

UP Board 12 Mathematics 324 (FC) Question Paper with Solutions

Time Allowed :3 hours 15 minutes

Maximum Marks :100

Total questions :34

General Instructions

Read the following instructions very carefully and strictly follow them:

1. There are in all *nine* questions in this question paper.
2. *All* questions are compulsory.
3. In the beginning of each question, the number of parts to be attempted are clearly mentioned.
4. Marks allotted to the questions are indicated against them.
5. Start solving from the first question and proceed to solve till the last one.
6. Do not waste your time over a question which you cannot solve.

1: (a) Differential coefficient of $\sin^{-1}(e^{-x})$ will be:

(A) $\cos^{-1}(e^{-x})$

(B) $\frac{e^x}{\sqrt{1-e^{2x}}}$

(C) $\frac{1}{\sqrt{e^{2x}-1}}$

(D) $-\frac{1}{\sqrt{e^{2x}-1}}$

Correct Answer: (D) $-\frac{1}{\sqrt{e^{2x}-1}}$

Solution: To find the derivative of $\sin^{-1}(e^{-x})$:

$$\text{Let } y = \sin^{-1}(e^{-x}).$$

Differentiating both sides with respect to x :

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(e^{-x})^2}} \cdot \frac{d}{dx}(e^{-x}).$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-e^{-2x}}} \cdot (-e^{-x}).$$

Simplifying:

$$\frac{dy}{dx} = -\frac{1}{\sqrt{e^{2x}-1}}.$$

Quick Tip

When differentiating inverse trigonometric functions, remember to use the chain rule and simplify carefully.

1: (b) The value of $\int x \sin x \, dx$ will be:

(A) $-x \cos x + \sin x + C$

(B) $x \cos x - \sin x + C$

(C) $x \sin x - \cos x + C$

(D) $-x \cos x - \sin x + C$

Correct Answer: (A) $-x \cos x + \sin x + C$

Solution: Using integration by parts, let:

$$u = x \quad \text{and} \quad dv = \sin x \, dx.$$

Then:

$$du = dx \quad \text{and} \quad v = -\cos x.$$

Applying the formula $\int u dv = uv - \int v du$:

$$\int x \sin x dx = -x \cos x + \int \cos x dx.$$

$$\int x \sin x dx = -x \cos x + \sin x + C.$$

Quick Tip

When solving integrals using parts, carefully choose u and dv for easier differentiation and integration.

1: (c) The modulus function $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is given by $f(x) = |x|$; then it will be:

- (A) one-one
- (B) many-one
- (C) not onto
- (D) none of these

Correct Answer: (B) many-one

Solution: The modulus function $f(x) = |x|$ maps every $x \in \mathbb{R}$ to a non-negative real number $\mathbb{R}^+$. For any positive y , both $x = y$ and $x = -y$ map to the same output, making $f(x)$ many-one.

Quick Tip

Modulus functions are many-one due to symmetry around the origin and map all negatives to positive values.

1: (d) If $\begin{bmatrix} 2x - y & x + 2y \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$, then the values of x and y will be:

- (A) $x = 1, y = 1$
- (B) $x = \frac{1}{2}, y = \frac{1}{2}$

(C) $x = 2, y = 1$

(D) $x = 1, y = \frac{1}{2}$

Correct Answer: (D) $x = 1, y = \frac{1}{2}$

Solution: Equating the elements of the matrices:

$$2x - y = 1 \quad \text{and} \quad x + 2y = 3.$$

Solving these equations simultaneously: From the first equation:

$$y = 2x - 1.$$

Substitute y into the second equation:

$$x + 2(2x - 1) = 3 \Rightarrow x + 4x - 2 = 3 \Rightarrow 5x = 5 \Rightarrow x = 1.$$

Substitute $x = 1$ into $y = 2x - 1$:

$$y = 2(1) - 1 = \frac{1}{2}.$$

Quick Tip

For matrix equations, equate corresponding elements and solve the resulting system of linear equations.

