UP Board 12 Mathematics 324 (FB) Question Paper with Solutions

Time Allowed : 3 hours 15 minutes	**Maximum Marks :** 100	**Total questions :** 34

General Instructions

Read the following instructions very carefully and strictly follow them:

1. There are in all *nine* questions in this question paper.
2. *All* questions are compulsory.
3. In the beginning of each question, the number of parts to be attempted are clearly mentioned.
4. Marks allotted to the questions are indicated against them.
5. Start solving from the first question and proceed to solve till the last one.
6. Do not waste your time over a question which you cannot solve.

1. (a)Function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 5x, \forall x \in \mathbb{R}$. Select the correct answer:

(A) f is onto

(B) f is many-one

(C) f is not onto

(D) f is not one-one

Correct Answer: (A) f is onto

Solution: Step 1: The function $f(x) = 5x$ maps each real number to a unique value in $\mathbb{R}$, ensuring it is a one-to-one mapping.

Step 2: Since the function f covers the entire real number set $\mathbb{R}$, it is onto.

Step 3: f is not many-one because no two distinct values of x map to the same value of $f(x)$. Hence, $f(x) = 5x$ is onto and one-one.

Quick Tip

To determine if a function is onto, check whether every element in the codomain has a preimage in the domain.

(b) Order of the differential equation:

$$5x^3\frac{d^3y}{dx^3} - 3\left(\frac{dy}{dx}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^4 + y = 0$$

(A) 2

(B) 1

(C) 3

(D) 4

Correct Answer: (A) 2

Solution: The order of a differential equation is determined by the highest derivative present in the equation. In this case, the highest derivative is $\frac{d^3y}{dx^3}$, which indicates that the order of the equation is 3.

Quick Tip

The order of a differential equation is always the highest derivative, regardless of its power or coefficients.

(c) The value of the integral:

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$$

(A) $\frac{\pi}{3}$
(B) $\frac{2\pi}{3}$
(C) $\frac{\pi}{6}$
(D) $\frac{\pi}{12}$

Correct Answer: (A) $\frac{\pi}{3}$

Solution: The given integral is a standard form:

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x).$$

Applying limits:

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

$$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}, \quad \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

$$\text{Result: } \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.$$

Quick Tip

Use standard trigonometric integrals and apply limits step by step.

(d) The value of the expression:

$$\hat{i}\cdot\hat{i} + \hat{j}\cdot\hat{j} + \hat{k}\cdot\hat{k}$$

(A) 0

(B) 1

(C) 2

(D) 3

Correct Answer: (D) 3

Solution: The dot product of a unit vector with itself is equal to 1:

$$\hat{i} \cdot \hat{i} = 1, \quad \hat{j} \cdot \hat{j} = 1, \quad \hat{k} \cdot \hat{k} = 1.$$

Adding these values:

$$1 + 1 + 1 = 3.$$

Quick Tip

The dot product of a unit vector with itself is always 1, and with others, it's 0.

(e) If A and B are two invertible matrices of order n, then:

(A) $(AB)^{-1} = B^{-1}A^{-1}$

(B) $(AB)^{-1} = A^{-1}B^{-1}$

(C) $(AB)^{-1} = A^{-1}B$

(D) $(AB)^{-1} = AB^{-1}$

Correct Answer: (A) $(AB)^{-1} = B^{-1}A^{-1}$

Solution: The inverse of the product of two invertible matrices follows the rule:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

The order of inversion is reversed due to matrix multiplication properties.

Quick Tip

For invertible matrices, the product of inverses is taken in the reverse order of multiplication.

2: (a) If $A = \{1, 2, 3\}$, $B = \{4, 5\}$, then find the number of relations from A to B:

Solution: The number of elements in A is $|A| = 3$ and in B is $|B| = 2$.

The total number of relations from A to B is given by:

$$2^{|A|\cdot|B|} = 2^{3\cdot 2} = 2^6 = 64.$$

Quick Tip

The total number of relations is $2^{m\cdot n}$, where m and n are the sizes of the sets A and B, respectively.

(b) Two coins are tossed together. Find the probability of getting both tails:

Solution: When two coins are tossed, the sample space is:

$$S = \{\text{HH, HT, TH, TT}\}.$$

The event of getting both tails is $E = \{\text{TT}\}$. The probability is given by:

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{1}{4}.$$

Quick Tip

The probability of an event is calculated as the ratio of favorable outcomes to the total outcomes.

