thus, the integral becomes:

$$\int \frac{1}{x^2 + 25} \, dx = \frac{1}{5} \tan^{-1} \left(\frac{x}{5} \right) + C.$$

Quick Tip

The integral $\int \frac{1}{x^2+a^2} \, dx$ is evaluated using:

$$\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C.$$

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(4) A particle is moving along the X-axis. Its acceleration at time t is proportional to its velocity at that time. Find the differential equation of the motion of the particle.

Solution:

Step 1: Relating Acceleration to Velocity

Given that acceleration a is proportional to velocity v, we have:

$$a = kv$$

Since acceleration is the rate of change of velocity, we can write:

$$\frac{dv}{dt} = kv.$$

Step 2: Formulating the Differential Equation

Rearrange the equation to separate the variables:

$$\frac{dv}{v} = k \, dt.$$

This is the desired first-order differential equation.

Quick Tip

When acceleration is proportional to velocity, the differential equation is:

$$\frac{dv}{dt} = kv.$$

SECTION - B

3. Construct the truth table for the statement pattern:

$$[(p \to q) \wedge q] \to p$$

Solution:

Step 1: Understanding Logical Operators

- $p \to q$ means "If p, then q". - $\wedge$ represents logical AND. - $\to$ represents logical implication.

Step 2: Constructing the Truth Table

p	q	$p \to q$	$(p \to q) \wedge q$	$[(p \to q) \wedge q] \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	F

Quick Tip

The truth table helps determine whether a logical statement is always true (tautology) or false under some conditions (contingency).

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4. Check whether the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is invertible or not.

Solution:

Step 1: Calculate the Determinant

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is given by:

$$\det(A) = (ad - bc).$$

Substituting the values from matrix A:

$$\det(A) = (\cos\theta \cdot \cos\theta - \sin\theta \cdot (-\sin\theta))$$

$$= \cos^2\theta + \sin^2\theta.$$

Step 2: Check the Invertibility Condition

Since $\cos^2\theta + \sin^2\theta = 1$, we have:

$$\det(A) = 1 \neq 0.$$

As the determinant is nonzero, the matrix is invertible.

Quick Tip

A matrix is invertible if and only if its determinant is nonzero.

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5. In $\triangle ABC$, if $a = 18$, $b = 24$, and $c = 30$, then find the value of

$$\sin\left(\frac{A}{2}\right).$$

Solution:

Step 1: Calculate the Semi-Perimeter

The semi-perimeter s of the triangle is computed as:

$$s = \frac{a+b+c}{2} = \frac{18+24+30}{2} = 36.$$

Step 2: Apply the Half-Angle Formula for Sine

The half-angle formula for sine is:

$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Step 3: Substituting the Given Values

Substitute the known values into the formula:

$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(36-24)(36-30)}{24 \times 30}}.$$

$$= \sqrt{\frac{12 \times 6}{720}} = \sqrt{\frac{72}{720}} = \sqrt{\frac{1}{10}}.$$

$$= \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}.$$

Quick Tip

To find the sine of half an angle in triangle problems, use the formula:

$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

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6. Find k, if the sum of the slopes of the lines represented by

$$x^2 + kxy - 3y^2 = 0$$

is twice their product.

Solution:

Step 1: Determine the Slopes of the Lines

The general form of a homogeneous second-degree equation:

$$Ax^2 + 2Hxy + By^2 = 0$$

represents two straight lines. The slopes of these lines are given by:

$$m_1, m_2 = \frac{-(H \pm \sqrt{H^2 - AB})}{B}.$$

By comparing this with the equation $x^2 + kxy - 3y^2 = 0$, we identify the coefficients:

$$A = 1, \quad H = \frac{k}{2}, \quad B = -3.$$

Step 2: Apply the Given Condition

The sum of the slopes is:

$$m_1 + m_2 = -\frac{2H}{B} = -\frac{k}{-3} = \frac{k}{3}.$$

The product of the slopes is:

$$m_1 m_2 = \frac{A}{B} = \frac{1}{-3} = -\frac{1}{3}.$$

Using the condition $m_1 + m_2 = 2(m_1 m_2)$, we substitute the known values:

$$\frac{k}{3} = 2 \times \left(-\frac{1}{3}\right).$$

$$\frac{k}{3} = -\frac{2}{3}.$$

Step 3: Solve for k

$$k = -2.$$

Quick Tip

For homogeneous second-degree equations, the slopes satisfy:

$$m_1 + m_2 = -\frac{2H}{B}, \quad m_1 m_2 = \frac{A}{B}.$$

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7. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the position vectors of the points A, B, C respectively and

$$5\mathbf{a} - 3\mathbf{b} - 2\mathbf{c} = \mathbf{0},$$

then find the ratio in which the point C divides the line segment BA externally.