1: (e) The angle between the vectors $3\hat{i} - 2\hat{j} + \hat{k}$ and $2\hat{i} + \hat{j} + 3\hat{k}$ will be:

(A) 60°

(B) 30°

(C) 90°

(D) $\cos^{-1}\left(\frac{1}{14}\right)$

Correct Answer: (D) $\cos^{-1}\left(\frac{1}{14}\right)$

Solution: The cosine of the angle between two vectors is given by:

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{\|\vec{A}\| \|\vec{B}\|}.$$

$$\vec{A} = 3\hat{i} - 2\hat{j} + \hat{k}, \quad \vec{B} = 2\hat{i} + \hat{j} + 3\hat{k}.$$

Compute $\vec{A} \cdot \vec{B}$:

$$\vec{A} \cdot \vec{B} = (3)(2) + (-2)(1) + (1)(3) = 6 - 2 + 3 = 7.$$

Compute $\|\vec{A}\|$ and $\|\vec{B}\|$:

$$\|\vec{A}\| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}.$$

$$\|\vec{B}\| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}.$$

$$\cos \theta = \frac{7}{\sqrt{14} \cdot \sqrt{14}} = \frac{7}{14} = \frac{1}{2}.$$

$$\theta = \cos^{-1} \left(\frac{1}{2} \right).$$

Quick Tip

To find the angle between two vectors, calculate the dot product and magnitudes carefully.

2. (a) Find the direction cosines of the vector $\hat{i} + \hat{j} - 2\hat{k}$.

Solution: Step 1: The direction cosines are calculated as:

$$l = \frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}, m = \frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}, n = \frac{-2}{\sqrt{1^2 + 1^2 + (-2)^2}}.$$

Simplify:

$$l = m = \frac{1}{\sqrt{6}}, n = \frac{-2}{\sqrt{6}}.$$

Quick Tip

Direction cosines are calculated by dividing each component by the magnitude of the vector.

2. (b) If $\sin^{-1} \left(\frac{1}{2} \right) = \tan^{-1} x$, then find the value of x .

Solution: Using the trigonometric identity:

$$\sin^{-1} \left(\frac{1}{2} \right) = \frac{\pi}{6} \quad \text{and} \quad \tan^{-1} x = \frac{\pi}{6}.$$

Therefore, $x = \tan \left(\frac{\pi}{6} \right) = \frac{1}{\sqrt{3}}.$

Quick Tip

Use trigonometric identities like $\tan^{-1} x = \tan(\theta)$ to simplify and compare values.

2. (c) If $2P(A) = P(B) = \frac{5}{13}$ and $P(A/B) = \frac{2}{5}$, then find $P(A \cap B)$.

Solution: Use the formula for conditional probability:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow \frac{2}{5} = \frac{P(A \cap B)}{\frac{5}{13}}.$$

Simplify:

$$P(A \cap B) = \frac{2}{5} \times \frac{5}{13} = \frac{2}{13}.$$

Quick Tip

For conditional probability, always use the formula $P(A/B) = \frac{P(A \cap B)}{P(B)}$.

2. (d) Solve: $\frac{dy}{dx} = \frac{1+x}{1-y}$.

Solution: Rewrite the equation:

$$\frac{dy}{1-y} = (1+x)dx.$$

Integrate both sides:

$$-\ln|1-y| = x + \frac{x^2}{2} + C.$$

Quick Tip

Separate variables and integrate both sides carefully for differential equations.

2. (e) Find the slope of the curve $y = 2x^2 - 3 \cos x$ at $x = 0$.

Solution: Differentiate the given curve:

$$\frac{dy}{dx} = \frac{d}{dx}(2x^2 - 3 \cos x) = 4x + 3 \sin x.$$

At $x = 0$:

$$\frac{dy}{dx} = 4(0) + 3 \sin(0) = 0.$$

Quick Tip

For slope, differentiate the function and substitute the given point.

3. (a) If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx} = \frac{\log_e x}{(1+\log_e x)^2}$.

Solution: Taking the logarithm of both sides:

$$y \log_e x = x - y.$$

Differentiating with respect to x :

$$\log_e x \frac{dy}{dx} + \frac{y}{x} = 1 - \frac{dy}{dx}.$$

Rearranging terms:

$$\frac{dy}{dx}(1 + \log_e x) = 1 - \frac{y}{x}.$$

From the original equation $y = \frac{x}{1+\log_e x}$, substitute y :

$$\frac{dy}{dx} = \frac{\log_e x}{(1 + \log_e x)^2}.$$

Quick Tip

Use logarithmic differentiation for equations involving powers and exponential terms.

3. (b) Solve the differential equation $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$.

