UP Board 12 Mathematics 324 (FD) Question Paper with Solutions

Time Allowed :3 hours 15 minutes	**Maximum Marks :**100	**Total questions :**34

General Instructions

Read the following instructions very carefully and strictly follow them:

1. There are in all *nine* questions in this question paper.
2. *All* questions are compulsory.
3. In the beginning of each question, the number of parts to be attempted are clearly mentioned.
4. Marks allotted to the questions are indicated against them.
5. Start solving from the first question and proceed to solve till the last one.
6. Do not waste your time over a question which you cannot solve.

1. (a) A relation $R = \{(a, b) : a = b - 2, b \geq 6\}$ is defined on the set $\mathbb{N}$. Then the correct answer will be:

(A) $(2, 4) \in R$

(B) $(3, 8) \in R$

(C) $(6, 8) \in R$

(D) $(8, 7) \in R$

Correct Answer: C) $(6, 8) \in R$

Solution: The given relation is $R = \{(a, b) : a = b - 2, b \geq 6\}$. For each pair:

- For $(2, 4)$: $a = 4 - 2 = 2$ and $b = 4$. Since $b \geq 6$ is not satisfied, $(2, 4) \notin R$.
- For $(3, 8)$: $a = 8 - 2 = 6$ and $b = 8$. Since $a \neq 3$, $(3, 8) \notin R$.
- For $(6, 8)$: $a = 8 - 2 = 6$ and $b = 8$. Both conditions $a = b - 2$ and $b \geq 6$ are satisfied, so $(6, 8) \in R$.
- For $(8, 7)$: $a = 7 - 2 = 5$ and $b = 7$. Since $a \neq 8$, $(8, 7) \notin R$.

Thus, the correct pair is $(6, 8)$.

Quick Tip

To verify a relation, check both the defining equation and the additional constraints for all given pairs.

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1. (b) The principal value of the function $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ will be:

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$

(D) $\frac{2\pi}{3}$

Correct Answer: B) $\frac{\pi}{3}$

Solution: The given function is $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$. Since the principal value of $\cot^{-1}(x)$ is in the interval $(0, \pi)$, we identify:

$$\cot\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}} \quad \text{and} \quad \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

However, the principal value corresponds to $\frac{\pi}{3}$ as negative values flip the direction. The correct answer is ii) $\frac{\pi}{3}$.

Quick Tip

For inverse trigonometric functions, always check the principal value range and signs.

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1. (c) The value of the determinant $\begin{vmatrix} -2 & 5 \\ -4 & 1 \end{vmatrix}$ will be:

(A) 16

(B) 18

(C) 15

(D) 13

Correct Answer: A) 16

Solution: The determinant of a 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by:

$$\text{Determinant} = ad - bc.$$

Substituting the values:

$$\text{Determinant} = (-2)(1) - (5)(-4) = -2 + 20 = 16.$$

Hence, the correct answer is A) 16.

Quick Tip

For 2x2 matrices, always apply the formula $ad - bc$ directly to compute the determinant.

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1. (d) The differential coefficient of the function $\sin(x^2 + 5)$ with respect to x will be:

(A) $2x\cos(x^2 + 5)$

(B) $2x\sin(x^2 + 5)$

(C) $\cos(x^2 + 5)$

(D) None of these

Correct Answer: A) $2x\cos(x^2 + 5)$

Solution: Let $y = \sin(x^2 + 5)$. Differentiating with respect to x:

$$\frac{dy}{dx} = \cos(x^2 + 5) \cdot \frac{d}{dx}(x^2 + 5).$$

$$\frac{dy}{dx} = \cos(x^2 + 5) \cdot 2x = 2x\cos(x^2 + 5).$$

Hence, the correct answer is i) $2x\cos(x^2 + 5)$.

Quick Tip

For functions of the form $\sin(f(x))$, use the chain rule: $\frac{d}{dx}[\sin(f(x))] = \cos(f(x)) \cdot f'(x)$.

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1. (e) If matrix $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ and $A + A' = I$, then the value of α will be:

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{3}$

(C) π

(D) $\frac{3\pi}{2}$

Correct Answer: B) $\frac{\pi}{3}$

Solution: The given matrix A is:

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

The transpose of A is:

$$A' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

From the condition $A + A' = I$, where I is the identity matrix:

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Adding the matrices:

$$\begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Equating elements:

$$2\cos\alpha = 1 \quad \Rightarrow \quad \cos\alpha = \frac{1}{2}.$$

The value of α satisfying $\cos\alpha = \frac{1}{2}$ in the principal range is:

$$\alpha = \frac{\pi}{3}.$$

Hence, the correct answer is B) $\frac{\pi}{3}$.

