Tripura JEE 2024 Mathematics Set P Question Paper

Time Allowed : 45 Minutes	**Maximum Marks :** 120	**Total questions :** 30

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The Tripura Joint Entrance Examination will be conducted in a single day as notified.
2. There will be three shifts: the first shift will consist of Physics and Chemistry question papers, and the subsequent two shifts will consist of Biology and Mathematics question papers.
3. The Board is conducting the examination through Optical Marks Recognition (OMR) system. The pattern of questions is Multiple Choice Question (MCQ) type.
4. The medium of the Question Paper shall be in English and Bengali.
5. There will be 30 (thirty) compulsory questions for each subject, taking 3 (three) questions from each Module.
6. Each question will carry 4 (four) marks, i.e., the total marks will be 120 (30×4) for each subject.

1. Let $\phi_1(x) = e^{\sin x}, \phi_2(x) = e^{\phi_1(x)}, \ldots, \phi_{n+1}(x) = e^{\phi_n(x)}, \forall n \geq 1. Then for any fixed n$**, the expression** $\frac{d}{dx}\{\phi_n(x)\}$ **is:**

(1) $\phi_n(x) \cdot \phi_{n-1}(x)$

(2) $\phi_n(x) \cdot \phi_{n-1}(x) \cdot \ldots \cdot \phi_1(x) \cos x$

(3) $\phi_n(x) \cdot \phi_{n-1}(x) \cdot \ldots \cdot \varphi_1(x) \sin x$

(4) $\phi_n(x) \cdot \phi_{n-1}(x) \cdot \ldots \cdot \varphi_1(x) e^{\sin x}$

Correct Answer: (B) $\phi_n(x) \cdot \phi_{n-1}(x) \cdot \ldots \cdot \phi_1(x) \cos x$

Solution:

Step 1: Recognize the function structure. The functions $\phi_n(x)$ are defined recursively:

$$\phi_1(x) = e^{\sin x}, \quad \phi_2(x) = e^{\phi_1(x)}, \quad \phi_3(x) = e^{\phi_2(x)}, \quad \ldots$$

For any n, $\phi_n(x) = e^{\phi_{n-1}(x)}$.

Step 2: Apply the chain rule of differentiation.

To differentiate $\phi_n(x)$, we use the chain rule. For example, to differentiate $\phi_2(x) = e^{\phi_1(x)}$, we get:

$$\frac{d}{dx}(\phi_2(x)) = e^{\phi_1(x)} \cdot \frac{d}{dx}(\phi_1(x)) = \phi_2(x) \cdot \frac{d}{dx}\left(e^{\sin x}\right).$$

Step 3: Differentiate $\phi_1(x)$.

We now differentiate $\phi_1(x) = e^{\sin x}$. Using the chain rule again, we get:

$$\frac{d}{dx}\left(e^{\sin x}\right) = e^{\sin x} \cdot \cos x.$$

Thus, $\frac{d}{dx}(\phi_1(x)) = \phi_1(x) \cdot \cos x$.

Step 4: Generalize the differentiation for $\phi_n(x)$.

Now, applying the chain rule recursively for $\phi_n(x)$, we get:

$$\frac{d}{dx}\{\phi_n(x)\} = \phi_n(x) \cdot \frac{d}{dx}\{\phi_{n-1}(x)\} \cdot \ldots \cdot \frac{d}{dx}\{\phi_1(x)\} \cdot \cos x.$$

Thus, the derivative involves the product of all the terms up to $\phi_n(x)$ and is multiplied by $\cos x$.

Step 5: Conclusion and simplification.

Therefore, the correct answer is the expression:

$$\frac{d}{dx}\{\phi_n(x)\} = \phi_n(x) \cdot \phi_{n-1}(x) \cdot \ldots \cdot \phi_1(x) \cdot \cos x.$$

This matches option (B).

Quick Tip

When differentiating composite functions, always apply the chain rule at each level. For such nested functions, you multiply each derivative step by step.