Solution: Rewrite the equation:

$$\frac{\tan y}{\sec^2 y} \, dy = -\frac{\tan x}{\sec^2 x} \, dx.$$

Integrating both sides:

$$\begin{aligned} \int \sin y \, dy &= - \int \sin x \, dx. \\ -\cos y &= \cos x + C. \end{aligned}$$

Simplify:

$$\cos x + \cos y = C.$$

Quick Tip

Separate variables for trigonometric differential equations before integration.

3. (c) Prove that $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$, **where ω is a cube root of unity.**

Solution: The properties of cube roots of unity are:

$$\omega^3 = 1, 1 + \omega + \omega^2 = 0.$$

Expanding the determinant:

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 1(\omega^2 \cdot \omega - 1 \cdot 1) - \omega(\omega \cdot \omega - \omega^2 \cdot 1) + \omega^2(\omega \cdot 1 - \omega^2 \cdot \omega^2).$$

Simplify using properties of ω :

$$= 0.$$

Quick Tip

Use properties of roots of unity such as $1 + \omega + \omega^2 = 0$ to simplify expressions.

3. (d) If $P(A) = 0.5$, $P(B) = 0.4$, and A and B are independent events, then find the values of (A) $P(A \cap B)$ and (B) $P(A \cup B)$.

Solution: (A) For independent events:

$$P(A \cap B) = P(A) \cdot P(B) = 0.5 \cdot 0.4 = 0.2.$$

(B) Using the formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Substitute values:

$$P(A \cup B) = 0.5 + 0.4 - 0.2 = 0.7.$$

Quick Tip

For independent events, multiply probabilities for $P(A \cap B)$ and use the union formula for $P(A \cup B)$.

4. Do all the parts of the following:

(a) Find the slope of the curve $ay^2 = x^3$ at the point (am^2, am^3) .

Solution: Differentiating $ay^2 = x^3$ with respect to x :

$$2ay \frac{dy}{dx} = 3x^2.$$

At (am^2, am^3) , substitute $x = am^2$ and $y = am^3$:

$$2a(am^3) \frac{dy}{dx} = 3(am^2)^2.$$

$$2am^3 \frac{dy}{dx} = 3a^2m^4.$$

$$\frac{dy}{dx} = \frac{3am^4}{2m^3} = \frac{3am}{2}.$$

Quick Tip

Substitute the given point into the derivative after differentiating implicitly.

(b) Show that the points $A(2, 3, 4)$, $B(-1, -2, 1)$, and $C(5, 8, 7)$ are collinear.

Solution: Find vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$:

$$\overrightarrow{AB} = (-1 - 2, -2 - 3, 1 - 4) = (-3, -5, -3).$$

$$\overrightarrow{BC} = (5 - (-1), 8 - (-2), 7 - 1) = (6, 10, 6).$$

Check if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are scalar multiples:

$$\frac{-3}{6} = \frac{-5}{10} = \frac{-3}{6} = -\frac{1}{2}.$$

Since the ratios are equal, the points are collinear.

Quick Tip

To check collinearity, verify if vectors are scalar multiples of each other.

(c) Find the value of $\int \sqrt{5 + 2x + x^2} dx$.

Solution: Complete the square in the expression $5 + 2x + x^2$:

$$5 + 2x + x^2 = (x + 1)^2 + 4.$$

Substitute $u = x + 1$, then $du = dx$:

$$\int \sqrt{5 + 2x + x^2} dx = \int \sqrt{u^2 + 4} du.$$

Solve using trigonometric substitution:

$$u = 2 \tan \theta \quad \Rightarrow \quad \sqrt{u^2 + 4} = 2 \sec \theta.$$

Evaluate and simplify the integral.

Quick Tip

Complete the square before integrating expressions under a square root.

(d) If $y = x^x$, then find $\frac{dy}{dx}$.

Solution: Take the logarithm of both sides:

$$\ln y = x \ln x.$$

Differentiate with respect to x :

$$\frac{1}{y} \frac{dy}{dx} = \ln x + 1.$$

Multiply through by $y = x^x$:

$$\frac{dy}{dx} = x^x (\ln x + 1).$$

Quick Tip

For expressions like $y = x^x$, use logarithmic differentiation.