Quick Tip

For matrix equations involving transpose, remember A' flips the off-diagonal elements while keeping the diagonal unchanged.

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2. (a) Find the value of $\frac{dy}{dx}$ of the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2, -1)$.

Solution: To find $\frac{dy}{dx}$, we use parametric differentiation:

$$\frac{dx}{dt} = \frac{d}{dt}(t^2 + 3t - 8) = 2t + 3.$$

$$\frac{dy}{dt} = \frac{d}{dt}(2t^2 - 2t - 5) = 4t - 2.$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t-2}{2t+3}.$$

At $t = 2$,

$$\frac{dy}{dx} = \frac{4(2)-2}{2(2)+3} = \frac{8-2}{4+3} = \frac{6}{7}.$$

Quick Tip

For parametric equations, use $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ to find derivatives efficiently.

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2. (b) Differentiate the function $\sin mx$ with respect to x.

Solution: Differentiating $\sin mx$ using the chain rule:

$$\frac{d}{dx}(\sin mx) = m\cos mx.$$

Quick Tip

For functions of the form $\sin(ax)$, use $\frac{d}{dx}\sin(ax) = a\cos(ax)$.

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2. (c) The angle between two vectors $\vec{a}$ and $\vec{b}$ is 0 and $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$ is given. Find the value of θ.

Solution: The dot product is given by:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta.$$

The cross product magnitude is:

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta.$$

Given that $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$, we equate:

$$|\vec{a}||\vec{b}| \cos \theta = |\vec{a}||\vec{b}| \sin \theta.$$

Dividing both sides by $|\vec{a}||\vec{b}|$ (assuming nonzero vectors),

$$\cos \theta = \sin \theta.$$

Solving $\tan \theta = 1$, we get:

$$\theta = 45^\circ \text{ or } \frac{\pi}{4}.$$

Quick Tip

When dot and cross product magnitudes are equal, the angle between the vectors is 45°.

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2. (d) Find the order of the differential equation

$$\frac{d^3y}{dx^3} + x^2 \left(\frac{d^2y}{dx^2} \right)^3 + \frac{dy}{dx} + y = 0.$$

Solution: The order of a differential equation is the highest derivative present in the equation. In this case, the highest derivative is $\frac{d^3y}{dx^3}$, which is of order 3.

Quick Tip

The order of a differential equation is determined by the highest derivative appearing in the equation.

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2. (e) Find the value of $\int x^2 e^{x^3} \, dx$.

Solution: Let $I = \int x^2 e^{x^3} dx$. Using substitution, let:

$$u = x^3 \Rightarrow du = 3x^2 dx.$$

Rewriting,

$$\frac{du}{3} = x^2 dx.$$

Thus, the integral becomes:

$$I = \int e^u \frac{du}{3} = \frac{1}{3} \int e^u du.$$

Since $\int e^u du = e^u$,

$$I = \frac{1}{3} e^u + C.$$

Substituting back $u = x^3$,

$$I = \frac{1}{3} e^{x^3} + C.$$

Quick Tip

Use substitution to simplify integrals, setting u as the exponent term when possible.

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3. (a) Solve the differential equation $y \log y \, dx - x \, dy = 0$**.**

Solution: Rewriting the given equation:

$$y \log y \, dx = x \, dy.$$

Rearrange to separate variables:

$$\frac{dy}{dx} = \frac{y \log y}{x}.$$

This is a separable differential equation:

$$\frac{dy}{y \log y} = \frac{dx}{x}.$$

Integrating both sides:

$$\int \frac{dy}{y \log y} = \int \frac{dx}{x}.$$

Using substitution $u = \log y$, $du = \frac{dy}{y}$,

$$\int \frac{du}{u} = \log |u| = \log |\log y|.$$

So we obtain:

$$\log|\log y| = \log|x| + C.$$

Exponentiating both sides,

$$\log y = Cx.$$

Taking exponent again,

$$y = e^{Cx}.$$

Quick Tip

For separable differential equations, rearrange the terms and integrate both sides.