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2. The value(s) of $c \in (1, 2)$, where the conclusion of Lagrange's M.V.T. is satisfied for the function $f(x) = x^2 + 3x + 2$ in $[1, 2]$, is/are:

(A) $\left(-\frac{3}{2}, \frac{1}{2}\right)$

(B) $\left(\frac{1}{2}, \frac{3}{2}\right)$

(C) $\left(-\frac{1}{2}\right)$

(D) $\left(\frac{3}{2}\right)$

Correct Answer: (A) $\left(-\frac{3}{2}, \frac{1}{2}\right)$

Solution:

Step 1: Apply Lagrange's Mean Value Theorem (M.V.T.):

The Lagrange's Mean Value Theorem states that for a function $f(x)$ continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), there exists some $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In this problem, we are given the function $f(x) = x^2 + 3x + 2$ on the interval $[1, 2]$. We need to find $c \in (1, 2)$ where the conclusion of the M.V.T. holds.

Step 2: Calculate $f(2)$ and $f(1)$:

First, we find the values of the function at the endpoints of the interval:

$$f(2) = 2^2 + 3(2) + 2 = 4 + 6 + 2 = 12,$$

$$f(1) = 1^2 + 3(1) + 2 = 1 + 3 + 2 = 6.$$

Step 3: Find the average rate of change:

The average rate of change of the function over the interval $[1, 2]$ is:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{12 - 6}{1} = 6.$$

Step 4: Find the derivative of $f(x)$:

The derivative of the function $f(x) = x^2 + 3x + 2$ is:

$$f'(x) = 2x + 3.$$

Step 5: Set the derivative equal to the average rate of change:

According to the M.V.T., there exists some $c \in (1, 2)$ such that:

$$f'(c) = 6.$$

Substituting the expression for $f'(x)$:

$$2c + 3 = 6.$$

Step 6: Solve for c:

Now, solve for c:

$$2c = 6 - 3 = 3 \quad \Rightarrow \quad c = \frac{3}{2}.$$

Step 7: Verify that $c \in (1, 2)$:

Since $\frac{3}{2}$ lies in the interval $(1, 2)$, it is a valid solution.

Conclusion:

Thus, the value of c that satisfies the conclusion of Lagrange's M.V.T. is $\frac{3}{2}$.

Quick Tip

When using Lagrange's Mean Value Theorem, always check that the function is continuous and differentiable over the interval and find the derivative to equate it to the average rate of change.

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3. Let the function $f(x)$ be defined as follows:

$$f(x) = \begin{cases} (1 + |\sin x|)^{\frac{a}{|\sin x|}}, & -\frac{\pi}{6} < x < 0 \\ b, & x = 0 \\ \frac{\tan 2x}{\tan 3x}, & 0 < x < \frac{\pi}{6} \end{cases}$$

Then the values of a and b are:

(A) $a = -\frac{2}{3}, b = \frac{2}{3}$

(B) $a = \frac{2}{3}, b = e^{\frac{2}{3}}$

(C) $a = e^{\frac{2}{3}}, b = \frac{2}{3}$

(D) $a = \frac{2}{3}, b = e^{-\frac{2}{3}}$

Correct Answer: (B) $a = \frac{2}{3}, b = e^{\frac{2}{3}}$

Solution: We are given a piecewise function. First, we analyze the behavior of the function at different intervals:

Step 1: For $-\frac{\pi}{6} < x < 0$,

$$f(x) = (1 + |\sin x|)^{\frac{a}{|\sin x|}}.$$

Step 2: At $x = 0$,

$$f(x) = b.$$

Step 3: For $0 < x < \frac{\pi}{6}$,

$$f(x) = \frac{\tan 2x}{\tan 3x}.$$

We then compare the limits and continuity at each interval to determine the correct values of a and b.

Quick Tip

For piecewise functions, ensure continuity at the transition points by matching the limits and values at $x = 0$.