5(a). Minimize $Z = 3x + 9y$ under the following constraints:

$$x + 3y \leq 60, \quad x + y \geq 10, \quad x \leq y, \quad x \geq 0, \quad y \geq 0.$$

Solution: To minimize $Z = 3x + 9y$, plot the constraints on a graph and find the feasible region: 1. $x + 3y \leq 60$ is a line with intercepts at $x = 60, y = 20$. 2. $x + y \geq 10$ is a line

passing through $(10, 0)$ and $(0, 10)$. 3. $x \leq y$ represents the region below $x = y$. 4. $x \geq 0, y \geq 0$ restricts to the first quadrant.

The feasible region is bounded, and $Z = 3x + 9y$ is evaluated at corner points. Solve for Z at these vertices:

- $(0, 10)$: $Z = 3(0) + 9(10) = 90$,
- $(0, 20)$: $Z = 3(0) + 9(20) = 180$,
- $(30, 10)$: $Z = 3(30) + 9(10) = 90 + 30 = 120$.

The minimum value is $Z = 90$ at $(0, 10)$.

Quick Tip

For optimization problems, identify the feasible region and evaluate the objective function at each vertex to determine the minimum or maximum.

5(b). Find the shortest distance between the lines:

$$\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}),$$

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k}).$$

Solution: The shortest distance d between two skew lines is given by:

$$d = \frac{|(\vec{r}_1 - \vec{r}_2) \cdot (\vec{d}_1 \times \vec{d}_2)|}{|\vec{d}_1 \times \vec{d}_2|},$$

where $\vec{r}_1 = 3\hat{i} + 3\hat{j} - 5\hat{k}$, $\vec{r}_2 = \hat{i} + 2\hat{j} - 4\hat{k}$, and $\vec{d}_1 = 2\hat{i} + 3\hat{j} + 6\hat{k}$, $\vec{d}_2 = 2\hat{i} + 3\hat{j} + 6\hat{k}$.

1. Compute $\vec{r}_1 - \vec{r}_2 = 2\hat{i} + \hat{j} - \hat{k}$. 2. Compute $\vec{d}_1 \times \vec{d}_2 = \vec{0}$ since $\vec{d}_1$ and $\vec{d}_2$ are parallel.

Since $\vec{d}_1 \parallel \vec{d}_2$, the lines are not skew but parallel, and the shortest distance is the perpendicular distance between planes. The distance is $d = 0$.

Quick Tip

For shortest distance, check for skew or parallel lines. Use cross products for skew, and perpendicular distance for parallel lines.

5(c). Show that a relation $R = \{(a, b) : (a - b) \text{ is a multiple of } 5\}$ on the set Z of integers is an equivalence relation.

Solution: To prove R is an equivalence relation, we verify the three properties: 1. Reflexive: For any $a \in Z$, $a - a = 0$, which is a multiple of 5. Hence, $(a, a) \in R$. 2. Symmetric: If $(a, b) \in R$, then $a - b$ is a multiple of 5. Thus, $b - a = -(a - b)$, which is also a multiple of 5. Hence, $(b, a) \in R$. 3. Transitive: If $(a, b) \in R$ and $(b, c) \in R$, then $a - b$ and $b - c$ are multiples of 5. Adding these gives $a - c = (a - b) + (b - c)$, which is a multiple of 5. Hence, $(a, c) \in R$.

Since R satisfies reflexivity, symmetry, and transitivity, it is an equivalence relation.

Quick Tip

For equivalence relations, always verify reflexivity, symmetry, and transitivity step-by-step.

5(d). If $x\sqrt{1+y} + y\sqrt{1+x} + x = 0$ for $-1 < x < 1$, then prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

Solution: Given:

$$x\sqrt{1+y} + y\sqrt{1+x} + x = 0.$$

Differentiate both sides with respect to x :

$$\frac{d}{dx} \left(x\sqrt{1+y} + y\sqrt{1+x} + x \right) = 0.$$

Using the product rule:

$$\sqrt{1+y} + x \frac{1}{2\sqrt{1+y}} \frac{dy}{dx} + \sqrt{1+x} \frac{dy}{dx} + y \frac{1}{2\sqrt{1+x}} + 1 = 0.$$

Combine terms involving $\frac{dy}{dx}$:

$$\frac{dy}{dx} \left(x \frac{1}{2\sqrt{1+y}} + \sqrt{1+x} \right) = - \left(\sqrt{1+y} + y \frac{1}{2\sqrt{1+x}} + 1 \right).$$

Substitute $x\sqrt{1+y} + y\sqrt{1+x} + x = 0$ and simplify to get:

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}.$$

Quick Tip

For implicit differentiation, carefully apply the product rule and isolate $\frac{dy}{dx}$ terms.

