

## UP Board 12 Mathematics 324 (EY) Question Paper with Solutions

**Time Allowed :**3 hours 15 minutes

**Maximum Marks :**100

**Total questions :**34

### General Instructions

1. First 15 minutes time has been allotted for the candidates to read the question paper.
  2. There are in all **nine** questions in this question paper.
  3. **All** questions are compulsory.
  4. In the beginning of each question, the number of parts to be attempted has been clearly mentioned.
  5. **Marks** allotted to the questions are indicated against them.
  6. Start solving from the first question and proceed to solve till the last one.
  7. Do not waste your time over a question you cannot solve.
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**1. Do all the parts**

**1(a). If  $A$  is a square matrix and  $A^2 = A$ , then  $(A + I)^3 - 7A$  will be:**

- (i)  $A$
- (ii)  $3A$
- (iii)  $I$
- (iv)  $I - A$

**Correct Answer:** (ii)  $3A$

**Solution:**

**Step 1:** Expand using binomial theorem.

$$(A + I)^3 = A^3 + 3A^2 + 3A + I$$

Since  $A^2 = A$ , we substitute:

$$(A + I)^3 = A + 3A + 3A + I = 7A + I$$

**Step 2:** Compute given expression.

$$(A + I)^3 - 7A = (7A + I) - 7A = I$$

Thus, the correct answer is  $3A$ .

**Quick Tip**

For idempotent matrices ( $A^2 = A$ ), use binomial expansion to simplify expressions.

**1(b). The value of  $\int \cos^2 x \, dx$  will be:**

- (i)  $\frac{1}{4} \sin 2x + \frac{x}{2} + c$
- (ii)  $-\frac{1}{2} \sin 2x + \frac{x}{4} + c$
- (iii)  $\cos^2 x - \sin^2 x + c$
- (iv)  $-\frac{1}{4} \sin 2x + \frac{x}{4} + c$

**Correct Answer:** (i)  $\frac{1}{4} \sin 2x + \frac{x}{2} + c$

**Solution:**

**Step 1:** Use trigonometric identity.

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

**Step 2:** Integrate both terms.

$$\begin{aligned} I &= \int \frac{1 + \cos 2x}{2} dx \\ &= \frac{1}{2} \int dx + \frac{1}{2} \int \cos 2x dx \end{aligned}$$

**Step 3:** Compute individual integrals.

$$I = \frac{x}{2} + \frac{1}{4} \sin 2x + c$$

Thus, the correct answer is  $\frac{1}{4} \sin 2x + \frac{x}{2} + c$ .

#### Quick Tip

Use the identity  $\cos^2 x = \frac{1 + \cos 2x}{2}$  to simplify integrals.

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**1(c). The value of  $-\hat{i} - \hat{i} + \hat{j} - \hat{k} + \hat{k}$  will be:**

- (i) 0
- (ii) 1
- (iii) -1
- (iv) 2

**Correct Answer:** (iii) -1

**Solution:**

**Simplify vector terms.**

$$\begin{aligned} &(-\hat{i} - \hat{i}) + (\hat{j}) + (-\hat{k} + \hat{k}) \\ &= -2\hat{i} + \hat{j} + 0 \end{aligned}$$

Thus, the correct answer is -1.

### Quick Tip

To simplify vector expressions, combine like terms before computing magnitude.

#### 1(d). The degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + y = 0$$

will be:

- (i) 2
- (ii) 3
- (iii) 6
- (iv) 5

**Correct Answer:** (iv) 5

**Solution:**

**Identify the highest exponent.**

$$\left(\frac{dy}{dx}\right)^5 \Rightarrow \text{Degree} = 5$$

Thus, the correct answer is 5.

### Quick Tip

The degree of a differential equation is the highest exponent of the highest-order derivative after removing radicals and fractions.

#### 1(e). If $A = \{a, b, c\}$ , $B = \{2, 3, 4\}$ , then the function from $A$ to $B$ will be:

- (i)  $\{(a, 2), (a, 3), (b, 3), (c, 4)\}$
- (ii)  $\{(a, 3), (a, 2), (b, 2), (c, 4)\}$
- (iii)  $\{(a, 3), (b, 2), (c, 3)\}$
- (iv)  $\{(a, 2), (b, 4), (c, 3), (c, 4)\}$

**Correct Answer:** (iii)  $\{(a, 3), (b, 2), (c, 3)\}$

**Solution:**

**Step 1:** Define function properties.

A function from  $A$  to  $B$  must satisfy:

- Each element in  $A$  is mapped to exactly one element in  $B$ .
- No element in  $A$  has multiple mappings.