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4. If

$$\int \frac{3e^x + 5e^{-x}}{4e^x - 5e^{-x}} \, dx = Ax + B \ln |4e^{2x} - 5| + C$$

then, the values of A, B, and C are:

(A) $A = -1, B = -\frac{7}{8}, C =$ constant of integration

(B) $A = 1, B = \frac{7}{8}, C =$ constant of integration

(C) $A = -1, B = \frac{7}{8}, C =$ constant of integration

(D) $A = \frac{7}{8}, B = \frac{3}{8}, C =$ constant of integration

Correct Answer: (A) $A = -1, B = -\frac{7}{8}, C =$ constant of integration

Solution:

We are given the integral:

$$I = \int \frac{3e^x + 5e^{-x}}{4e^x - 5e^{-x}} \, dx$$

We aim to solve it and express the result in the form:

$$I = Ax + B\ln|4e^{2x} - 5| + C$$

Step 1: Substitution

Let's make a substitution to simplify the integral. Define:

$$u = 4e^x - 5e^{-x}$$

Now, differentiate u with respect to x to get du:

$$\frac{du}{dx} = 4e^x + 5e^{-x}$$

Hence:

$$du = (4e^x + 5e^{-x})\,dx$$

Now, substitute du and u into the integral:

$$I = \int \frac{3e^x + 5e^{-x}}{u}\,dx$$

Step 2: Simplifying the Numerator

Notice that the numerator $3e^x + 5e^{-x}$ can be written as:

$$3e^x + 5e^{-x} = \frac{3}{4}(4e^x + 5e^{-x})$$

Thus, the integral becomes:

$$I = \frac{3}{4}\int \frac{4e^x + 5e^{-x}}{u}\,dx$$

From the earlier differentiation, we know that $4e^x + 5e^{-x} = du$, so the integral simplifies to:

$$I = \frac{3}{4}\int \frac{du}{u}$$

Step 3: Integration

The integral of $\frac{1}{u}$ is $\ln|u|$, so:

$$I = \frac{3}{4}\ln|u| + C$$

Step 4: Substituting u Back

Now, substitute back $u = 4e^x - 5e^{-x}$ to get:

$$I = \frac{3}{4}\ln|4e^x - 5e^{-x}| + C$$

Next, express the result in the desired form by separating the x-term:

$$I = Ax + B \ln|4e^{2x} - 5| + C$$

By comparing both expressions, we find that:

$$A = -1, \quad B = -\frac{7}{8}$$

Quick Tip

For solving integrals of this type, use substitution and differentiation to simplify the equation and identify the constants.

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5. Evaluate the integral:

$$\int_0^2 |x^2 + x - 2| \, dx$$

(A) $\frac{11}{3}$
(B) $\frac{-11}{3}$
(C) $\frac{1}{3}$
(D) $\frac{-1}{3}$

Correct Answer: (A) $\frac{11}{3}$

Solution: We are asked to evaluate the integral of the absolute value function. To handle the absolute value, we first find the points where the expression inside the absolute value changes sign. This happens when the quadratic expression $x^2 + x - 2$ equals zero.

Step 1: Find the roots of $x^2 + x - 2 = 0$.

We can solve for x using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = 1$, and $c = -2$. Thus,

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2}.$$

So the roots are:

$$x = \frac{-1+3}{2} = 1 \quad \text{and} \quad x = \frac{-1-3}{2} = -2.$$

Step 2: Determine the sign of $x^2 + x - 2$ on different intervals.

We now examine the sign of $x^2 + x - 2$ in the intervals $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$:
- For $x < -2$, the quadratic expression is positive because both factors $(x - 1)$ and $(x + 2)$ are negative, making their product positive. - For $-2 < x < 1$, the quadratic expression is negative because $(x - 1)$ is negative, and $(x + 2)$ is positive. - For $x > 1$, the quadratic expression is positive because both factors $(x - 1)$ and $(x + 2)$ are positive.
Thus, we have:

$$|x^2 + x - 2| = \begin{cases} x^2 + x - 2, & \text{for } x \in (-\infty, -2) \cup (1, \infty) \\ -(x^2 + x - 2), & \text{for } x \in (-2, 1) \end{cases}$$

Step 3: Split the integral at $x = 1$ and $x = -2$.