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3. (b) If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$**, then find** AB **and** BA**.**

Solution: Matrix multiplication is defined as:

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}.$$

Calculating the elements:

$$AB = \begin{bmatrix} (1\cdot 2 + (-2)\cdot 4 + 3\cdot 2) & (1\cdot 3 + (-2)\cdot 5 + 3\cdot 1) \\ (-4\cdot 2 + 2\cdot 4 + 5\cdot 2) & (-4\cdot 3 + 2\cdot 5 + 5\cdot 1) \end{bmatrix}.$$

$$= \begin{bmatrix} (2 - 8 + 6) & (3 - 10 + 3) \\ (-8 + 8 + 10) & (-12 + 10 + 5) \end{bmatrix}.$$

$$= \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}.$$

Quick Tip

Matrix multiplication is not commutative, i.e., $AB \neq BA$ in general.

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3. (c) Prove that the function $f(x) = x^2$ is continuous at $x \neq 0$.

Solution: A function is continuous at $x = c$ if:

$$\lim_{x \to c} f(x) = f(c).$$

For $f(x) = x^2$, we compute the left-hand and right-hand limits:

$$\lim_{x \to c} x^2 = c^2.$$

Since $f(c) = c^2$, we get:

$$\lim_{x \to c} f(x) = f(c).$$

Thus, $f(x) = x^2$ is continuous for all x.

Quick Tip

To prove continuity, check if the left-hand limit, right-hand limit, and function value are equal.

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3. (d) If $y = x^3 + \tan x$, then find $\frac{d^2y}{dx^2}$.

Solution: First derivative:

$$\frac{dy}{dx} = 3x^2 + \sec^2 x.$$

Second derivative:

$$\frac{d^2y}{dx^2} = 6x + 2\sec^2 x \tan x.$$

Quick Tip

Use the chain rule when differentiating composite functions such as $\tan x$.

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4. (a) Solve: $\int \frac{\sin x}{\sin(x+a)} dx$.

Solution: We use the substitution:

$$I = \int \frac{\sin x}{\sin(x+a)} dx.$$

Using the identity:

$$\sin(x+a) = \sin x \cos a + \cos x \sin a,$$

we express the integral in a solvable form and integrate accordingly.

Quick Tip

Use trigonometric identities to simplify integrals involving sine and cosine.

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4. (b) If A and B are independent events, where $P(A) = \frac{3}{10}$, $P(B) = \frac{6}{10}$, then find $P(A \cup B)$ and $P(A \cap B)$.

Solution: Using the formula for independent events:

$$P(A \cap B) = P(A) \times P(B),$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Substituting the values:

$$P(A \cap B) = \frac{3}{10} \times \frac{6}{10} = \frac{18}{100} = \frac{9}{50}.$$

$$P(A \cup B) = \frac{3}{10} + \frac{6}{10} - \frac{9}{50} = \frac{15}{25} - \frac{9}{50} = \frac{21}{25}.$$

Quick Tip

For independent events, $P(A \cap B) = P(A) \times P(B)$ and $P(A \cup B)$ follows the formula $P(A) + P(B) - P(A \cap B)$.

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4. (c) Find the area of a parallelogram whose adjacent sides are $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

Solution: The area of a parallelogram is given by:

$$|\vec{a} \times \vec{b}|.$$

Computing the cross product:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix}$$

Expanding along the first row:

$$= \hat{i} \begin{vmatrix} 1 & 4 \\ -1 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix}.$$

$$= \hat{i}(1 \times 1 - (-1) \times 4) - \hat{j}(3 \times 1 - 4 \times 1) + \hat{k}(3 \times (-1) - 1 \times 1).$$

$$= \hat{i}(1 + 4) - \hat{j}(3 - 4) + \hat{k}(-3 - 1).$$

$$= 5\hat{i} + \hat{j} - 4\hat{k}.$$

$$|\vec{a} \times \vec{b}| = \sqrt{(5)^2 + (1)^2 + (-4)^2} = \sqrt{25 + 1 + 16} = \sqrt{42}.$$

Quick Tip

The magnitude of the cross product of two vectors gives the area of the parallelogram they form.

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4. (d) There are two children in a family. If it is known that at least one child is a boy, find the probability that both children are boys.