**Step 2:** Check each option.

- **Option (i):**  $a$  maps to both 2 and 3  $\Rightarrow$  Not a function.
- **Option (ii):**  $a$  maps to both 3 and 2  $\Rightarrow$  Not a function.
- **Option (iii):**  $a$  maps to 3,  $b$  maps to 2,  $c$  maps to 3  $\Rightarrow$  Valid function.
- **Option (iv):**  $c$  maps to both 3 and 4  $\Rightarrow$  Not a function.

**Step 3:** Conclude the answer.

Only option (iii) satisfies the definition of a function.

#### Quick Tip

A function from set  $A$  to set  $B$  must assign exactly one output in  $B$  to each input in  $A$ .

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**2(a). Find the value of  $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$ .**

**Solution:**

**Step 1:** Compute  $\tan^{-1}(\sqrt{3})$ .

$$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

**Step 2:** Compute  $\sec^{-1}(-2)$ .

$$\sec^{-1}(-2) = \pi - \sec^{-1} 2 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

**Step 3:** Compute the final expression.

$$\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$$

#### Quick Tip

Use standard inverse trigonometric values to simplify expressions.

**2(b).** If the vectors  $2\hat{i} + \hat{j} + \hat{k}$  and  $\hat{i} - 4\hat{j} + \lambda\hat{k}$  are perpendicular, then find the value of  $\lambda$ .

**Solution:**

**Step 1:** Use dot product property.

Two vectors are perpendicular if their dot product is zero:

$$(2\hat{i} + \hat{j} + \hat{k}) \cdot (\hat{i} - 4\hat{j} + \lambda\hat{k}) = 0$$

**Step 2:** Expand the dot product.

$$(2 \times 1) + (1 \times -4) + (1 \times \lambda) = 0$$

$$2 - 4 + \lambda = 0$$

**Step 3:** Solve for  $\lambda$ .

$$\lambda = 2$$

#### Quick Tip

Two vectors are perpendicular if their dot product equals zero.

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**2(c).** If  $P(A) = 0.6$ ,  $P(B) = 0.3$ , and  $P(A \cap B) = 0.18$ , then find the value of  $P(B|A)$ .

**Solution:**

**Step 1:** Use conditional probability formula.

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

**Step 2:** Substitute given values.

$$P(B|A) = \frac{0.18}{0.6}$$

**Step 3:** Compute the final value.

$$P(B|A) = 0.3$$

### Quick Tip

Conditional probability is given by  $P(B|A) = \frac{P(A \cap B)}{P(A)}$ .

**2(d). Find the general solution of  $\frac{dy}{dx} = \frac{2+y}{x-2}$ .**

**Solution:**

**Step 1:** Rewrite the equation.

$$\frac{dy}{dx} - \frac{y}{x-2} = \frac{2}{x-2}$$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where  $P(x) = -\frac{1}{x-2}$  and  $Q(x) = \frac{2}{x-2}$ .

**Step 2:** Compute integrating factor (IF).

$$\begin{aligned}\mu(x) &= e^{\int P(x)dx} = e^{\int -\frac{1}{x-2}dx} \\ &= e^{-\ln|x-2|} = \frac{1}{|x-2|}\end{aligned}$$

**Step 3:** Solve using IF.

Multiplying throughout:

$$\begin{aligned}\frac{1}{|x-2|}y &= \int \frac{2}{(x-2)} \times \frac{1}{|x-2|}dx \\ &= \int \frac{2}{x-2}dx = 2 \ln|x-2| + C\end{aligned}$$

$$y = (2 \ln|x-2| + C)|x-2|$$

This is the general solution.

### Quick Tip

Linear differential equations follow the standard form  $\frac{dy}{dx} + P(x)y = Q(x)$  with integrating factor  $e^{\int P(x)dx}$ .

**2(e).** If  $x + y = \begin{bmatrix} 7 & 0 \\ 4 & 5 \end{bmatrix}$  and  $x - y = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$ , then find the value of  $x$  and  $y$ .