Solution:

Step 1: Define the Position Vector of C

Let C divide BA externally in the ratio $5:3$. The position vector of C can be expressed as:

$$\mathbf{c} = \frac{5\mathbf{a} - 3\mathbf{b}}{5-3}.$$

Step 2: Use the Given Equation

Rearranging the given equation $5\mathbf{a} - 3\mathbf{b} - 2\mathbf{c} = \mathbf{0}$, we obtain:

$$2\mathbf{c} = 5\mathbf{a} - 3\mathbf{b}.$$

Step 3: Substitute the Expression for $\mathbf{c}$

Substitute the previously derived expression for $\mathbf{c}$:

$$2 \times \frac{5\mathbf{a} - 3\mathbf{b}}{5-3} = 5\mathbf{a} - 3\mathbf{b}.$$

Simplifying the expression gives:

$$\mathbf{c} = \frac{5\mathbf{a} - 3\mathbf{b}}{2}.$$

Thus, point C divides the line segment BA externally in the ratio $5:3$.

Quick Tip

For a point dividing a line segment BA externally in the ratio $m : n$, the position vector is given by:

$$\mathbf{c} = \frac{m\mathbf{a} - n\mathbf{b}}{m - n}.$$

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8. Find the vector equation of the line passing through the point having position vector

$$4\hat{i} - \hat{j} + 2\hat{k}$$

and parallel to the vector

$$-2\hat{i} - \hat{j} + \hat{k}.$$

Solution:

Step 1: Vector Equation of a Line

The vector equation of a line passing through $\mathbf{r_0}$ and parallel to $\mathbf{d}$ is:

$$\mathbf{r} = \mathbf{r_0} + \lambda\mathbf{d}.$$

Step 2: Substitute Given Vectors

$$\mathbf{r_0} = 4\hat{i} - \hat{j} + 2\hat{k}, \quad \mathbf{d} = -2\hat{i} - \hat{j} + \hat{k}.$$

Step 3: Write the Equation

$$\mathbf{r} = (4\hat{i} - \hat{j} + 2\hat{k}) + \lambda(-2\hat{i} - \hat{j} + \hat{k}).$$

Expanding:

$$\mathbf{r} = (4 - 2\lambda)\hat{i} + (-1 - \lambda)\hat{j} + (2 + \lambda)\hat{k}.$$

Quick Tip

The vector equation of a line through $\mathbf{r_0}$ parallel to $\mathbf{d}$ is:

$$\mathbf{r} = \mathbf{r_0} + \lambda\mathbf{d}.$$

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9. Find $\frac{dy}{dx}$, if $y = (\log x)^x$.

Solution:

Step 1: Take the Logarithm

Taking the natural logarithm on both sides:

$$\ln y = x \ln(\log x).$$

Step 2: Differentiate Both Sides

Differentiating using implicit differentiation:

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x \ln(\log x)).$$

Using the product rule:

$$\frac{d}{dx}(x \ln(\log x)) = x \times \frac{1}{\log x} \times \frac{1}{x} + \ln(\log x).$$

$$= \frac{1}{\log x} + \ln(\log x).$$

Step 3: Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = y\left(\frac{1}{\log x} + \ln(\log x)\right).$$

Substituting $y = (\log x)^x$:

$$\frac{dy}{dx} = (\log x)^x\left(\frac{1}{\log x} + \ln(\log x)\right).$$

Quick Tip

For differentiation of functions in the form $y = f(x)^{g(x)}$, take the logarithm first:

$$\ln y = g(x) \ln f(x).$$

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10. Evaluate:

$$\int \log x \, dx.$$

Solution:

Step 1: Use Integration by Parts

Using the formula:

$$\int u \, dv = uv - \int v \, du,$$

let:

$$u = \log x, \quad dv = dx.$$

Step 2: Compute du and v

$$du = \frac{1}{x}dx, \quad v = x.$$

Step 3: Apply Integration by Parts

$$\int \log x \, dx = x \log x - \int x \times \frac{1}{x} dx.$$

$$= x \log x - \int dx.$$

$$= x \log x - x + C.$$

Quick Tip

Use integration by parts where $u = \log x$ and $dv = dx$ for $\int \log x \, dx$.

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11. Evaluate:

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx.$$

Solution:

Step 1: Use the Power Reduction Formula

$$\cos^2 x = \frac{1 + \cos 2x}{2}.$$

Step 2: Substitute into the Integral

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2x}{2} dx.$$

Splitting:

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} dx + \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos 2x \, dx.$$

Step 3: Evaluate Each Integral

$$\frac{1}{2}[x]_0^{\frac{\pi}{2}} + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}.$$

$$= \frac{1}{2} \times \frac{\pi}{2} + \frac{1}{2} \times 0.$$

$$= \frac{\pi}{4}.$$

Quick Tip

Use the identity $\cos^2 x = \frac{1+\cos 2x}{2}$ for integrals involving even powers of trigonometric functions.

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12. Find the area of the region bounded by the curve $y = x^2$, and the lines $x = 1$, $x = 2$, and $y = 0$.