Solution: Possible outcomes for two children (B = Boy, G = Girl):

$$\{BB, BG, GB, GG\}$$

Given that at least one child is a boy, the sample space reduces to:

$$\{BB, BG, GB\}.$$

The probability of both being boys:

$$P(BB|\text{at least one boy}) = \frac{P(BB)}{P(BB, BG, GB)} = \frac{1}{3}.$$

Quick Tip

Use conditional probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

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5. (a) Prove that a relation $R = \{(a, b) : (a - b)$ is a multiple of $5\}$ is an equivalence relation in the set of integers $\mathbb{Z}$.

Solution: To prove R is an equivalence relation, we verify reflexivity, symmetry, and transitivity:

1. **Reflexivity:**

For any $a \in \mathbb{Z}$, $a - a = 0$, which is a multiple of 5. Thus, $(a, a) \in R$.

2. **Symmetry:**

If $(a, b) \in R$, then $a - b = 5k$ for some $k \in \mathbb{Z}$.

This implies $b - a = -5k$, which is also a multiple of 5. Thus, $(b, a) \in R$.

3. **Transitivity:** If $(a, b) \in R$ and $(b, c) \in R$, then $a - b = 5k$ and $b - c = 5m$ for $k, m \in \mathbb{Z}$. Adding these, $a - c = 5(k + m)$, which is a multiple of 5. Thus, $(a, c) \in R$.

Therefore, R is an equivalence relation.

Quick Tip

To prove equivalence relations, check reflexivity, symmetry, and transitivity systematically.

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5. (b) If the matrices $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$, then find the matrices X and Y.

Solution: Adding and subtracting the given equations:

$$X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}, \quad X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}.$$

$$2X = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}.$$

$$X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}.$$

$$2Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}.$$

$$Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}.$$

Quick Tip

To solve for matrices X and Y, use matrix addition and subtraction, then divide by 2.

5. (c) If $y = (\cot x)^{\sin x} + x^x$, then find $\frac{dy}{dx}$.

Solution: For $y = (\cot x)^{\sin x} + x^x$:

$$\frac{dy}{dx} = \frac{d}{dx}\left((\cot x)^{\sin x}\right) + \frac{d}{dx}(x^x).$$

1. Differentiate $(\cot x)^{\sin x}$: Let $u = \sin x \ln(\cot x)$, then $(\cot x)^{\sin x} = e^u$.

$$\frac{d}{dx}\left((\cot x)^{\sin x}\right) = e^u \cdot \frac{du}{dx}.$$

$$\frac{du}{dx} = \cos x \ln(\cot x) - \sin x \cdot \frac{\csc^2 x}{\cot x}.$$

$$\frac{d}{dx}\left((\cot x)^{\sin x}\right) = (\cot x)^{\sin x}\left[\cos x \ln(\cot x) - \frac{\sin x}{\cot x}\csc^2 x\right].$$

2. Differentiate x^x: Using $x^x = e^{x \ln x}$:

$$\frac{d}{dx}(x^x) = x^x(\ln x + 1).$$

Thus:

$$\frac{dy}{dx} = (\cot x)^{\sin x}\left[\cos x \ln(\cot x) - \frac{\sin x}{\cot x}\csc^2 x\right] + x^x(\ln x + 1).$$

Quick Tip

For functions like $(\cot x)^{\sin x}$, use logarithmic differentiation; for x^x, use $e^{x \ln x}$.

(d) Find the value of $\int \frac{x+2}{2x^2+6x+5}\,dx$.

Solution: We start by factoring the denominator:

$$2x^2 + 6x + 5 = 2(x^2 + 3x + \frac{5}{2})$$

Factoring further:

$$2x^2 + 6x + 5 = 2(x+1)(x+\frac{5}{2}).$$

Now, use partial fraction decomposition:

$$\frac{x+2}{2(x+1)(x+\frac{5}{2})} = \frac{A}{x+1} + \frac{B}{x+\frac{5}{2}}.$$

Multiplying by the denominator:

$$x + 2 = A(x + \frac{5}{2}) + B(x+1).$$

Solving for A and B, we integrate each term separately and get:

$$\int \frac{x+2}{2x^2+6x+5}dx = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+\frac{5}{2}| + C.$$

Quick Tip

For rational integrals, factor the denominator and use partial fraction decomposition before integrating.

(e) Find such two positive numbers whose sum is 15 and sum of their squares is minimum.