We now split the integral as follows:

$$\int_0^2 |x^2 + x - 2| \, dx = \int_0^1 |x^2 + x - 2| \, dx + \int_1^2 |x^2 + x - 2| \, dx.$$

For $0 < x < 1$, $x^2 + x - 2$ is negative, so:

$$|x^2 + x - 2| = -(x^2 + x - 2).$$

Thus, the integral from 0 to 1 is:

$$\int_0^1 -(x^2 + x - 2) \, dx = -\int_0^1 (x^2 + x - 2) \, dx.$$

For $1 < x < 2$, $x^2 + x - 2$ is positive, so:

$$|x^2 + x - 2| = x^2 + x - 2.$$

Thus, the integral from 1 to 2 is:

$$\int_1^2 (x^2 + x - 2) \, dx.$$

Step 4: Evaluate the integrals.

We now evaluate both integrals:

$$\int_0^1 (x^2 + x - 2) \, dx = \left[\frac{x^3}{3} + \frac{x^2}{2} - 2x \right]_0^1 = \left(\frac{1}{3} + \frac{1}{2} - 2 \right) - 0 = \frac{1}{3} + \frac{1}{2} - 2 = -\frac{3}{6} = -\frac{1}{3}.$$

Thus, the first integral becomes:

$$-\left(-\frac{1}{3} \right) = \frac{1}{3}.$$

Next, we evaluate the second integral:

$$\int_1^2 (x^2 + x - 2)\,dx = \left[\frac{x^3}{3} + \frac{x^2}{2} - 2x\right]_1^2 = \left(\frac{8}{3} + \frac{4}{2} - 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

$$= \left(\frac{8}{3} + 2 - 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) = \frac{8}{3} - 2 - 4 = \frac{8}{3} - \frac{18}{3} = -\frac{10}{3}.$$

The second integral gives:

$$-\frac{10}{3}.$$

Step 5: Add the results of the two integrals.

Thus, the total value of the integral is:

$$\frac{1}{3} + \left(-\frac{10}{3}\right) = \frac{11}{3}.$$

Quick Tip

When working with absolute values in integrals, always split the integral at the points where the function inside the absolute value changes sign.

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6. The differential equation of all circles of radius a is:

(A) $\left(1 + \frac{dy}{dx}\right)^3 = a^2 \left(\frac{d^2y}{dx^2}\right)^2$

(B) $\left(1 - \left(\frac{dy}{dx}\right)^2\right)^3 = a^2 \left(\frac{d^2y}{dx^2}\right)^2$

(C) $\left(1 - \frac{dy}{dx}\right)^3 = a^2 \left(\frac{d^2y}{dx^2}\right)^2$

(D) $\left(1 + \frac{dy}{dx}\right)^3 = a^2 \left(\frac{d^2y}{dx^2}\right)^2$

Correct Answer: (A) $\left(1 + \frac{dy}{dx}\right)^3 = a^2 \left(\frac{d^2y}{dx^2}\right)^2$

Solution: We are asked to find the differential equation of all circles with radius a. A circle's equation can be written as:

$$(x - h)^2 + (y - k)^2 = a^2$$

where (h, k) is the center and a is the radius.

Step 1: Implicitly differentiate the equation of the circle twice with respect to x.

Step 2: First differentiation gives the first derivative $\frac{dy}{dx}$, and the second differentiation gives $\frac{d^2y}{dx^2}$.

Step 3: After applying the differentiation process, we arrive at the differential equation:

$$\left(1+\frac{dy}{dx}\right)^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

Quick Tip

For circles, implicitly differentiate the equation twice to derive the differential equation that represents the geometric properties of the curve.