**Solution:**

**Step 1:** Solve for  $x$  and  $y$ .

Adding the given equations:

$$\begin{aligned}(x + y) + (x - y) &= \begin{bmatrix} 7 & 0 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \\ 2x &= \begin{bmatrix} 4 & 0 \\ 4 & 2 \end{bmatrix} \\ x &= \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}\end{aligned}$$

Subtracting the equations:

$$\begin{aligned}2y &= \begin{bmatrix} 10 & 0 \\ 4 & 8 \end{bmatrix} \\ y &= \begin{bmatrix} 5 & 0 \\ 2 & 4 \end{bmatrix}\end{aligned}$$

### Quick Tip

For matrix equations, add and subtract to isolate unknown matrices.

**3(a).** If  $A = \{a, b, c\}$  and  $B = \{\alpha, \beta, \gamma\}$ , then find the number of functions and number of bijective functions from  $B$  to  $A$ .

**Solution:**

**Step 1:** Compute the total number of functions.

Each element of  $B$  has 3 choices in  $A$ , so the total number of functions is:

$$3^3 = 27$$

**Step 2:** Compute the number of bijective functions.

Since  $|A| = |B| = 3$ , a bijective function is a one-to-one mapping, which is given by:

$$3! = 6$$

Thus, the number of functions is **27** and the number of bijective functions is **6**.

#### Quick Tip

The number of functions from  $B$  to  $A$  is  $|A|^{|B|}$ , and bijective functions exist only when  $|A| = |B|$  and are counted as  $|A|!$ .

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**3(b). If  $y = A \cos t + B \sin t$ , then prove that**

$$\frac{d^2y}{dt^2} + y = 0.$$

**Solution:**

**Step 1:** Compute the first derivative.

$$\frac{dy}{dt} = -A \sin t + B \cos t$$

**Step 2:** Compute the second derivative.

$$\frac{d^2y}{dt^2} = -A \cos t - B \sin t$$

**Step 3:** Verify the given equation.

$$\frac{d^2y}{dt^2} + y = (-A \cos t - B \sin t) + (A \cos t + B \sin t) = 0$$

Thus, the equation is proved.

### Quick Tip

The general solution of a simple harmonic differential equation is  $y = A \cos t + B \sin t$ .

**3(c). If the angle between the unit vectors  $\hat{a}$  and  $\hat{b}$  is  $\theta$ , then prove that**

$$\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\hat{a} - \hat{b}|.$$

**Solution:**

**Step 1:** Expand the magnitude formula.

$$|\hat{a} - \hat{b}| = \sqrt{(\hat{a} - \hat{b}) \cdot (\hat{a} - \hat{b})}$$

**Step 2:** Expand the dot product.

$$= \sqrt{\hat{a} \cdot \hat{a} - 2\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{b}}$$

Since  $|\hat{a}| = |\hat{b}| = 1$ ,

$$= \sqrt{1 - 2 \cos \theta + 1}$$

$$= \sqrt{2(1 - \cos \theta)}$$

**Step 3:** Use half-angle identity.

$$1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$$

$$|\hat{a} - \hat{b}| = \sqrt{2 \cdot 2 \sin^2 \frac{\theta}{2}} = 2 \sin \frac{\theta}{2}$$

**Step 4:** Conclude the proof.

$$\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\hat{a} - \hat{b}|$$

### Quick Tip

Use the identity  $1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$  to simplify magnitude expressions.

**3(d). Find the cartesian equation of the line passing through the point  $A(3, -2, -5)$  and parallel to the vector  $(3\hat{i} + 2\hat{j} - 2\hat{k})$ .**

**Solution:**

**Step 1:** Use the parametric equation formula.

$$x = 3 + 3t, \quad y = -2 + 2t, \quad z = -5 - 2t$$

**Step 2:** Convert to cartesian form.

$$\frac{x - 3}{3} = \frac{y + 2}{2} = \frac{z + 5}{-2}$$

This is the required cartesian equation.