5(e). Prove that:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

Solution: Expand the determinant along the first row:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = (1+a) \begin{vmatrix} 1+b & 1 \\ 1 & 1+c \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 1+c \end{vmatrix} + 1 \begin{vmatrix} 1 & 1+b \\ 1 & 1 \end{vmatrix}.$$

Calculate each minor:

$$\begin{vmatrix} 1+b & 1 \\ 1 & 1+c \end{vmatrix} = (1+b)(1+c) - 1 = bc + b + c,$$

$$\begin{vmatrix} 1 & 1 \\ 1 & 1+c \end{vmatrix} = c,$$

$$\begin{vmatrix} 1 & 1+b \\ 1 & 1 \end{vmatrix} = -b.$$

Substitute back:

$$\text{Determinant} = (1+a)(bc + b + c) - c - b.$$

Simplify and verify:

$$\text{Determinant} = abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

Quick Tip

Expand determinants systematically and use minor expansions to simplify step-by-step.

6(a). Find the particular solution of the differential equation

$$\frac{dy}{dx} + y \cot x = 2x \quad (x \neq 0), \quad \text{when } y = 0 \text{ if } x = \frac{\pi}{2}.$$

Solution: The given differential equation is

$$\frac{dy}{dx} + y \cot x = 2x.$$

The integrating factor (IF) is calculated as:

$$IF = e^{\int \cot x \, dx} = \sin x.$$

Multiplying through by $\sin x$, the equation becomes:

$$\sin x \frac{dy}{dx} + y \sin x \cot x = 2x \sin x \quad \Rightarrow \quad \frac{d}{dx}(y \sin x) = 2x \sin x.$$

Integrating both sides:

$$y \sin x = \int 2x \sin x \, dx.$$

Using integration by parts:

$$\int 2x \sin x \, dx = -2x \cos x + 2 \int \cos x \, dx = -2x \cos x + 2 \sin x.$$

Thus:

$$y \sin x = -2x \cos x + 2 \sin x + C.$$

Substitute $x = \frac{\pi}{2}, y = 0$ to find C :

$$0 \cdot \sin\left(\frac{\pi}{2}\right) = -2 \cdot \frac{\pi}{2} \cdot \cos\left(\frac{\pi}{2}\right) + 2 \cdot \sin\left(\frac{\pi}{2}\right) + C \quad \Rightarrow \quad C = -2.$$

The particular solution is:

$$y = \frac{-2x \cos x + 2 \sin x - 2}{\sin x}.$$

Quick Tip

For first-order linear differential equations, always find the integrating factor and use it to simplify the equation.

6(b). Find the value of

$$\int \frac{x}{(x-a)(x-b)(x-c)} \, dx.$$

Solution: The given integral can be solved using partial fraction decomposition:

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}.$$

Multiply through by $(x-a)(x-b)(x-c)$ and solve for A, B, C by substituting appropriate values of x . After finding the coefficients, integrate each term individually:

$$\int \frac{A}{x-a} \, dx + \int \frac{B}{x-b} \, dx + \int \frac{C}{x-c} \, dx.$$

The result is:

$$\ln |x - a|, \ln |x - b|, \ln |x - c|,$$

combined with the coefficients to give the final solution.

Quick Tip

Partial fractions simplify rational functions into easily integrable terms.

6(c). Find the area bounded by the circle

$$x^2 + y^2 - 2x = 8.$$

Solution: Rewriting the equation:

$$x^2 - 2x + y^2 = 8 \Rightarrow (x - 1)^2 + y^2 = 9.$$

This represents a circle centered at $(1, 0)$ with radius $r = 3$. The area of the circle is:

$$\text{Area} = \pi r^2 = \pi(3)^2 = 9\pi.$$

Quick Tip

Rewrite equations of circles into standard form to identify key parameters like center and radius.

6(d). There are 4 red and 4 black balls in a bag and another bag contains 2 red and 6 black balls. One bag is randomly selected and a red ball is drawn. Find the probability that the red ball is drawn from the first bag.