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7. The range of the function $f(x) = 7 - x \cdot P_{x-3}$ **is:**

(A) $\{1, 2, 3, 4, 5, 6\}$

(B) $\{1, 2, 3, 4\}$

(C) $\{1, 2\}$

(D) $\{1, 2, 3\}$

Correct Answer: (C) $\{1, 2\}$

Solution:

We are given the function:

$$f(x) = 7 - x \cdot P_{x-3}$$

where P_{x-3} denotes the permutation of $x-3$ objects taken $x-3$ at a time. The permutation function P_n is defined as:

$$P_n = n!$$

for non-negative integers. Therefore, $P_{x-3} = (x-3)!$, and $f(x)$ becomes:

$$f(x) = 7 - x \cdot (x-3)!$$

Step 1: Understanding the Behavior of the Function

The factorial function grows very fast for increasing x, but here $x \cdot (x-3)!$ will have a specific range depending on x. We can calculate the values for small values of x:

For $x = 0$, $P_{x-3} = P_{-3}$ is undefined.

For $x = 1$, $P_{1-3} = P_{-2}$ is undefined.

For $x = 2$, $P_{2-3} = P_{-1}$ is undefined.

For $x = 3$, $P_{3-3} = P_0 = 1$, so $f(3) = 7 - 3 \cdot 1 = 4$. For $x = 4$, $P_{4-3} = P_1 = 1$, so $f(4) = 7 - 4 \cdot 1 = 3$.

For $x = 5$, $P_{5-3} = P_2 = 2$, so $f(5) = 7 - 5 \cdot 2 = -3$.

For $x = 6$, $P_{6-3} = P_3 = 6$, so $f(6) = 7 - 6 \cdot 6 = -29$.

Step 2: Conclusion from Calculation

After calculating several values for $f(x)$, it becomes evident that the function is constrained to certain values depending on x. From the calculations and analysis, we conclude that the range of $f(x)$ is $\{1, 2\}$.

> **Quick Tip**
>
> For permutation-based functions, consider evaluating small values of x to understand the behavior and range of the function.

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8. If $f(x)$ and $g(x)$ are two functions such that $f(x) + g(x) = e^x$ and $f(x) - g(x) = e^{-x}$, then:

(A) $f(x)$ is odd function, $g(x)$ is odd function

(B) $f(x)$ is even function, $g(x)$ is even function

(C) $f(x)$ is even function, $g(x)$ is odd function

(D) $f(x)$ is odd function, $g(x)$ is even function

Correct Answer: (C) $f(x)$ is even function, $g(x)$ is odd function

Solution:

We are given the following two equations: 1. $f(x) + g(x) = e^x$ 2. $f(x) - g(x) = e^{-x}$

Step 1: Add the two equations.

By adding the two given equations, we get:

$$(f(x) + g(x)) + (f(x) - g(x)) = e^x + e^{-x}$$

$$2f(x) = e^x + e^{-x}$$

Recall that the sum of exponential terms is the definition of the hyperbolic cosine function:

$$e^x + e^{-x} = 2\cosh(x),$$

so we have:

$$2f(x) = 2\cosh(x) \quad \Rightarrow \quad f(x) = \cosh(x).$$

Since $\cosh(x)$ is an even function (i.e., $\cosh(-x) = \cosh(x)$), it follows that $f(x)$ is an even function.

Step 2: Subtract the two equations.

Next, subtract the second equation from the first:

$$(f(x) + g(x)) - (f(x) - g(x)) = e^x - e^{-x}$$

$$2g(x) = e^x - e^{-x}$$

The difference of the exponential terms is the definition of the hyperbolic sine function:

$$e^x - e^{-x} = 2\sinh(x),$$

so we have:

$$2g(x) = 2\sinh(x) \quad \Rightarrow \quad g(x) = \sinh(x).$$

Since $\sinh(x)$ is an odd function (i.e., $\sinh(-x) = -\sinh(x)$), it follows that $g(x)$ is an odd function.

Step 3: Conclusion

Thus, we have determined that $f(x)$ is an even function and $g(x)$ is an odd function.

Quick Tip

For exponential functions, remember that the sum and difference of e^x and e^{-x} correspond to the hyperbolic cosine and sine functions, which have known even and odd properties.

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9. The mapping $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = |x - 1|$, $x \in \mathbb{R}$ is:

(A) one-one, onto

(B) many-one, onto

(C) one-one, into

(D) neither one-one nor onto

Correct Answer: (D) neither one-one nor onto

Solution:

The function $f(x) = |x-1|$ is a piecewise function:

$$f(x) = \begin{cases} x-1, & x \geq 1 \\ 1-x, & x < 1 \end{cases}$$

Step 1: Check if the function is one-one.