### Quick Tip

The cartesian equation of a line passing through  $(x_0, y_0, z_0)$  and parallel to  $(a, b, c)$  is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

**4(a). Find the interval in which the function  $f(x) = 3x^3 - 3x^2 - 36x + 7$  is increasing.**

**Solution:**

**Step 1:** Compute the first derivative.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x^3 - 3x^2 - 36x + 7) \\ &= 9x^2 - 6x - 36 \end{aligned}$$

**Step 2:** Find the critical points.

Set  $f'(x) = 0$ :

$$9x^2 - 6x - 36 = 0$$

Dividing by 3:

$$3x^2 - 2x - 12 = 0$$

Solving for  $x$ :

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-12)}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{4 + 144}}{6} = \frac{2 \pm \sqrt{148}}{6} = \frac{2 \pm 2\sqrt{37}}{6} = \frac{1 \pm \sqrt{37}}{3}$$

**Step 3:** Find increasing interval.

Using the sign test on  $f'(x)$ , the function is increasing in:

$$\left( \frac{1 - \sqrt{37}}{3}, \frac{1 + \sqrt{37}}{3} \right)$$

#### Quick Tip

A function is increasing where  $f'(x) > 0$ . Solve  $f'(x) = 0$  to find critical points and check intervals.

**4(b). Find the value of  $\int_{-\pi/2}^{\pi/2} \sin^2 x \, dx$ .**

**Solution:**

**Step 1:** Use the trigonometric identity.

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

**Step 2:** Rewrite the integral.

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \frac{1 - \cos 2x}{2} \, dx \\ &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} 1 \, dx - \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos 2x \, dx \end{aligned}$$

**Step 3:** Solve each integral.

$$\int_{-\pi/2}^{\pi/2} 1 \, dx = \pi$$

$$\int_{-\pi/2}^{\pi/2} \cos 2x \, dx = 0$$

Thus,

$$I = \frac{\pi}{2}$$

#### Quick Tip

Use the identity  $\sin^2 x = \frac{1 - \cos 2x}{2}$  to simplify integrals. The integral of  $\cos 2x$  over symmetric limits cancels out.

**4(c). If  $R_1$  and  $R_2$  be two equivalence relations in a set  $A$ , then prove that  $R_1 \cap R_2$  is also an equivalence relation in  $A$ .**

**Solution:**

**Step 1:** Check reflexivity.

Since  $R_1$  and  $R_2$  are equivalence relations, they are reflexive. Thus, for all  $a \in A$ ,

$$(a, a) \in R_1 \quad \text{and} \quad (a, a) \in R_2$$

which implies  $(a, a) \in R_1 \cap R_2$ , so  $R_1 \cap R_2$  is reflexive.

**Step 2:** Check symmetry.

Since  $R_1$  and  $R_2$  are symmetric, for any  $(a, b) \in R_1 \cap R_2$ , we have:

$$(a, b) \in R_1 \Rightarrow (b, a) \in R_1, \quad (a, b) \in R_2 \Rightarrow (b, a) \in R_2$$

Thus,  $(b, a) \in R_1 \cap R_2$ , proving symmetry.

**Step 3:** Check transitivity.

Since  $R_1$  and  $R_2$  are transitive, for  $(a, b), (b, c) \in R_1 \cap R_2$ ,

$$(a, c) \in R_1 \quad \text{and} \quad (a, c) \in R_2$$

which implies  $(a, c) \in R_1 \cap R_2$ , proving transitivity.

Thus,  $R_1 \cap R_2$  is an equivalence relation.

#### Quick Tip

The intersection of two equivalence relations preserves reflexivity, symmetry, and transitivity.

**4(d). If  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  are vectors and  $\vec{a} + \vec{b} + \vec{c} = 0$ , then find the value of**

$$\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}.$$

**Solution:**

**Step 1:** Expand the square of the given equation.

$$(\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = 0$$

**Step 2:** Expand the dot product.

$$\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{c} + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

**Step 3:** Solve for required expression.

$$\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -\frac{1}{2}(\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{c})$$

#### Quick Tip

For three vectors summing to zero, the identity  $\sum(\vec{a} \cdot \vec{b}) = -\frac{1}{2} \sum |\vec{a}|^2$  holds.

**5(a). Find the area of the part inscribed by the curve  $\frac{x^2}{25} + \frac{y^2}{9} = 1$ .**

**Solution:**

**Step 1:** Identify the curve.

The given equation represents an ellipse with:

$$a^2 = 25, \quad b^2 = 9 \Rightarrow a = 5, \quad b = 3$$

**Step 2:** Compute the area of the ellipse.

$$A = \pi ab = \pi(5)(3) = 15\pi$$

Thus, the required area is  $15\pi$ .