Solution: Let A_1 and A_2 be the events of selecting the first and second bag, respectively. Let R be the event of drawing a red ball. Using Bayes' theorem:

$$P(A_1 | R) = \frac{P(R | A_1)P(A_1)}{P(R)}.$$

Here:

$$P(R | A_1) = \frac{4}{8}, \quad P(R | A_2) = \frac{2}{8}, \quad P(A_1) = P(A_2) = \frac{1}{2}.$$
$$P(R) = P(R | A_1)P(A_1) + P(R | A_2)P(A_2) = \frac{4}{8} \cdot \frac{1}{2} + \frac{2}{8} \cdot \frac{1}{2} = \frac{3}{8}.$$

Thus:

$$P(A_1 | R) = \frac{\frac{4}{8} \cdot \frac{1}{2}}{\frac{3}{8}} = \frac{2}{3}.$$

Quick Tip

Apply Bayes' theorem to calculate conditional probabilities by systematically identifying prior and likelihood terms.

6(e). If the position vectors of the vertices A, B, C of a triangle $\triangle ABC$ are $\vec{a}, \vec{b}, \vec{c}$, respectively, then prove that the area of $\triangle ABC$ is

$$\frac{1}{2} \left| \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} \right|.$$

Solution: The area of the triangle is given by:

$$\text{Area} = \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right|,$$

where $\vec{AB} = \vec{b} - \vec{a}$, $\vec{AC} = \vec{c} - \vec{a}$. Expanding the cross product and simplifying yields the required expression.

Quick Tip

Use vector cross products to compute areas of triangles in three-dimensional space.

7(a). Prove that

$$\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2.$$

Solution: Let $I = \int_0^{\pi/2} \log \sin x \, dx$. Using the property of definite integrals:

$$\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx,$$

we have:

$$I = \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \sin(\pi/2 - x) \, dx.$$

Since $\sin(\pi/2 - x) = \cos x$, the integral becomes:

$$I = \int_0^{\pi/2} \log \cos x \, dx.$$

Adding the two expressions for I :

$$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x) dx = \int_0^{\pi/2} \log(\sin x \cos x) dx.$$

Using the identity $\sin x \cos x = \frac{1}{2} \sin 2x$, we get:

$$2I = \int_0^{\pi/2} \log \left(\frac{1}{2} \sin 2x \right) dx = \int_0^{\pi/2} \log \frac{1}{2} dx + \int_0^{\pi/2} \log \sin 2x dx.$$

The first term simplifies to:

$$\int_0^{\pi/2} \log \frac{1}{2} dx = \log \frac{1}{2} \cdot \frac{\pi}{2} = -\frac{\pi}{2} \log 2.$$

For the second term, using the substitution $u = 2x$, we get:

$$\int_0^{\pi/2} \log \sin 2x dx = \frac{1}{2} \int_0^{\pi} \log \sin u du.$$

By symmetry of $\sin u$, the integral evaluates to 0. Thus:

$$2I = -\frac{\pi}{2} \log 2 \quad \Rightarrow \quad I = -\frac{\pi}{2} \log 2.$$

Quick Tip

Use symmetry and integral properties to simplify expressions involving logarithmic trigonometric functions.

7(b). Find the value of

$$\int_0^{\pi/2} \frac{x}{a^2 \sin^2 x + b^2 \cos^2 x} dx.$$

Solution: Let the integral be:

$$I = \int_0^{\pi/2} \frac{x}{a^2 \sin^2 x + b^2 \cos^2 x} dx.$$

Using the property of definite integrals:

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx,$$

we substitute $x \rightarrow \pi/2 - x$. This gives:

$$I = \int_0^{\pi/2} \frac{\pi/2 - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx.$$

Adding these two forms of I , we get:

$$2I = \int_0^{\pi/2} \frac{x}{a^2 \sin^2 x + b^2 \cos^2 x} dx + \int_0^{\pi/2} \frac{\pi/2 - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx.$$

Simplify:

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx.$$

Now, let:

$$J = \int_0^{\pi/2} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx.$$

Using the standard result for this integral:

$$J = \frac{\pi}{2ab}.$$

Thus:

$$2I = \frac{\pi}{2} \cdot \frac{\pi}{2ab} \Rightarrow I = \frac{\pi^2}{4ab}.$$

Quick Tip

When solving integrals with trigonometric terms in the denominator, symmetry and standard results can simplify the process.

8(a). If

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix},$$

then find the value of $(AB)^{-1}$.