Solution:

Step 1: Apply Definite Integration for Area

The area under the curve is given by the integral:

$$A = \int_1^2 x^2\,dx.$$

Step 2: Compute the Integral

Evaluating the integral, we get:

$$A = \left[\frac{x^3}{3}\right]_1^2.$$

$$= \frac{2^3}{3} - \frac{1^3}{3}.$$

$$= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

Quick Tip

To calculate areas under curves, use the formula $\int_a^b f(x)\,dx$, where $f(x)$ is the function representing the curve.

13. Solve:

$$1 + \frac{dy}{dx} = \text{cosec}(x + y); \quad \text{put } x + y = u.$$

Solution:

Step 1: Substituting $u = x + y$

$$du = dx + dy \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}.$$

Step 2: Rewrite the Equation

$$\frac{du}{dx} = \text{cosec}\, u.$$

Step 3: Solve the Differential Equation

$$\int \sin u \, du = \int dx.$$

$$-\cos u = x + C.$$

$$-\cos(x + y) = x + C.$$

Quick Tip

Use substitution to simplify differential equations of the form $f(x + y)$.

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14. If two coins are tossed simultaneously, write the probability distribution of the number of heads.

Solution:

Step 1: Determine Sample Space

When two fair coins are tossed, the possible outcomes are:

$$S = \{HH, HT, TH, TT\}.$$

Step 2: Count the Number of Heads

Define X as the number of heads in each outcome:

Outcome	Number of Heads(X)
HH	2
HT	1
TH	1
TT	0

Step 3: Find the Probability Distribution

The probability of each value of X is calculated as follows:

$$P(X = x) = \frac{\text{Number of favorable cases}}{\text{Total outcomes}}.$$

X	$P(X)$
0	$\frac{1}{4}$
1	$\frac{2}{4} = \frac{1}{2}$
2	$\frac{1}{4}$

Thus, the probability distribution is:

$$P(X = 0) = \frac{1}{4}, \quad P(X = 1) = \frac{1}{2}, \quad P(X = 2) = \frac{1}{4}.$$

Quick Tip

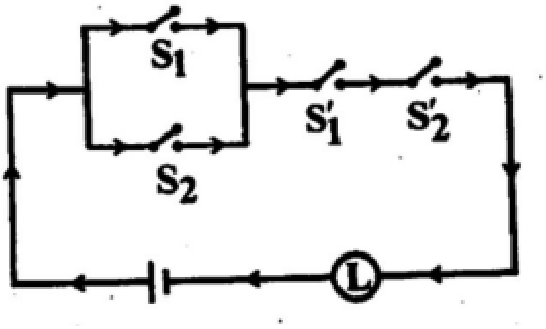
For a binomial experiment with n trials and probability p, the probability of X successes follows:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

SECTION - C

Attempt any EIGHT of the following questions:

15. Express the following switching circuit in symbolic form of logic. Construct the switching table.



Solution:

Step 1: Identifying the Switches and Logic Representation

The given circuit consists of switches S_1, S_2, S_1', S_2', arranged in both series and parallel configurations.

- Switches S_1 and S_1' are controlled by input A. - Switches S_2 and S_2' are controlled by input B.

Step 2: Deriving the Boolean Expression

- The parallel arrangement corresponds to the logical OR operation. - The series arrangement corresponds to the logical AND operation.

Hence, the Boolean expression for the circuit is:

$$Y = (S_1 + S_1') \cdot (S_2 + S_2').$$

Step 3: Constructing the Truth Table

S_1	S_2	$Output(Y)$
0	0	0
0	1	1
1	0	1
1	1	1

Quick Tip

In switching circuits: - Series connections represent the AND operation. - Parallel connections represent the OR operation.

16. Prove that:

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}.$$

Solution:

Step 1: Use the Identity

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\frac{a+b}{1-ab}\right), \quad \text{if } ab < 1.$$

Step 2: Apply Given Values

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right).$$

$$= \tan^{-1}\left(\frac{\frac{3+2}{6}}{1-\frac{1}{6}}\right).$$

$$= \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}(1).$$

$$= \frac{\pi}{4}.$$

Quick Tip

Use the identity:

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\frac{a+b}{1-ab}\right).$$

for sum of inverse tangent functions.

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17. In $\triangle ABC$, prove that:

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}.$$

Solution:

Step 1: Apply the Cosine Rule

The cosine of the angles in a triangle can be expressed as:

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad \cos B = \frac{a^2+c^2-b^2}{2ac}, \quad \cos C = \frac{a^2+b^2-c^2}{2ab}.$$

Step 2: Set Up the Summation

The desired summation is:

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}.$$

Substituting the cosine formulas:

$$= \frac{1}{a} \cdot \frac{b^2 + c^2 - a^2}{2bc} + \frac{1}{b} \cdot \frac{a^2 + c^2 - b^2}{2ac} + \frac{1}{c} \cdot \frac{a^2 + b^2 - c^2}{2ab}.$$

Step 3: Simplify the Expression

$$= \frac{b^2 + c^2 - a^2}{2abc} + \frac{a^2 + c^2 - b^2}{2abc} + \frac{a^2 + b^2 - c^2}{2abc}.$$

Now, combine the terms:

$$= \frac{(b^2 + c^2 - a^2) + (a^2 + c^2 - b^2) + (a^2 + b^2 - c^2)}{2abc}.$$

Simplifying further:

$$= \frac{a^2 + b^2 + c^2}{2abc}.$$

Quick Tip

To derive summation identities in triangles, use the cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

---

18. Prove by vector method, the angle subtended on a semicircle is a right angle.

Solution:

Step 1: Define the Points

Let the semicircle be centered at O with radius r, and points $A(-r, 0)$, $B(r, 0)$, and $P(x, y)$ on the semicircle.