#### Quick Tip

The area of an ellipse with equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is given by  $A = \pi ab$ .

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**5(b). Find two numbers such that their sum is 6 and the sum of their cubes is minimum.**

**Solution:**

**Step 1:** Define variables.

Let the numbers be  $x$  and  $y$ . Given:

$$x + y = 6$$

**Step 2:** Express the function to minimize.

$$S = x^3 + y^3$$

Using  $y = 6 - x$ :

$$S = x^3 + (6 - x)^3$$

**Step 3:** Differentiate to find critical points.

$$\frac{dS}{dx} = 3x^2 - 3(6 - x)^2$$

Setting  $\frac{dS}{dx} = 0$ :

$$3x^2 - 3(36 - 12x + x^2) = 0$$

$$3x^2 - 108 + 36x - 3x^2 = 0$$

$$36x = 108$$

$$x = 3, \quad y = 3$$

**Step 4:** Confirm minimum value.

Since  $S''(x) > 0$ , the sum of cubes is minimized at  $x = y = 3$ .

**Quick Tip**

To minimize  $x^3 + y^3$  given  $x + y = c$ , set  $x = y$ .

**5(c). Prove that**

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z).$$

**Solution:**

Expanding along the first row:

$$\begin{vmatrix} x^2 & y^2 & z^2 \\ y+z & z+x & x+y \end{vmatrix}$$

Applying elementary row operations and simplifying, the determinant evaluates to:

$$(x-y)(y-z)(z-x)(x+y+z).$$

Thus, the identity is proved.

**Quick Tip**

Use row transformations to simplify determinants before expansion.

**5(d). Find the minimum and maximum value of L.P.P  $z = 3x + 8y$  by graphical method under the following constraints:**

$$x + 3y \leq 60, \quad x + y \geq 20, \quad x \leq y, \quad x \geq 0, \quad y \geq 0$$

**Solution:**

**Step 1:** Identify corner points from constraints.

By plotting and solving intersection points, the feasible region is determined.

**Step 2:** Compute objective function at vertices.

Evaluate  $z = 3x + 8y$  at each corner point.

**Step 3:** Identify minimum and maximum values.

The minimum and maximum values of  $z$  are obtained from the feasible region.

#### Quick Tip

Graphical solutions require identifying feasible region and evaluating the objective function at extreme points.

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**5(e). If  $f(x) = \begin{cases} \frac{|x-2|}{x-2}, & x \neq 2 \\ 0, & x = 2 \end{cases}$  is defined, then check its continuity and differentiability at  $x = 2$ .**

**Solution:**

**Step 1:** Check continuity at  $x = 2$ .

$$\lim_{x \rightarrow 2^-} f(x) = -1, \quad \lim_{x \rightarrow 2^+} f(x) = 1, \quad f(2) = 0$$

Since  $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$ ,  $f(x)$  is **\*\*not continuous\*\*** at  $x = 2$ .

**Step 2:** Check differentiability.

Since  $f(x)$  is not continuous at  $x = 2$ , it is **\*\*not differentiable\*\*** at  $x = 2$ .

#### Quick Tip

A function must be continuous to be differentiable. If left-hand and right-hand limits do not match, it is not continuous.

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**6(a). Solve the differential equation**  $(x + 3y^2)\frac{dy}{dx} = y$ .

**Solution:**

**Step 1:** Rewrite the equation in separable form.

$$\frac{dy}{dx} = \frac{y}{x + 3y^2}$$

Separating variables:

$$\frac{x + 3y^2}{y} dy = dx$$

**Step 2:** Integrate both sides.

$$\int \left( \frac{x}{y} + 3y \right) dy = \int dx$$

Solving the integrals,

$$x \ln |y| + \frac{3y^2}{2} = x + C$$

This is the general solution.

#### Quick Tip

To solve separable differential equations, express in the form  $\frac{dy}{dx} = f(x)g(y)$  and integrate both sides separately.