(c) If the vectors $\vec{v_1} = 3\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{v_2} = \hat{i} - 4\hat{j} + \lambda\hat{k}$ are perpendicular, find the value of λ:

Solution: Two vectors are perpendicular if their dot product is zero:

$$\vec{v_1} \cdot \vec{v_2} = 0.$$

Substituting the given vectors:

$$(3)(1) + (2)(-4) + (1)(\lambda) = 0.$$

$$3 - 8 + \lambda = 0 \quad \Rightarrow \quad \lambda = 5.$$

Quick Tip

For perpendicular vectors, their dot product is always zero.

(d) If $P(A) = \frac{3}{13}$, $P(B) = \frac{5}{13}$, and $P(A \cap B) = \frac{2}{13}$, find the value of $P(B/A)$:

Solution: The conditional probability is given by:

$$P(B/A) = \frac{P(A \cap B)}{P(A)}.$$

Substituting the given values:

$$P(B/A) = \frac{\frac{2}{13}}{\frac{3}{13}} = \frac{2}{3}.$$

Quick Tip

Conditional probability is calculated as $P(B/A) = \frac{P(A \cap B)}{P(A)}$.

(e) If $y = \log_e(\tan x)$, find $\frac{dy}{dx}$:

Solution: Given $y = \log_e(\tan x)$, differentiating with respect to x:

$$\frac{dy}{dx} = \frac{1}{\tan x} \cdot \frac{d}{dx}(\tan x).$$

Since $\frac{d}{dx}(\tan x) = \sec^2 x$, we have:

$$\frac{dy}{dx} = \frac{\sec^2 x}{\tan x}.$$

Quick Tip

Apply the chain rule carefully when differentiating logarithmic functions.

3: (a) Prove that the function $f(x) = |x - 1|$ is continuous at $x = 1$:

Solution: The function $f(x) = |x-1|$ can be written as:

$$f(x) = \begin{cases} x-1 & \text{if } x \geq 1, \\ 1-x & \text{if } x < 1. \end{cases}$$

To check continuity at $x = 1$, we evaluate:

1. $f(1) = |1-1| = 0$.

2. Left-hand limit ($\lim_{x\to 1^-} f(x)$):

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x) = 0.$$

3. Right-hand limit ($\lim_{x\to 1^+} f(x)$):

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1) = 0.$$

Since $f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$, the function is continuous at $x = 1$.

Quick Tip

For continuity, ensure $f(a) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.

(b) Find the value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$:

Solution: We know:

$$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}, \quad \cot^{-1}(-\sqrt{3}) = \tan^{-1}(-\frac{1}{\sqrt{3}}).$$

Since $\tan^{-1}(-x) = -\tan^{-1}(x)$:

$$\cot^{-1}(-\sqrt{3}) = -\tan^{-1}(\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}.$$

Thus:

$$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3}) = \frac{\pi}{3} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}.$$

Quick Tip

Use trigonometric identities and inverse function properties for simplifications.

(c) If the unit vectors $\vec{a}, \vec{b}, \vec{c}$ are such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$:

Solution: Given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, we have:

$$\vec{c} = -(\vec{a} + \vec{b}).$$

Now calculate the dot products:

$$\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \vec{a} \cdot \vec{b} + \vec{b} \cdot (-\vec{a} - \vec{b}) + (-\vec{a} - \vec{b}) \cdot \vec{a}.$$

Simplify:

$$\vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b} - \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} = -(\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b}).$$

Since $\vec{a}, \vec{b}, \vec{c}$ are unit vectors:

$$\vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b} = 1, \quad \text{so } \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -2.$$

Quick Tip

Use unit vector properties and dot product identities for simplifications.

(d) Find the value of $\int \log x \, dx$:

Solution: Using integration by parts: Let $u = \log x$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$.

$$\int \log x \, dx = x \log x - \int x \cdot \frac{1}{x} dx = x \log x - \int dx.$$

$$\int \log x \, dx = x \log x - x + C.$$

Quick Tip

For $\int \log x \, dx$, apply integration by parts with $u = \log x$ and $dv = dx$.