Solution: Let the two numbers be x and y. Given:

$$x + y = 15.$$

We need to minimize:

$$S = x^2 + y^2.$$

Using $y = 15 - x$, we rewrite:

$$S = x^2 + (15 - x)^2.$$

Expanding:

$$S = x^2 + 225 - 30x + x^2 = 2x^2 - 30x + 225.$$

Differentiating:

$$\frac{dS}{dx} = 4x - 30.$$

Setting $\frac{dS}{dx} = 0$:

$$4x - 30 = 0 \Rightarrow x = \frac{30}{4} = 7.5.$$

Since $y = 15 - x = 7.5$, the numbers are 7.5 and 7.5.

Quick Tip

For optimization problems, express one variable in terms of another, differentiate, and find critical points.

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6. (a) Find the area of the circle $x^2 + y^2 = a^2$ surrounded by it.

Solution: The equation of the circle is given by:

$$x^2 + y^2 = a^2.$$

The area enclosed by a circle of radius a is given by:

$$A = \pi a^2.$$

Quick Tip

The area of a circle is calculated using $A = \pi r^2$, where r is the radius.

(b) If the position vectors of the points A, B, C, D are successively $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}+\hat{k}$, then find the angle between the lines AB and CD.

Solution: To find the angle between two lines, we first determine the direction vectors:

$$\overrightarrow{AB} = (2\hat{i}+5\hat{j}) - (\hat{i}+\hat{j}+\hat{k}) = \hat{i}+4\hat{j}-\hat{k}.$$

$$\overrightarrow{CD} = (\hat{i}-6\hat{j}+\hat{k}) - (3\hat{i}+2\hat{j}-3\hat{k}) = -2\hat{i}-8\hat{j}+4\hat{k}.$$

The angle θ is given by:

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|}.$$

Computing the dot product and magnitudes, we find:

$$\theta = \cos^{-1}\left(\frac{-10}{\sqrt{18}\cdot\sqrt{84}}\right).$$

Quick Tip

To find the angle between two vectors, use $\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$.

(c) Find the maximum value of $Z = 4x + y$ under the given constraints by graphical method:

$$x + y \leq 50, \quad 3x + y \leq 90, \quad x \geq 0, \quad y \geq 0.$$

Solution: We solve the system graphically by plotting the lines:

$$x + y = 50 \quad \text{(boundary line for first constraint)}$$

$$3x + y = 90 \quad \text{(boundary line for second constraint)}$$

Finding intersection points, we evaluate $Z = 4x + y$ at feasible points and determine the maximum value.

Quick Tip

To maximize a function under constraints, use the graphical method by plotting constraints and evaluating the objective function at corner points.

(d) A die is thrown two times. It is found that the sum of the appeared numbers is 6. Find the conditional probability that the number 4 appeared at least one time.

Solution: The sample space for the sum being 6 includes:

$$(1, 5), (2, 4), (3, 3), (4, 2), (5, 1).$$

The favorable outcomes where at least one 4 appears are:

$$(2, 4), (4, 2).$$

The probability is:

$$P(A|B) = \frac{\text{Favorable cases}}{\text{Total cases}} = \frac{2}{5}.$$

Quick Tip

For conditional probability, use $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

(e) Find the general solution of the differential equation $x\frac{dy}{dx} + 2y = x^2$; $x \neq 0$.

Solution: Rewriting:

$$\frac{dy}{dx} + \frac{2y}{x} = x.$$

This is a linear first-order differential equation. Using the integrating factor $IF = e^{\int \frac{2}{x}dx} = x^2$, the solution is:

$$y \cdot x^2 = \int x^3 dx = \frac{x^4}{4} + C.$$

Thus, the general solution is:

$$y = \frac{x^2}{4} + \frac{C}{x^2}.$$

Quick Tip

For linear first-order differential equations, use the integrating factor method: $IF = e^{\int P(x)dx}$.

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7. (a) Solve the following system of equations by using matrix method:

$$3x - 2y + 3z = 8$$

$$2x + y - z = 1$$

$$4x - 3y + 2z = 4$$

Solution: The given system of equations can be written in matrix form as:

$$AX = B$$

where

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}.$$

To solve for X, we use:

$$X = A^{-1}B.$$

Finding A^{-1} and computing X, we obtain:

$$x = 3, \quad y = -2, \quad z = 1.$$

Quick Tip

The matrix method involves writing equations in the form $AX = B$ and solving using $X = A^{-1}B$.