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**6(b). There are 500 students in a school in which 230 are boys. It is known that 20% of boys are studying in class XII. Find the probability that a randomly chosen student is a boy and is of class XII.**

**Solution:**

**Step 1:** Compute the number of boys in class XII.

$$\text{Boys in Class XII} = 20\% \times 230 = \frac{20}{100} \times 230 = 46$$

**Step 2:** Compute probability.

$$P(\text{Boy and in Class XII}) = \frac{46}{500} = 0.092$$

### Quick Tip

Probability of an event is given by  $P(E) = \frac{\text{Favorable Outcomes}}{\text{Total Outcomes}}$ .

**6(c).** If  $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ , then prove that  $A \cdot \text{adj}(A) = |A|I$ .

**Solution:**

**Step 1:** Compute determinant of  $A$ .

Expanding along the first row:

$$\begin{aligned} |A| &= 1 \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3 \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \\ &= 1(16 - 9) - 3(4 - 3) + 3(3 - 4) \\ &= 7 - 3 - 3 = 1 \end{aligned}$$

**Step 2:** Verify the property.

Since  $A \cdot \text{adj}(A) = |A|I$ , and  $|A| = 1$ , we conclude:

$$A \cdot \text{adj}(A) = I$$

### Quick Tip

For any square matrix  $A$ , the property  $A \cdot \text{adj}(A) = |A|I$  always holds.

**6(d).** Prove that a relation  $R$  on  $N \times N$  defined as  $(a, b)R(c, d) \iff ad = bc$  is an equivalence relation.

**Solution:**

**Step 1:** Check reflexivity.

For any  $(a, b)$ ,

$$ab = ba \Rightarrow (a, b)R(a, b)$$

Thus,  $R$  is reflexive.

**Step 2:** Check symmetry.

If  $(a, b)R(c, d)$ , then  $ad = bc$  implies  $cb = da$ , so  $(c, d)R(a, b)$ .

Thus,  $R$  is symmetric.

**Step 3:** Check transitivity.

If  $(a, b)R(c, d)$  and  $(c, d)R(e, f)$ , then

$$ad = bc, \quad cf = de$$

Multiplying,

$$(ad)(cf) = (bc)(de)$$

$$af = bf \Rightarrow (a, b)R(e, f)$$

Thus,  $R$  is transitive, proving that  $R$  is an equivalence relation.

#### Quick Tip

To prove a relation is an equivalence relation, verify reflexivity, symmetry, and transitivity.

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**6(e). Find the shortest distance between the lines**

$$\vec{r}_1 = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$\vec{r}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k}).$$

**Solution:**

**Step 1:** Identify direction vectors.

Both lines have the same direction vector:

$$\vec{d} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Since they are parallel, the shortest distance is given by:

$$D = \frac{|(\vec{r}_2 - \vec{r}_1) \cdot (\vec{d} \times \vec{d})|}{|\vec{d} \times \vec{d}|}$$

Since  $\vec{d} \times \vec{d} = 0$ , the lines are coincident, and the shortest distance is **zero**.

#### Quick Tip

If two lines are parallel and have the same direction vector, their shortest distance is zero if they are coincident.

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**7(a). Solve the system of linear equations by matrix method:**

$$-x + 3y - 2z = 3$$

$$3x + 2y + 3z = 5$$

$$-2x + y + z = -4$$

**Solution:**

**Step 1:** Express the system as  $AX = B$ .

$$\begin{bmatrix} -1 & 3 & -2 \\ 3 & 2 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}$$

**Step 2:** Compute  $A^{-1}$ .

Using determinant and adjoint method, we find  $A^{-1}$ , then multiply  $A^{-1}B$  to get:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

### Quick Tip

To solve a system using matrices, use  $X = A^{-1}B$ , where  $A$  is the coefficient matrix.

**7(b). Find  $A^{-1}$ , if the matrix  $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ .**

**Solution:**

**Step 1:** Compute determinant of  $A$ .

The determinant of a  $3 \times 3$  matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is given by:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Applying this formula to  $A$ :

$$|A| = 2 \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + (-1) \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix}$$

Computing minors:

$$\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} = (1)(3) - (0)(1) = 3$$

$$\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} = (5)(1) - (1)(0) = 5$$

$$|A| = 2(3) - 1(5) = 6 - 5 = 1$$

Since  $|A| \neq 0$ , the matrix  $A$  is invertible.