Step 2: Write the Vectors

Vectors from A and B to P:

$$\mathbf{AP} = (x + r, y), \quad \mathbf{BP} = (x - r, y).$$

Step 3: Find the Dot Product

$$\mathbf{AP} \cdot \mathbf{BP} = (x + r, y) \cdot (x - r, y).$$

$$= (x + r)(x - r) + y^2.$$

$$= x^2 - r^2 + y^2.$$

Using the equation of the semicircle:

$$x^2 + y^2 = r^2.$$

$$\mathbf{AP} \cdot \mathbf{BP} = r^2 - r^2 = 0.$$

Since dot product is zero, AP is perpendicular to BP, proving a right angle.

Quick Tip

If $\mathbf{A} \cdot \mathbf{B} = 0$, then the vectors are perpendicular.

---

19. Find the shortest distance between the lines

$$\mathbf{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$$

and

$$\mathbf{r} = (\hat{i} - \hat{j} - 2\hat{k}) + \mu(\hat{i} + \hat{j} - 5\hat{k}).$$

Solution:

We are given two skew lines, whose equations are:

$$\mathbf{r}_1 = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k}),$$

$$\mathbf{r}_2 = (\hat{i} - \hat{j} - 2\hat{k}) + \mu(\hat{i} + \hat{j} - 5\hat{k}).$$

Step 1: Identify points and direction vectors.

The lines pass through the points:

$$\mathbf{a}_1 = (4\hat{i} - \hat{j}), \quad \mathbf{a}_2 = (\hat{i} - \hat{j} - 2\hat{k}),$$

and are parallel to the direction vectors:

$$\mathbf{b}_1 = \hat{i} + 2\hat{j} - 3\hat{k}, \quad \mathbf{b}_2 = \hat{i} + \hat{j} - 5\hat{k}.$$

Step 2: Find the vector between the points.

The vector between the two points $\mathbf{a}_1$ and $\mathbf{a}_2$ is:

$$\mathbf{a}_2 - \mathbf{a}_1 = (\hat{i} - \hat{j} - 2\hat{k}) - (4\hat{i} - \hat{j}) = -3\hat{i} + 2\hat{k}.$$

Step 3: Calculate the cross product of the direction vectors.

Now, calculate the cross product of the direction vectors $\mathbf{b}_1$ and $\mathbf{b}_2$:

$$\mathbf{b}_1 \times \mathbf{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 1 & 1 & -5 \end{vmatrix} = 2\hat{k}.$$

Step 4: Apply the shortest distance formula.

The formula for the shortest distance d between two skew lines is given by:

$$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

Substituting the values:

$$d = \frac{|(-3\hat{i} + 2\hat{k}) \cdot 2\hat{k}|}{|2\hat{k}|} = \frac{|-6+4|}{2\sqrt{3}} = \frac{2}{2\sqrt{3}}.$$

Final Answer: Thus, the shortest distance between the two lines is:

$$d = \frac{1}{\sqrt{3}} \text{ units}.$$

Quick Tip

For skew lines, the shortest distance is given by:

$$d = \frac{|(\mathbf{B} - \mathbf{A}) \cdot (\mathbf{d_1} \times \mathbf{d_2})|}{|\mathbf{d_1} \times \mathbf{d_2}|}.$$

---

20. Find the angle between the line

$$\mathbf{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} + \hat{j} + \hat{k})$$

and the plane

$$\mathbf{r} \cdot (2\hat{i} + \hat{j} + \hat{k}) = 8.$$

Solution:

Step 1: Identify the Direction and Normal Vectors

- The direction vector of the line is $\mathbf{d} = (1, 1, 1)$.

- The normal vector of the plane is $\mathbf{n} = (2, 1, 1)$.

Step 2: Apply the Formula for the Angle Between Line and Plane

The angle θ between the line and the plane can be calculated using the angle between the direction vector of the line and the normal vector of the plane:

$$\sin\theta = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}.$$

$$= \frac{|(1, 1, 1) \cdot (2, 1, 1)|}{\sqrt{1^2 + 1^2 + 1^2} \times \sqrt{2^2 + 1^2 + 1^2}}.$$

$$= \frac{|2 + 1 + 1|}{\sqrt{3} \times \sqrt{6}}.$$

$$= \frac{4}{\sqrt{18}} = \frac{2\sqrt{2}}{3}.$$

$$\theta = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right).$$

Quick Tip

To find the angle between a line and a plane, use the formula:

$$\sin\theta = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}.$$

---

21. If $y = \sin^{-1} x$, then show that:

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0.$$

Solution:

Step 1: First Derivative

Since $y = \sin^{-1} x$, the first derivative of y with respect to x is:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

Step 2: Second Derivative

To find the second derivative, differentiate $\frac{dy}{dx}$ with respect to x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{\sqrt{1-x^2}}\right).$$

Using the chain rule:

$$\frac{d^2y}{dx^2} = -\frac{x}{(1-x^2)^{3/2}}.$$

Step 3: Verify the Given Identity

We need to show that:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0.$$

Substitute the expressions for $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$:

$$(1-x^2)\left(-\frac{x}{(1-x^2)^{3/2}}\right) - x\left(\frac{1}{\sqrt{1-x^2}}\right).$$

Simplify the first term:

$$= -\frac{x(1-x^2)}{(1-x^2)^{3/2}}.$$

Simplify the second term:

$$= -\frac{x}{\sqrt{1-x^2}}.$$

Now, combine the terms:

$$-\frac{x(1-x^2)}{(1-x^2)^{3/2}} - \frac{x}{\sqrt{1-x^2}}.$$

Factor out $-\frac{x}{\sqrt{1-x^2}}$:

$$= -\frac{x}{\sqrt{1-x^2}}\left(\frac{(1-x^2)}{(1-x^2)^{3/2}} + 1\right).$$

Simplify the expression inside the parentheses:

$$= -\frac{x}{\sqrt{1-x^2}}\left(\frac{1}{(1-x^2)^{1/2}} + 1\right).$$

This simplifies to:

$$= -\frac{x}{\sqrt{1-x^2}} \times 0 = 0.$$

Thus, we have shown that:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0.$$

Quick Tip

For inverse trigonometric functions, differentiate stepwise and use algebraic simplifications.

---

22. Find the approximate value of $\tan^{-1}(1.002)$.

Given: $\pi = 3.1416$.

Solution:

Step 1: Use Approximation Formula

For small x, $\tan^{-1}(1+x) \approx \frac{\pi}{4} + x$.

Step 2: Apply Given Values

$$\tan^{-1}(1.002) \approx \frac{\pi}{4} + 0.002.$$

$$= \frac{3.1416}{4} + 0.002.$$

$$= 0.7854 + 0.002 = 0.7874.$$

Quick Tip

For small x, use the approximation:

$$\tan^{-1}(1+x) \approx \frac{\pi}{4} + x.$$

---

23. Prove that:

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right) + C.$$

Solution:

Step 1: Use Standard Integral Formula

The standard formula for the given integral is:

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C.$$

Step 2: Proof via Partial Fractions

Rewriting:

$$\frac{1}{a^2 - x^2} = \frac{1}{(a-x)(a+x)}.$$

Using partial fractions:

$$\frac{1}{(a-x)(a+x)} = \frac{A}{a-x} + \frac{B}{a+x}.$$

Solving for A and B, we obtain:

$$A = \frac{1}{2a}, \quad B = -\frac{1}{2a}.$$

Step 3: Integrate

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \int \frac{dx}{a+x} - \frac{1}{2a} \int \frac{dx}{a-x}.$$

$$= \frac{1}{2a} \log |a+x| - \frac{1}{2a} \log |a-x|.$$

$$= \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C.$$

Quick Tip

Use the identity:

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C.$$

for integrals of this form.

---

24. Solve the differential equation:

$$x\frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0.$$

Solution:

Step 1: Use Substitution

Let $v = \frac{y}{x}$, so that:

$$y = vx \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Step 2: Rewrite the Equation

$$x(v + x\frac{dv}{dx}) - vx + x\sin v = 0.$$

$$xv + x^2\frac{dv}{dx} - vx + x\sin v = 0.$$

$$x^2\frac{dv}{dx} = -x\sin v.$$

Step 3: Solve the Separable Equation

$$\frac{dv}{\sin v} = -\frac{dx}{x}.$$

Step 4: Integrate Both Sides

$$\int \frac{dv}{\sin v} = -\int \frac{dx}{x}.$$

$$\log|\csc v - \cot v| = -\log|x| + C.$$

Step 5: Substitute Back $v = \frac{y}{x}$

$$\log|\csc(y/x) - \cot(y/x)| = -\log|x| + C.$$

Taking exponentials on both sides:

$$|\csc(y/x) - \cot(y/x)| = \frac{C}{x}.$$

Quick Tip

Use the substitution $v = \frac{y}{x}$ in equations of the form:

$$x\frac{dy}{dx} - y = f(y/x).$$

---

25. Find k, if the probability density function is given by:

$$f(x) = kx^2(1-x), \quad \text{for } 0 < x < 1,$$

$$= 0, \quad \text{otherwise.}$$

Solution:

Step 1: Use the Probability Density Function Property

The total probability must be 1:

$$\int_0^1 f(x)dx = 1.$$

Step 2: Compute the Integral

$$\int_0^1 kx^2(1-x)dx = 1.$$

Expanding:

$$\int_0^1 k(x^2 - x^3)dx = 1.$$

Step 3: Evaluate the Integrals

$$k\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 1.$$

$$k\left(\frac{1}{3} - \frac{1}{4}\right) = 1.$$

$$k\left(\frac{4-3}{12}\right) = 1.$$

$$k \times \frac{1}{12} = 1.$$

$$k = 12.$$

Quick Tip

The total probability for a probability density function (p.d.f.) must always satisfy:

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

---

26. A die is thrown 6 times. If "getting an odd number" is a success, find the probability of 5 successes.

Solution:

Step 1: Define the Probability of Success

A die has numbers {1,2,3,4,5,6}. The odd numbers are {1,3,5}, so the probability of success (getting an odd number) is:

$$p = \frac{3}{6} = \frac{1}{2}.$$

Step 2: Use the Binomial Probability Formula

The probability of exactly k successes in n trials is:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Substituting $n = 6$, $k = 5$, and $p = \frac{1}{2}$:

$$P(X = 5) = \binom{6}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^1.$$

Step 3: Compute the Probability

$$\binom{6}{5} = \frac{6!}{5!(6-5)!} = 6.$$

$$P(X = 5) = 6 \times \left(\frac{1}{2}\right)^6.$$

$$= 6 \times \frac{1}{64}.$$

$$= \frac{6}{64} = \frac{3}{32}.$$

Quick Tip

For binomial probability:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

SECTION - D

Attempt any FIVE of the following questions:

27. Solve the following system of equations by the method of reduction:

$$x + y + z = 6, \quad y + 3z = 11, \quad x + z = 2y.$$

Solution:

Step 1: Convert to Standard Form

Rewriting the third equation:

$$x + z = 2y \quad \Rightarrow \quad x - 2y + z = 0.$$

Thus, the system of equations becomes:

$$x + y + z = 6, \quad y + 3z = 11, \quad x - 2y + z = 0.$$

Step 2: Solve for the Variables

- From the equation $y + 3z = 11$, we get:

$$y = 11 - 3z.$$

- Substituting this expression for y into the first equation:

$$x + (11 - 3z) + z = 6.$$

$$x + 11 - 2z = 6 \quad \Rightarrow \quad x = 2z - 5.$$

Step 3: Solve for z

Now, substituting $x = 2z - 5$ and $y = 11 - 3z$ into the third equation $x - 2y + z = 0$:

$$(2z - 5) - 2(11 - 3z) + z = 0.$$

$$2z - 5 - 22 + 6z + z = 0.$$

$$9z - 27 = 0 \quad \Rightarrow \quad z = 3.$$

Step 4: Calculate x and y

Substitute $z = 3$ into the equations for y and x:

$$y = 11 - 3(3) = 2, \quad x = 2(3) - 5 = 1.$$

Final Answer: $x = 1, y = 2, z = 3$.

> **Quick Tip**
>
> When solving linear systems, using substitution and elimination makes the process straightforward and manageable.

---

28. Prove that the acute angle θ between the lines represented by the equation

$$ax^2 + 2hxy + by^2 = 0$$

is

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|.$$

Hence find the condition that the lines are coincident.

Solution:

Step 1: Find Slopes of the Lines

The given equation represents two straight lines passing through the origin. The slopes are given by:

$$m_1, m_2 = \frac{-(h \pm \sqrt{h^2 - ab})}{b}.$$

Step 2: Use the Angle Formula

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

Substituting values:

$$\tan\theta = \left|\frac{\frac{-h+\sqrt{h^2-ab}}{b} - \frac{-h-\sqrt{h^2-ab}}{b}}{1 + \left(\frac{-h+\sqrt{h^2-ab}}{b}\right)\left(\frac{-h-\sqrt{h^2-ab}}{b}\right)}\right|.$$

$$= \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|.$$

Step 3: Condition for Coincidence

For coincident lines, $\theta = 0$, which means:

$$h^2 - ab = 0 \quad \Rightarrow \quad h^2 = ab.$$

Quick Tip

For conic sections, coincident lines satisfy $h^2 = ab$.

---

29. Find the volume of the parallelepiped whose vertices are $A(3, 2, -1), B(-2, 2, -3), C(3, 5, -2)$ **and** $D(-2, 5, 4)$.

Solution:

Step 1: Find Three Vectors

$$\mathbf{AB} = (-2-3, 2-2, -3+1) = (-5, 0, -2).$$

$$\mathbf{AC} = (3-3, 5-2, -2+1) = (0, 3, -1).$$

$$\mathbf{AD} = (-2-3, 5-2, 4+1) = (-5, 3, 5).$$

Step 2: Compute the Volume

$$V = |\mathbf{AB} \cdot (\mathbf{AC} \times \mathbf{AD})|.$$

$$\mathbf{AC} \times \mathbf{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & -1 \\ -5 & 3 & 5 \end{vmatrix}.$$

$$= (15+3)\hat{i} - (0+5)\hat{j} + (0+15)\hat{k}.$$

$$= (18, -5, 15).$$

$$V = |(-5, 0, -2) \cdot (18, -5, 15)|.$$

$$= |(-5 \times 18) + (0 \times -5) + (-2 \times 15)|.$$

$$= | -90 - 30| = |120|.$$

Final Answer: $V = 120$.