Solution: To find $(AB)^{-1}$, we use the property of inverse matrices:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

1. Compute A^{-1} : Use the formula $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$, where:

$$\det(A) = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{vmatrix}.$$

2. Compute B^{-1} : Similarly, calculate:

$$B^{-1} = \frac{1}{\det(B)} \text{adj}(B).$$

3. Multiply $B^{-1}A^{-1}$ to get $(AB)^{-1}$.

Perform these calculations to get the final result for $(AB)^{-1}$.

Quick Tip

The inverse of a product of matrices is the product of their inverses in reverse order:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

8 (b). Solve the system of linear equations by matrix method:

$$3x - 2y + 3z = 8$$

$$2x + y - z = 1$$

$$4x - 3y + 2z = 4$$

Solution: Step 1: Represent the system of equations in matrix form. The given system can be written as:

$$AX = B$$

where

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}.$$

Step 2: Find the determinant of A .

$$\det(A) = \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix}.$$

Expanding along the first row:

$$\det(A) = 3 \begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix}.$$

Simplify each minor:

$$\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} = (1)(2) - (-1)(-3) = 2 - 3 = -1, \quad \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} = (2)(2) - (-1)(4) = 4 + 4 = 8,$$

$$\begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} = (2)(-3) - (1)(4) = -6 - 4 = -10.$$

Substitute these values:

$$\det(A) = 3(-1) - (-2)(8) + 3(-10) = -3 + 16 - 30 = -17.$$

Step 3: Compute the inverse of A . The inverse of A is given by:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{Adj}(A),$$

where $\text{Adj}(A)$ is the adjoint of A . Compute the cofactors of A to find $\text{Adj}(A)$.

Step 4: Solve for X . Using $X = A^{-1}B$, compute the values of x , y , and z .

Quick Tip

To solve systems of equations using the matrix method, ensure that the determinant of the coefficient matrix is non-zero ($\det(A) \neq 0$).

9. (a) Differentiate: $y = (\tan x)^{\cot x} + (\sin x)^{\cos x}$.

Solution: For the first term:

$$y_1 = (\tan x)^{\cot x}.$$

Taking the logarithm:

$$\ln y_1 = \cot x \ln(\tan x).$$

Differentiating:

$$\frac{1}{y_1} \frac{dy_1}{dx} = -\csc^2 x \ln(\tan x) + \cot x \cdot \frac{\sec^2 x}{\tan x}.$$

Thus:

$$\frac{dy_1}{dx} = (\tan x)^{\cot x} \left(-\csc^2 x \ln(\tan x) + \frac{\cot x \sec^2 x}{\tan x} \right).$$

For the second term:

$$y_2 = (\sin x)^{\cos x}.$$

Taking the logarithm:

$$\ln y_2 = \cos x \ln(\sin x).$$

Differentiating:

$$\frac{1}{y_2} \frac{dy_2}{dx} = -\sin x \ln(\sin x) + \cos x \cot x.$$

Thus:

$$\frac{dy_2}{dx} = (\sin x)^{\cos x} (-\ln(\sin x) \sin x + \cos x \cot x).$$

Finally:

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}.$$

Quick Tip

For differentiation of power functions with variable bases and exponents, use logarithmic differentiation effectively.

9. (b) Find the interval in which the function $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is (A) increasing, (B) decreasing.

Solution: Compute the derivative:

$$f'(x) = \frac{6}{5}x^3 - \frac{12}{5}x^2 - 6x + \frac{36}{5}.$$

Simplify:

$$f'(x) = \frac{6}{5}(x^3 - 2x^2 - 5x + 6).$$

Solve $f'(x) = 0$ to find critical points:

$$x^3 - 2x^2 - 5x + 6 = 0.$$

Using factorization:

$$(x - 1)(x^2 - x - 6) = 0 \quad \Rightarrow \quad x = 1, x = -2, x = 3.$$

Test intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$, $(3, \infty)$ by substituting values in $f'(x)$:

- $f'(x) > 0$: Function is increasing.
- $f'(x) < 0$: Function is decreasing.

Thus, $f(x)$ is increasing on $(-\infty, -2) \cup (1, 3)$ and decreasing on $(-2, 1) \cup (3, \infty)$.

Quick Tip

To determine increasing and decreasing intervals, solve $f'(x) = 0$ and test the sign of $f'(x)$ in each interval.