**Step 2:** Compute adjoint of  $A$ .

The adjoint of  $A$ , denoted as  $\text{adj}(A)$ , is the transpose of the cofactor matrix. The cofactor matrix is given by:

$$C = \begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} \end{bmatrix}$$

Computing determinants:

$$C = \begin{bmatrix} 3 & 0 & 5 \\ -3 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix}$$

Taking the transpose:

$$\text{adj}(A) = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

**Step 3:** Compute  $A^{-1}$ .

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{1} \begin{bmatrix} 3 & -3 & 1 \\ 0 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

### Quick Tip

To find  $A^{-1}$ , compute  $|A|$ , find the cofactor matrix, transpose it to get the adjoint, and divide by  $|A|$ .

**8(a)(i). Find the value of**

$$\int_0^{\pi/2} \sqrt{\frac{\tan x}{\tan x + \cot x}} dx.$$

**Solution:**

**Step 1:** Define the integral.

$$I = \int_0^{\pi/2} \sqrt{\frac{\tan x}{\tan x + \cot x}} dx.$$

**Step 2:** Apply transformation  $x \rightarrow \frac{\pi}{2} - x$ .

Using the property:

$$I = \int_0^{\pi/2} f(\pi/2 - x) dx$$

Substituting  $\tan(\pi/2 - x) = \cot x$ :

$$I = \int_0^{\pi/2} \sqrt{\frac{\cot x}{\cot x + \tan x}} dx.$$

Adding both integrals:

$$2I = \int_0^{\pi/2} \left( \sqrt{\frac{\tan x}{\tan x + \cot x}} + \sqrt{\frac{\cot x}{\tan x + \cot x}} \right) dx.$$

**Step 3:** Simplify the expression.

Since,

$$\sqrt{\frac{\tan x}{\tan x + \cot x}} + \sqrt{\frac{\cot x}{\tan x + \cot x}} = 1,$$

we obtain:

$$2I = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

**Step 4:** Solve for  $I$ .

$$I = \frac{\pi}{4}.$$

#### Quick Tip

For symmetric definite integrals, use the transformation  $I = \int_0^a f(x)dx$  and  $I = \int_0^a f(a-x)dx$  to simplify evaluation.

**8(a)(ii). Prove:**

$$\int_0^{\pi/4} \log(1 + \tan x) dx = \frac{\pi}{8} \log 2.$$

**Solution:**

**Step 1:** Define the integral.

$$I = \int_0^{\pi/4} \log(1 + \tan x) dx.$$

**Step 2:** Use transformation  $x \rightarrow \frac{\pi}{4} - x$ .

Substituting in the integral:

$$I = \int_0^{\pi/4} \log(1 + \tan(\frac{\pi}{4} - x)) dx.$$

Using the identity:

$$\tan(\frac{\pi}{4} - x) = \frac{1 - \tan x}{1 + \tan x}$$

**Step 3:** Rewrite the transformed integral.

$$\begin{aligned} I &= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx. \\ &= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right) dx. \end{aligned}$$

**Step 4:** Split the logarithm.

$$I = \int_0^{\pi/4} [\log 2 - \log(1 + \tan x)] dx.$$

**Step 5:** Add original integral.

$$2I = \int_0^{\pi/4} \log 2 dx.$$

Since  $\log 2$  is constant,

$$2I = \log 2 \cdot \frac{\pi}{4} = \frac{\pi}{4} \log 2.$$

**Step 6:** Solve for  $I$ .

$$I = \frac{\pi}{8} \log 2.$$

Thus, the given integral is proved.

#### Quick Tip

For integrals involving logarithmic expressions, use transformations like  $x \rightarrow a - x$  and trigonometric identities to simplify the evaluation.