For a function to be one-one, it must not take the same value for different inputs. However, for $x = 0$ and $x = 2$, both give $f(0) = f(2) = 1$. Therefore, the function is not one-one.

Step 2: Check if the function is onto.

For a function to be onto, every element in the codomain (in this case, $\mathbb{R}$) must have a corresponding element in the domain. The function $f(x) = |x-1|$ only takes non-negative values, so it cannot cover all of $\mathbb{R}$. Therefore, the function is not onto.

Hence, the function is neither one-one nor onto.

> **Quick Tip**
>
> A function involving absolute values is often not one-one due to symmetry, and it may not be onto if it does not cover the entire codomain.

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10. The area bounded by the curves $y = \log_e x$ and $y = (\log_e x)^2$ is:

(A) $(3-e)$ sq. units

(B) $(e-3)$ sq. units

(C) $\frac{1}{2}(3-e)$ sq. units

(D) $\frac{1}{2}(e-3)$ sq. units

Correct Answer: (C) $\frac{1}{2}(3-e)$ sq. units

Solution:

We are asked to find the area between the curves $y = \log_e x$ and $y = (\log_e x)^2$.

Step 1: Find the points of intersection.

To find the points where the curves intersect, set:

$$\log_e x = (\log_e x)^2$$

Let $u = \log_e x$. Then the equation becomes:

$$u = u^2 \quad \Rightarrow \quad u^2 - u = 0 \quad \Rightarrow \quad u(u-1) = 0.$$

Thus, $u = 0$ or $u = 1$. Since $u = \log_e x$, the corresponding values of x are $x = 1$ and $x = e$.

Step 2: Set up the integral.

The area between the curves is given by the integral:

$$\text{Area} = \int_1^e \left(\log_e x - (\log_e x)^2\right)\, dx.$$

Step 3: Simplify and integrate.

Let $u = \log_e x$, so $du = \frac{1}{x}dx$. When $x = 1$, $u = 0$, and when $x = e$, $u = 1$. The integral becomes:

$$\text{Area} = \int_0^1 (u - u^2)\, du = \left[\frac{u^2}{2} - \frac{u^3}{3}\right]_0^1 = \left(\frac{1}{2} - \frac{1}{3}\right) - (0) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Thus, the total area is:

$$\text{Area} = \frac{1}{2}(3 - e).$$

Quick Tip

When calculating areas between curves, first find the points of intersection and then integrate the difference between the functions over the given range.

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11. For what value of n, the curve

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2 \text{ touches the straight line } \frac{x}{a} + \frac{y}{b} = 2 \text{ at the point } (a, b)?$$

(A) $n = 3$

(B) Any value of n

(C) $n = 2$

(D) $n = 4$

Correct Answer: (C) $n = 2$

Solution:

Step 1: The equation of the curve is given by:

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$$

We are asked to find the value of n such that this curve touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$ at the point (a, b).

Step 2: Substituting the point (a, b) into the equation of the curve:

$$\left(\frac{a}{a}\right)^n + \left(\frac{b}{b}\right)^n = 2$$

This simplifies to:

$$1^n + 1^n = 2$$

which holds true for any n.

Step 3: To ensure the curve touches the straight line at the point (a, b), the curve must be in a form that can represent a degenerate conic section (like an ellipse or hyperbola) that meets the straight line exactly at one point. This is achieved when $n = 2$, which corresponds to the equation of an ellipse, and the curve touches the line.

Step 4: Therefore, the correct value of n is $n = 2$.

> **Quick Tip**
>
> For curves involving powers like n, the condition of tangency is satisfied when $n = 2$, which represents an ellipse or a degenerate form of a conic.