(b) If

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 15 & 6 & 5 \\ 5 & 2 & 2 \end{bmatrix}$$

then find A^{-1}.

Solution: The inverse of a 3×3 matrix is given by:

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A).$$

Computing $\det(A)$:

$$\det(A) = 3(6\cdot2-5\cdot2)-1(15\cdot2-5\cdot5)+1(15\cdot2-6\cdot5) = 3(12-10)-(30-25)+(30-30) = 6-5 = 1.$$

Since $\det(A) = 1$, we compute adj(A) and find:

$$A^{-1} = \begin{bmatrix} 4 & -1 & -1 \\ -5 & 2 & 1 \\ 5 & -2 & -1 \end{bmatrix}.$$

Quick Tip

To find A^{-1}, use $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$, where adjoint is the transpose of the cofactor matrix.

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8. (a)(i) Find the value of p such that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$$

and

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

are mutually perpendicular.

Solution: To find p, we extract the direction ratios of the two given lines.

For the first line:

$$\frac{x-1}{-3} = \frac{y-14/7}{2p} = \frac{z-3}{2}.$$

The direction ratios are:

$$(-3, 2p, 2).$$

For the second line:

$$\frac{x-7}{-3p} = \frac{y-5}{1} = \frac{z-6}{-5}.$$

The direction ratios are:

$$(-3p, 1, -5).$$

Since the lines are perpendicular, the dot product of their direction ratios must be zero:

$$(-3)(-3p) + (2p)(1) + (2)(-5) = 0.$$

$$9p + 2p - 10 = 0.$$

$$11p = 10 \Rightarrow p = \frac{10}{11}.$$

Quick Tip

For mutually perpendicular lines, their direction ratios should satisfy $a_1a_2+b_1b_2+c_1c_2 = 0$.

(a)(ii) Show that the points $(2, -1, 1)$, $(1, -3, -5)$ and $(3, -4, -4)$ are the vertices of a right-angled triangle.

Solution: We calculate the squared distances between the given points:

$$AB^2 = (1-2)^2 + (-3+1)^2 + (-5-1)^2$$

$$= (-1)^2 + (-2)^2 + (-6)^2 = 1 + 4 + 36 = 41.$$

$$BC^2 = (3-1)^2 + (-4+3)^2 + (-4+5)^2$$

$$= (2)^2 + (-1)^2 + (1)^2 = 4 + 1 + 1 = 6.$$

$$CA^2 = (3-2)^2 + (-4+1)^2 + (-4-1)^2$$

$$= (1)^2 + (-3)^2 + (-5)^2 = 1 + 9 + 25 = 35.$$

Since $AB^2 = BC^2 + CA^2$ (i.e., $41 = 6 + 35$), the triangle is right-angled at B.

Quick Tip

For checking a right-angled triangle, verify if the sum of squares of two sides equals the square of the third side using the Pythagorean theorem.

(b) Find the shortest distance between two lines l_1 and l_2 whose vector equations are:

$$\mathbf{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$$

and

$$\mathbf{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k}).$$

Solution: To find the shortest distance between two skew lines, we use the formula:

$$d = \frac{|(\mathbf{b_1} \times \mathbf{b_2}) \cdot (\mathbf{a_2} - \mathbf{a_1})|}{|\mathbf{b_1} \times \mathbf{b_2}|}$$

where:

- $\mathbf{a_1}$ and $\mathbf{a_2}$ are position vectors of any points on the two lines.

- $\mathbf{b_1}$ and $\mathbf{b_2}$ are the direction vectors of the two lines.

From the given equations:

$$\mathbf{a_1} = (1, 1, 0), \quad \mathbf{a_2} = (2, 1, -1)$$

$$\mathbf{b_1} = (2, -1, 1), \quad \mathbf{b_2} = (3, -5, 2).$$

First, we compute $\mathbf{b_1} \times \mathbf{b_2}$:

$$\mathbf{b_1} \times \mathbf{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

$$= \hat{i}((-1)(2) - (1)(-5)) - \hat{j}((2)(2) - (1)(3)) + \hat{k}((2)(-5) - (-1)(3)).$$

$$= \hat{i}(-2 + 5) - \hat{j}(4 - 3) + \hat{k}(-10 + 3).$$

$$= 3\hat{i} - \hat{j} - 7\hat{k}.$$

Now, compute the magnitude:

$$|\mathbf{b_1} \times \mathbf{b_2}| = \sqrt{3^2 + (-1)^2 + (-7)^2} = \sqrt{9 + 1 + 49} = \sqrt{59}.$$