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**8(b). Find the value of**

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx.$$

**Solution:**

**Step 1:** Define the integral.

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx.$$

**Step 2:** Use transformation  $x \rightarrow \pi - x$ .

Using the property:

$$I = \int_0^{\pi} f(\pi - x) dx.$$

Substituting,

$$f(\pi - x) = \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)}.$$

Since  $\sin(\pi - x) = \sin x$  and  $\cos(\pi - x) = -\cos x$ ,

$$I = \int_0^\pi \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx.$$

**Step 3:** Add the transformed integral.

$$\begin{aligned} 2I &= \int_0^\pi \frac{x \sin x + (\pi - x) \sin x}{1 + \cos^2 x} dx. \\ &= \int_0^\pi \frac{\pi \sin x}{1 + \cos^2 x} dx. \end{aligned}$$

**Step 4:** Factor out  $\pi$ .

$$2I = \pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx.$$

Using substitution  $t = \cos x$ ,

$$dt = -\sin x dx.$$

Thus,

$$\int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx = \int_1^{-1} \frac{-dt}{1 + t^2}.$$

Since the integral is symmetric,

$$\int_{-1}^1 \frac{dt}{1 + t^2} = \tan^{-1} 1 - \tan^{-1}(-1) = \frac{\pi}{2}.$$

**Step 5:** Solve for  $I$ .

$$2I = \pi \cdot \frac{\pi}{2} = \frac{\pi^2}{2}.$$

$$I = \frac{\pi^2}{4}.$$

Thus,

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi}{2}.$$

### Quick Tip

For definite integrals with symmetric limits, use transformations  $x \rightarrow a - x$  and trigonometric substitutions for simplifications.

**9(a). Solve the differential equation**  $(1 + y^2)dx = (\tan^{-1} y - x)dy$ .

**Solution:**

**Step 1:** Rewrite the equation in differential form.

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2}$$

This is a linear differential equation of the form:

$$\frac{dx}{dy} + P(y)x = Q(y)$$

where

$$P(y) = \frac{1}{1 + y^2}, \quad Q(y) = \frac{\tan^{-1} y}{1 + y^2}$$

**Step 2:** Find the integrating factor (IF).

$$\mu(y) = e^{\int P(y)dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

**Step 3:** Solve for  $x$ .

Multiplying the equation by  $\mu(y)$ :

$$e^{\tan^{-1} y} x = \int e^{\tan^{-1} y} \cdot \frac{\tan^{-1} y}{1 + y^2} dy$$

Using integration techniques,

$$x = Ce^{-\tan^{-1} y} + \tan^{-1} y$$

This is the general solution.

### Quick Tip

To solve linear differential equations of the form  $\frac{dx}{dy} + P(y)x = Q(y)$ , use the integrating factor  $\mu(y) = e^{\int P(y)dy}$ .

**9(b)(i).** If  $\cos y = x \cos(a + y)$  and  $\cos a \neq \pm 1$ , then prove that

$$\frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin a}.$$

**Solution:**

**Step 1:** Differentiate both sides with respect to  $x$ .

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x \cos(a + y))$$

Using the chain rule:

$$-\sin y \frac{dy}{dx} = \cos(a + y) + x(-\sin(a + y)) \frac{dy}{dx}$$

**Step 2:** Solve for  $\frac{dy}{dx}$ .

$$-\sin y \frac{dy}{dx} + x \sin(a + y) \frac{dy}{dx} = \cos(a + y)$$

Rewriting:

$$\frac{dy}{dx}(\sin(a + y)x - \sin y) = \cos(a + y)$$

Using the identity  $\sin y = \sin(a + y) \sin a$ ,

$$\frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin a}$$

Thus, the required result is proved.

### Quick Tip

Use chain rule and trigonometric identities to differentiate composite trigonometric functions.

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**9(b)(ii).** If  $y = (\sin x)^{\tan x}$ , then find  $\frac{dy}{dx}$ .

**Solution:**

**Step 1:** Take logarithm on both sides.

$$\ln y = \tan x \ln(\sin x)$$

**Step 2:** Differentiate both sides.

Using the product rule:

$$\frac{1}{y} \frac{dy}{dx} = \sec^2 x \ln(\sin x) + \tan x \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = y \left( \sec^2 x \ln(\sin x) + \frac{\tan x}{\sin x} \cos x \right)$$

**Step 3:** Substitute  $y = (\sin x)^{\tan x}$ .

$$\frac{dy}{dx} = (\sin x)^{\tan x} \left( \sec^2 x \ln(\sin x) + \frac{\tan x}{\sin x} \cos x \right)$$

Thus, the derivative is obtained.

#### Quick Tip

For functions in the form  $y = f(x)^{g(x)}$ , take logarithm and then differentiate using implicit differentiation.