4: (a) If $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, find $\frac{dy}{dx}$:

Solution: We are given:

$$x = a(\theta + \sin\theta), \quad y = a(1 - \cos\theta).$$

Differentiating x and y with respect to θ:

$$\frac{dx}{d\theta} = a(1 + \cos\theta), \quad \frac{dy}{d\theta} = a\sin\theta.$$

Using the chain rule, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$:

$$\frac{dy}{dx} = \frac{a \sin\theta}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta}.$$

Quick Tip

For parametric equations, use $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ and simplify carefully.

(b) Find the differential coefficient of the function x^x with respect to x:

Solution: Let $y = x^x$. Taking the natural logarithm on both sides:

$$\ln y = x \ln x.$$

Differentiating both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1.$$

Multiply by y to get $\frac{dy}{dx}$:

$$\frac{dy}{dx} = x^x(\ln x + 1).$$

Quick Tip

For x^x, take the natural logarithm to simplify differentiation.

(c) Find the value of $\int_{-\pi/2}^{\pi/2} \sin^2 x \, dx$:

Solution: Using the identity $\sin^2 x = \frac{1-\cos 2x}{2}$:

$$\int_{-\pi/2}^{\pi/2} \sin^2 x \, dx = \int_{-\pi/2}^{\pi/2} \frac{1-\cos 2x}{2} \, dx.$$

Separate the terms:

$$\int_{-\pi/2}^{\pi/2} \sin^2 x \, dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} 1 \, dx - \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos 2x \, dx.$$

The first term evaluates to:

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2} 1 \, dx = \frac{1}{2}[x]_{-\pi/2}^{\pi/2} = \frac{1}{2}(\pi - (-\pi)) = \frac{\pi}{2}.$$

The second term evaluates to 0 because $\cos 2x$ is an odd function. Thus:

$$\int_{-\pi/2}^{\pi/2} \sin^2 x\, dx = \frac{\pi}{2}.$$

Quick Tip

Use trigonometric identities like $\sin^2 x = \frac{1-\cos 2x}{2}$ to simplify integrals.

(d) Prove that the function $f(x) = |x|$ is not differentiable at $x = 0$:

Solution: The function $f(x) = |x|$ can be written as:

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The left-hand derivative at $x = 0$:

$$f'(0^-) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h - 0}{h} = -1.$$

The right-hand derivative at $x = 0$:

$$f'(0^+) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h - 0}{h} = 1.$$

Since $f'(0^-) \neq f'(0^+)$, $f(x)$ is not differentiable at $x = 0$.

Quick Tip

Check both left-hand and right-hand derivatives to determine differentiability at a point.

5: (a) Prove that for the two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$:

Solution: The dot product of two vectors is given by:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta,$$

where θ is the angle between the vectors. The absolute value of $\cos\theta$ is always less than or equal to 1, i.e., $|\cos\theta| \leq 1$. Therefore:

$$|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}||\cos\theta| \leq |\vec{a}||\vec{b}|.$$

Quick Tip

For proving inequalities involving vectors, use the properties of the dot product and trigonometric bounds.

(b) Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$**:**

Solution: Expand the determinant using row or column operations. First, subtract the first column from the second and third columns:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} \to \begin{vmatrix} 1+a & b & c \\ 1 & b & c \\ 1 & b & c \end{vmatrix}.$$

Using the properties of determinants and simplifying yields:

$$abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Quick Tip

For determinant proofs, use row and column operations to simplify the matrix.

(c) Solve the differential equation $(\tan^{-1} y - x)dy = (1+y^2)dx$**:**

Solution: Rewriting the equation:

$$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1} y - x}.$$

Use substitution $z = \tan^{-1} y - x$, then differentiate and solve. The solution is:

$$z = C, \quad \text{where } z = \tan^{-1} y - x.$$

Thus:

$$\tan^{-1} y - x = C.$$

Quick Tip

Substitution is a powerful tool for solving first-order differential equations.

(d) If $(\cos x)^y = (\cos y)^x$, find $\frac{dy}{dx}$:

Solution: Take the natural logarithm of both sides:

$$\ln(\cos x)^y = \ln(\cos y)^x.$$

Differentiating both sides:

$$y\ln(\cos x) + \frac{y}{\cos x}\frac{dy}{dx} = x\ln(\cos y) + \frac{x}{\cos y}.$$

Simplify to find $\frac{dy}{dx}$.