Next, compute $(\mathbf{a_2} - \mathbf{a_1})$:

$$\mathbf{a_2} - \mathbf{a_1} = (2-1, 1-1, -1-0) = (1, 0, -1).$$

Now, compute the dot product:

$$(\mathbf{b_1} \times \mathbf{b_2}) \cdot (\mathbf{a_2} - \mathbf{a_1})$$

$$= (3\hat{i} - \hat{j} - 7\hat{k}) \cdot (1, 0, -1).$$

$$= (3 \times 1) + (-1 \times 0) + (-7 \times -1).$$

$$= 3 + 0 + 7 = 10.$$

Finally, compute the shortest distance:

$$d = \frac{|10|}{\sqrt{59}} = \frac{10}{\sqrt{59}}.$$

Quick Tip

For the shortest distance between two skew lines, use the formula:

$$d = \frac{|(\mathbf{b_1} \times \mathbf{b_2}) \cdot (\mathbf{a_2} - \mathbf{a_1})|}{|\mathbf{b_1} \times \mathbf{b_2}|}.$$

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9. (a) Find the value of the integral:

$$I = \int \frac{x^2 + x + 1}{(x+2)(x^2+1)} dx.$$

Solution: We use partial fraction decomposition to split the given function. Assume:

$$\frac{x^2 + x + 1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx + C}{x^2+1}.$$

Multiplying both sides by $(x+2)(x^2+1)$, we get:

$$x^2 + x + 1 = A(x^2+1) + (Bx + C)(x+2).$$

Expanding the right-hand side:

$$x^2 + x + 1 = Ax^2 + A + Bx^2 + 2Bx + Cx + 2C.$$

$$= (A + B)x^2 + (2B + C)x + (A + 2C).$$

Equating coefficients, we get:

1. $A + B = 1$

2. $2B + C = 1$

3. $A + 2C = 1$

Solving these equations:

- From (1), $A = 1 - B$.
- Substituting into (3): $(1 - B) + 2C = 1 \Rightarrow 2C = B \Rightarrow C = \frac{B}{2}$.
- Substituting into (2): $2B + \frac{B}{2} = 1 \Rightarrow \frac{4B+B}{2} = 1 \Rightarrow 5B = 2 \Rightarrow B = \frac{2}{5}$.
- So, $A = 1 - \frac{2}{5} = \frac{3}{5}$.
- And, $C = \frac{1}{5}$.

Thus, we rewrite:

$$I = \int \left(\frac{3/5}{x + 2} + \frac{(2/5)x + (1/5)}{x^2 + 1} \right) dx.$$

$$= \frac{3}{5} \int \frac{dx}{x + 2} + \frac{2}{5} \int \frac{xdx}{x^2 + 1} + \frac{1}{5} \int \frac{dx}{x^2 + 1}.$$

Solving these integrals:

$$I = \frac{3}{5} \ln |x + 2| + \frac{2}{5} \cdot \frac{1}{2} \ln |x^2 + 1| + \frac{1}{5} \tan^{-1} x + C.$$

$$= \frac{3}{5} \ln |x + 2| + \frac{1}{5} \ln |x^2 + 1| + \frac{1}{5} \tan^{-1} x + C.$$

Quick Tip

For integrating rational functions, always attempt partial fraction decomposition. Look for linear and quadratic factors to decompose and simplify.

(b) Find the value of the integral:

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx.$$

Solution: Using the integration by parts method, let:

- $u = x \Rightarrow du = dx$.

- $dv = \frac{\sin x}{1+\cos^2 x} dx$.

To integrate dv, substitute $t = 1 + \cos^2 x \Rightarrow dt = -2 \cos x \sin x dx$.

Thus, rewriting the integral:

$$I = \int x d \left(\tan^{-1}(\cos x) \right).$$

Using integration by parts:

$$I = x \tan^{-1}(\cos x) \Big|_0^{\pi} - \int_0^{\pi} \tan^{-1}(\cos x) dx.$$

From symmetry properties, the integral simplifies to:

$$I = \frac{\pi}{4}\pi - \frac{\pi^2}{4} = \frac{\pi^2}{4}.$$

Quick Tip

For definite integrals with trigonometric functions, use substitutions and symmetry properties to simplify the computation.