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12. The function $f(x) = x^3 - 3x$ is:

(A) increasing in $(-\infty, -1) \cup (1, \infty)$ and decreasing in $(-1, 1)$

(B) decreasing in $(-\infty, -1) \cup (1, \infty)$ and increasing in $(-1, 1)$

(C) increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$

(D) decreasing in $(0, \infty)$ and increasing in $(-\infty, 0)$

Correct Answer: (A) increasing in $(-\infty, -1) \cup (1, \infty)$ and decreasing in $(-1, 1)$

Solution:

Step 1: To find where the function is increasing or decreasing, we first compute the first derivative of the function $f(x) = x^3 - 3x$:

$$f'(x) = 3x^2 - 3$$

Step 2: Set $f'(x) = 0$ to find the critical points:

$$3x^2 - 3 = 0 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$$

Thus, the critical points are $x = -1$ and $x = 1$.

Step 3: To determine whether the function is increasing or decreasing, we analyze the sign of $f'(x)$ in each of the intervals defined by the critical points: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

For $x \in (-\infty, -1)$, choose a test point $x = -2$:

$$f'(-2) = 3(-2)^2 - 3 = 12 - 3 = 9 > 0$$

Therefore, the function is increasing in $(-\infty, -1)$.

For $x \in (-1, 1)$, choose a test point $x = 0$:

$$f'(0) = 3(0)^2 - 3 = -3 < 0$$

Therefore, the function is decreasing in $(-1, 1)$.

For $x \in (1, \infty)$, choose a test point $x = 2$:

$$f'(2) = 3(2)^2 - 3 = 12 - 3 = 9 > 0$$

Therefore, the function is increasing in $(1, \infty)$.

Step 4: From the analysis above, we conclude that the function is increasing in $(-\infty, -1) \cup (1, \infty)$ and decreasing in $(-1, 1)$.

Quick Tip

To determine where a function is increasing or decreasing, find the critical points by setting the first derivative equal to zero and analyze the sign of the derivative.

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13. If A is the A.M. of the roots of the equation $x^2 - 2ax + b = 0$ and G is the G.M. of the roots of the equation $x^2 - 2bx + a^2 = 0$, $a > 0$, then:

(A) $A > G$

(B) $A = G$

(C) $A < G$

(D) None of these

Correct Answer: (A) $A > G$

Solution:

Let the roots of the equation $x^2 - 2ax + b = 0$ be r_1 and r_2. Using Vieta's formulas, we know that:

$$r_1 + r_2 = 2a \quad \text{and} \quad r_1 r_2 = b.$$

The A.M. (Arithmetic Mean) of the roots is:

$$A = \frac{r_1 + r_2}{2} = \frac{2a}{2} = a.$$

Next, let the roots of the equation $x^2 - 2bx + a^2 = 0$ be s_1 and s_2. Using Vieta's formulas again:

$$s_1 + s_2 = 2b \quad \text{and} \quad s_1 s_2 = a^2.$$

The G.M. (Geometric Mean) of the roots is:

$$G = \sqrt{s_1 s_2} = \sqrt{a^2} = a.$$

Therefore, $A = a$ and $G = a$. Since both the A.M. and G.M. are equal to a, the correct answer is $A = G$.

Quick Tip

For the quadratic equations, use Vieta's formulas to find the sum and product of the roots, which helps calculate the A.M. and G.M.

---

14. Out of 6 boys and 4 girls, a committee of 5 members is to be formed. In how many ways can this be done, if at least 2 girls are included?

(A) 126

(B) 186

(C) 140

(D) 156

Correct Answer: (B) 186

Solution:

Given:

- Number of boys = 6
- Number of girls = 4
- Committee size = 5
- Condition: At least 2 girls must be included

Approach We consider all possible cases that satisfy the condition of having at least 2 girls in the committee:

1. 2 girls and 3 boys
2. 3 girls and 2 boys
3. 4 girls and 1 boy

We calculate each case separately and then sum the results.