> **Quick Tip**
>
> Logarithmic differentiation simplifies equations involving powers of variables.

(e) If $y = x\cos(a+y)$ and $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$:

Solution: Differentiating both sides with respect to x:

$$\frac{dy}{dx} = \cos(a+y) - x\sin(a+y)\frac{dy}{dx}.$$

Rearranging terms:

$$\frac{dy}{dx} = \frac{\cos(a+y)}{1 + x\sin(a+y)}.$$

> **Quick Tip**
>
> Implicit differentiation is essential when the dependent variable appears inside a trigonometric function.

6: (a) If a die is thrown three times, find the probability of getting one appearing number in them will be odd:

Solution: For a die, the numbers that are odd are $1, 3, 5$. The probability of rolling an odd number is:

$$P(\text{odd}) = \frac{3}{6} = \frac{1}{2}.$$

The probability of rolling an even number is also $\frac{1}{2}$. Using the binomial probability formula, for exactly one odd number in three rolls:

$$P(\text{one odd}) = \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = 3 \cdot \frac{1}{8} = \frac{3}{8}.$$

Quick Tip

Use the binomial theorem for probabilities involving repeated independent trials.

(b) Minimize $Z = x + 2y$ under the following constraints:

$$2x + y \geq 3, \quad x + 2y \geq 6, \quad x \geq 0, \quad y \geq 0.$$

Solution: Plot the constraints on a graph to form the feasible region. The vertices of the feasible region are determined by solving the intersection points:

$$2x + y = 3 \quad \text{and} \quad x + 2y = 6.$$

Substitute the vertices into $Z = x + 2y$:

$$Z(0,3) = 0 + 2(3) = 6, \quad Z(1,2) = 1 + 2(2) = 5, \quad Z(3,0) = 3 + 2(0) = 3.$$

The minimum value of Z is 3.

Quick Tip

Linear programming problems are solved by evaluating the objective function at the vertices of the feasible region.

(c) Solve:

$$\int \frac{3x+5}{x^3 - x^2 - x + 1}\, dx.$$

Solution: Factorize the denominator:

$$x^3 - x^2 - x + 1 = (x-1)(x^2+1).$$

Use partial fractions:

$$\frac{3x+5}{x^3 - x^2 - x + 1} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1}.$$

Find A, B, and C, then integrate:

$$\int \frac{A}{x-1}\,dx + \int \frac{Bx+C}{x^2+1}\,dx.$$

Quick Tip

Partial fractions simplify the integration of rational functions.

(d) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$ and find A^{-1}:

Solution: First, compute A^2:

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}.$$

Then verify:

$$A^2 - 5A + 7I = 0.$$

For A^{-1}, use:

$$A^{-1} = \frac{1}{\text{det}(A)}\text{adj}(A).$$

Quick Tip

Matrix inverses can be found using the adjoint formula or Gaussian elimination.

(e) Show that the semi-vertical angle of the right circular cone of maximum volume and given slant height is $\tan^{-1}(\sqrt{2})$:

Solution: For a cone, volume $V = \frac{1}{3}\pi r^2 h$. Using the slant height l:

$$r^2 + h^2 = l^2.$$

Substitute and differentiate to maximize V. Solve for θ where $\tan\theta = \frac{r}{h}$. The result is:

$$\theta = \tan^{-1}(\sqrt{2}).$$

Quick Tip

Optimization problems often involve substituting constraints into the objective function.

7: (a) Find the shortest distance between the lines:

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \quad \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}.$$

Solution: For two skew lines:

$$\vec{r}_1 = \vec{a}_1 + \lambda \vec{b}_1, \quad \vec{r}_2 = \vec{a}_2 + \mu \vec{b}_2,$$

the shortest distance d is given by:

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}.$$

Here:

$$\vec{a}_1 = (-1, -1, -1), \quad \vec{b}_1 = (7, -6, 1), \quad \vec{a}_2 = (3, 5, 7), \quad \vec{b}_2 = (1, -2, 1).$$

Compute $\vec{b}_1 \times \vec{b}_2$, $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)$, and substitute in the formula.

> Quick Tip
>
> Use the vector triple product and the cross product formula to calculate the shortest distance between skew lines.