Quick Tip

The volume of a parallelepiped is given by:

$$V = |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})| .$$

---

30. Solve the following L.P.P. by graphical method: Maximize:

$$z = 10x + 25y.$$

Subject to:

$$0 \leq x \leq 3, \quad 0 \leq y \leq 3, \quad x + y \leq 5.$$

Solution:

The given constraints are:

1. $0 \leq x \leq 3$ (This defines the range for x.) 2. $0 \leq y \leq 3$ (This defines the range for y.) 3. $x + y \leq 5$ (This defines a line that we will use to find the feasible region.)

We will now graph the constraints.

1. The constraint $x \geq 0$ and $x \leq 3$ limits x between 0 and 3.
2. The constraint $y \geq 0$ and $y \leq 3$ limits y between 0 and 3.
3. The constraint $x + y \leq 5$ is a line. We can rewrite it as:

   $$y = 5 - x$$

   and plot this line.

Now, we need to find the feasible region by plotting these constraints. The feasible region is the area where all the constraints are satisfied simultaneously.

Next, we plot the graph of the constraints and find the vertices of the feasible region.

[scale=1.5] [very thin,color=gray] (-0.2,-0.2) grid (3.5,3.5); [-¿] (-0.5,0) – (3.5,0) node[right] x; [-¿] (0,-0.5) – (0,3.5) node[above] y;

[domain=0:3] plot (, 5 -) node[right] $x + y = 5$;

[black] (0,3) circle (2pt); [black] (3,0) circle (2pt); [black] (2,3) circle (2pt); [black] (0,0) circle (2pt);

at (-0.2,3) [left] $(0, 3)$; at (3,0.2) [below] $(3, 0)$; at (2,3.2) [above] $(2, 3)$; at (-0.2,-0.2) [below left] $(0, 0)$;

The feasible region is the triangle formed by the points $(0, 0)$, $(0, 3)$, and $(2, 3)$.

Step 1: Evaluate the Objective Function at the Vertices

We now evaluate the objective function $z = 10x + 25y$ at the vertices of the feasible region:

- At $(0, 0)$:

$$z = 10(0) + 25(0) = 0$$

- At $(0, 3)$:

$$z = 10(0) + 25(3) = 75$$

- At $(2, 3)$:

$$z = 10(2) + 25(3) = 20 + 75 = 95$$

Step 2: Determine the Maximum Value

The maximum value of z occurs at $(2, 3)$, and the maximum value of z is 95.

Quick Tip

For L.P.P, the optimal solution is found by evaluating the objective function at the vertices of the feasible region. The maximum (or minimum) value occurs at one of the corner points of the feasible region.

---

31. If $x = f(t)$ and $y = g(t)$ are differentiable functions of t, so that y is a function of x and $\frac{dx}{dt} \neq 0$, then prove that:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Hence find $\frac{dy}{dx}$, if $x = at^2$, $y = 2at$.

Solution:

Step 1: Proof of Chain Rule

Since $y = g(t)$ and $x = f(t)$, we differentiate both functions with respect to t:

$$\frac{dx}{dt} = f'(t), \quad \frac{dy}{dt} = g'(t).$$

Using the chain rule for derivatives:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Step 2: Compute $\frac{dy}{dx}$ for Given Functions

Given:

$$x = at^2, \quad y = 2at.$$

Differentiate with respect to t:

$$\frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a.$$

Using the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}.$$

Quick Tip

For parametric differentiation:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

---

32. A box with a square base is to have an open top. The surface area of the box is 147 cm^2. What should its dimensions be in order that the volume is largest?

Solution:

Step 1: Define the Variables

Let: - x be the side length of the square base, - h be the height of the box.

The total surface area of the box is expressed as:

$$x^2 + 4(xh) = 147.$$

Step 2: Solve for h in Terms of x

From the surface area equation:

$$h = \frac{147 - x^2}{4x}.$$

Step 3: Maximize the Volume

The volume V of the box is given by:

$$V = x^2 h = x^2 \times \frac{147 - x^2}{4x}.$$

$$V = \frac{x(147 - x^2)}{4}.$$

Differentiate the volume function with respect to x:

$$\frac{dV}{dx} = \frac{147 - 3x^2}{4}.$$

Set the derivative equal to zero to find the critical points:

$$147 - 3x^2 = 0 \Rightarrow x^2 = 49 \Rightarrow x = 7.$$

Step 4: Calculate h

Substitute $x = 7$ into the equation for h:

$$h = \frac{147 - 49}{4(7)} = \frac{98}{28} = 3.5.$$

Final Answer: $x = 7$ cm, $h = 3.5$ cm.

Quick Tip

To maximize volume for a given surface area, express height as a function of the base side length and differentiate the volume function.

---

33. Evaluate:

$$I = \int \frac{5e^x}{(e^x + 1)(e^{2x} + 9)}\, dx.$$

Solution:

Step 1: Use Substitution

Let $t = e^x$, so that $dt = e^x dx$. Hence, we have:

$$dx = \frac{dt}{t}.$$

Now, substitute into the integral:

$$I = \int \frac{5e^x}{(e^x + 1)(e^{2x} + 9)}\, dx = \int \frac{5t}{(t + 1)(t^2 + 9)} \cdot \frac{dt}{t}.$$

Simplifying, we get:

$$I = \int \frac{5}{(t+1)(t^2+9)}\, dt.$$

Step 2: Decompose Using Partial Fractions

We need to decompose the integrand using partial fractions:

$$\frac{5}{(t+1)(t^2+9)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+9}.$$

To solve for A, B, and C, multiply both sides by $(t+1)(t^2+9)$ to get:

$$5 = A(t^2+9) + (Bt+C)(t+1).$$

Expanding the right-hand side:

$$5 = A(t^2+9) + Bt(t+1) + C(t+1),$$

$$5 = A(t^2+9) + Bt^2 + Bt + Ct + C.$$

Now, group the terms based on powers of t:

$$5 = (A+B)t^2 + (B+C)t + (9A+C).$$

Equate the coefficients of like terms: - For t^2: $A+B=0$, - For t: $B+C=0$, - For the constant: $9A+C=5$.

Solving this system of equations: 1. $A+B=0$ gives $B=-A$, 2. $B+C=0$ gives $C=-B=A$, 3. Substituting $B=-A$ and $C=A$ into $9A+C=5$:

$$9A + A = 5 \quad \Rightarrow \quad 10A = 5 \quad \Rightarrow \quad A = \frac{1}{2}.$$

Hence, $B = -\frac{1}{2}$ and $C = \frac{1}{2}$.

Step 3: Substitute the Values of A, B, and C

Substituting these values into the partial fraction decomposition:

$$\frac{5}{(t+1)(t^2+9)} = \frac{1/2}{t+1} + \frac{-\frac{1}{2}t + \frac{1}{2}}{t^2+9}.$$

Thus, we can rewrite the integral as:

$$I = \int \left(\frac{1/2}{t+1} + \frac{-\frac{1}{2}t + \frac{1}{2}}{t^2+9} \right) dt.$$

Step 4: Break the Integral Into Separate Terms

Now, split the integral into two parts:

$$I = \frac{1}{2}\int \frac{dt}{t+1} - \frac{1}{2}\int \frac{t\,dt}{t^2+9} + \frac{1}{2}\int \frac{dt}{t^2+9}.$$

Step 5: Evaluate Each Integral

1. $\int \frac{dt}{t+1} = \ln|t+1|$, 2. $\int \frac{t\,dt}{t^2+9} = \frac{1}{2}\ln|t^2+9|$ (standard result), 3. $\int \frac{dt}{t^2+9} = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)$ (standard result).

So, the total integral becomes:

$$I = \frac{1}{2}\ln|t+1| - \frac{1}{4}\ln|t^2+9| + \frac{1}{6}\tan^{-1}\left(\frac{t}{3}\right) + C.$$

Step 6: Substitute $t = e^x$ Back Into the Solution

Finally, substitute $t = e^x$ back into the result:

$$I = \frac{1}{2}\ln|e^x+1| - \frac{1}{4}\ln|e^{2x}+9| + \frac{1}{6}\tan^{-1}\left(\frac{e^x}{3}\right) + C.$$

Quick Tip

For integrals involving rational functions with exponential terms, a substitution like $t = e^x$ can simplify the expression. Partial fractions can then be used to decompose the integrand.

---

34. Prove that:

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx.$$

Hence show that:

$$\int_0^{\pi} \sin x\,dx = 2\int_0^{\pi/2} \sin x\,dx.$$

Solution:

Step 1: Proof of Integral Identity

Using the substitution $u = 2a - x$, we get:

$$du = -dx.$$

Changing limits:

$$\int_0^a f(2a-x)dx = \int_{2a}^a f(u)(-du) = \int_a^{2a} f(u)du.$$

Since:

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx,$$

we conclude:

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx.$$

Step 2: Apply to Sine Function

For $f(x) = \sin x$, let $a = \frac{\pi}{2}$, so $2a = \pi$:

$$\int_0^{\pi} \sin x dx = \int_0^{\pi/2} \sin x dx + \int_0^{\pi/2} \sin(\pi - x)dx.$$

Since $\sin(\pi - x) = \sin x$:

$$\int_0^{\pi} \sin x dx = 2\int_0^{\pi/2} \sin x dx.$$

Final Verification:

$$\int_0^{\pi} \sin x dx = [-\cos x]_0^{\pi} = -\cos\pi + \cos 0 = 1 + 1 = 2.$$

$$\int_0^{\pi/2} \sin x dx = [-\cos x]_0^{\pi/2} = -\cos(\pi/2) + \cos 0 = 0 + 1 = 1.$$

$$2 \times 1 = 2.$$

Quick Tip

For integrals of symmetric functions, use the property:

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx.$$