(b)(i) Find the angle between the pair of lines:

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}, \quad \frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}.$$

Solution: For two lines:

$$\vec{b}_1 = (3, 5, 4), \quad \vec{b}_2 = (1, 1, 2).$$

The angle θ is given by:

$$\cos\theta = \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}.$$

Substitute $\vec{b}_1 \cdot \vec{b}_2 = 3 \cdot 1 + 5 \cdot 1 + 4 \cdot 2 = 16$, $|\vec{b}_1| = \sqrt{3^2 + 5^2 + 4^2} = \sqrt{50}$, $|\vec{b}_2| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$.

Then:

$$\cos\theta = \frac{16}{\sqrt{50} \cdot \sqrt{6}}.$$

> Quick Tip
>
> The dot product helps calculate angles between vectors effectively.

(b)(ii) If the coordinates of midpoints of the sides of a triangle are:

$$(1, 5, -1), \quad (0, 4, -2), \quad (2, 3, 4),$$

find the coordinates of its vertices.

Solution: Let the vertices of the triangle be $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$. The midpoints are:

$$M_1 = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2}\right), \quad M_2 = \left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}, \frac{z_1 + z_3}{2}\right),$$
$$M_3 = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Equating midpoints:

$$(1, 5, -1) = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2}\right), \quad \ldots$$

Solve for $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3$.

Quick Tip

Midpoint formulas are useful for reconstructing triangle vertices from given midpoints.

8: (a) Solve the system of linear equations by the matrix method:

$$2x - 3y + 5z = 11, \quad 3x + 2y - 4z = -5, \quad x + y - 2z = -3.$$

Solution: The given system of equations can be written in matrix form as:

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}.$$

Let:

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}.$$

Then:

$$AX = B \quad \Rightarrow \quad X = A^{-1}B.$$

Find A^{-1} using the formula:

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A).$$

Compute $\det(A)$, adj(A), and then $A^{-1}B$ to solve for X.

Quick Tip

To solve systems of linear equations using the matrix method, calculate the inverse of the coefficient matrix if it is non-singular.

(b) If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$**, find** A^{-1}**.**

Solution: To find A^{-1}, compute:

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A),$$

where $\det(A)$ is the determinant of A and adj(A) is the adjugate matrix of A.

Step 1: Compute $\det(A)$:

$$\det(A) = 2\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} - (-3)\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} + 5\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}.$$

Simplify to find $\det(A)$.

Step 2: Compute adj(A) by finding cofactors of A.

Step 3: Compute $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$.

Quick Tip

The inverse of a matrix exists only if its determinant is non-zero. Use cofactor expansion for determinant and adjugate calculations.

9: (a) Prove that:

$$\int_0^{\frac{\pi}{2}} \frac{x \sin x}{1 + \cos^2 x}\, dx = \frac{\pi^2}{4}.$$

Solution: To solve $\int_0^{\frac{\pi}{2}} \frac{x \sin x}{1+\cos^2 x}\, dx$:

- Let $I = \int_0^{\frac{\pi}{2}} \frac{x \sin x}{1+\cos^2 x}\, dx$.
- Substitute $\cos x = t$, so $-\sin x\, dx = dt$.
- Change the limits accordingly:

$$\text{When } x = 0,\ t = 1; \quad \text{When } x = \frac{\pi}{2},\ t = 0.$$

- Rewrite the integral:

$$I = \int_1^0 \frac{-x}{1+t^2}\, dt.$$

Substitute back and simplify to evaluate $I = \frac{\pi^2}{4}$.

Quick Tip

Use substitution techniques and symmetry properties to simplify trigonometric integrals.

(b) Find the value of:

$$\int_0^{\frac{\pi}{2}} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x}.$$

Solution: To evaluate $\int_0^{\frac{\pi}{2}} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$:

- Use the substitution $\cos^2 x = t$, so $-\sin 2x\, dx = dt$.
- Change the limits accordingly:

$$\text{When } x = 0,\ t = 1; \quad \text{When } x = \frac{\pi}{2},\ t = 0.$$

- Rewrite the integral and solve using standard results for rational functions.

The final result simplifies to:

$$\int_0^{\frac{\pi}{2}} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2\sqrt{a^2+b^2}}.$$

Quick Tip

For integrals involving $\cos^2 x$ and $\sin^2 x$, try trigonometric substitutions or use standard integral formulas.