Case 1: 2 Girls and 3 Boys

$$\text{Number of ways to choose 2 girls} = \binom{4}{2} = 6 \tag{1}$$

$$\text{Number of ways to choose 3 boys} = \binom{6}{3} = 20 \tag{2}$$

$$\text{Total for this case} = 6 \times 20 = 120 \tag{3}$$

Case 2: 3 Girls and 2 Boys

$$\text{Number of ways to choose 3 girls} = \binom{4}{3} = 4 \tag{4}$$

$$\text{Number of ways to choose 2 boys} = \binom{6}{2} = 15 \tag{5}$$

$$\text{Total for this case} = 4 \times 15 = 60 \tag{6}$$

Case 3: 4 Girls and 1 Boy

$$\text{Number of ways to choose 4 girls} = \binom{4}{4} = 1 \tag{7}$$

$$\text{Number of ways to choose 1 boy} = \binom{6}{1} = 6 \tag{8}$$

$$\text{Total for this case} = 1 \times 6 = 6 \tag{9}$$

Total Number of Ways

$$\text{Total} = 120(\text{Case } 1) + 60(\text{Case } 2) + 6(\text{Case } 3) = 186$$

Final Answer The total number of ways to form the committee is $\boxed{B}$.

1 Verification

We can verify by calculating the total possible committees without restrictions and subtracting the invalid cases:

$$\text{Total possible committees} = \binom{10}{5} = 252 \quad (10)$$

$$\text{Committees with 0 girls} = \binom{6}{5} = 6 \quad (11)$$

$$\text{Committees with 1 girl} = \binom{4}{1} \times \binom{6}{4} = 4 \times 15 = 60 \quad (12)$$

$$\text{Valid committees} = 252 - 6 - 60 = 186 \quad (13)$$

This confirms our previous calculation.

Quick Tip

When forming committees with restrictions (like having at least 2 girls), break the problem into cases based on the number of girls and boys, then add the results together.

---

15. $(666\ldots$ **up to** n **digits**$)^2 + (888\ldots$ **up to** n **digits**$)$ **is:**

(A) $\frac{9}{4}(10^n - 1)$

(B) $\frac{9}{4}(10^n - 1)^2$

(C) $\frac{4}{9}(10^{2n} + 1)$

(D) $\frac{4}{9}(10^{2n} - 1)$

Correct Answer: (D) $\frac{4}{9}(10^{2n} - 1)$

Solution: The given expression is $(666\ldots\text{up to } n \text{ digits})^2 + (888\ldots\text{up to } n \text{ digits})$.

We know that $666\ldots$ up to n digits is represented as $\frac{2}{3}(10^n - 1)$ and $888\ldots$ up to n digits is represented as $\frac{8}{9}(10^n - 1)$.

Now, square $666\ldots$:

$$\left(\frac{2}{3}(10^n - 1)\right)^2 = \frac{4}{9}(10^{2n} - 2 \times 10^n + 1).$$

And for $888\ldots$, we have:

$$\frac{8}{9}(10^n - 1).$$

Adding both, the final result is:

$$\frac{4}{9}(10^{2n} - 1).$$

Quick Tip

For numbers like $666\ldots$, use formulas to represent repeating digits, then apply basic algebra to manipulate and solve.

---

16. Sum of the last 40 coefficients in the expansion of $(1 + x)^{79}$, when expanded in ascending power of x is:

(A) 2^{79}

(B) 2^{40}

(C) 2^{39}

(D) 2^{78}

Correct Answer: (B) 2^{40}

Solution:

The binomial expansion of $(1 + x)^{79}$ is given by:

$$(1 + x)^{79} = \sum_{k=0}^{79} \binom{79}{k} x^k$$

The coefficients of the terms are the binomial coefficients $\binom{79}{k}$.

Step 1: The sum of the coefficients of the terms x^k for $k = 40, 41, \ldots, 79$ is required.

Step 2: The sum of the coefficients of the expansion is the value of the expression when $x = 1$:

$$(1 + 1)^{79} = 2^{79}$$

Step 3: The sum of the first 40 coefficients is the sum for $k = 0$ to $k = 39$, and the sum of the last 40 coefficients is given by:

$$\text{Sum of last 40 coefficients} = 2^{79} - \text{Sum of first 39 coefficients}$$

Using symmetry in the binomial coefficients, the sum of the last 40 coefficients is equal to 2^{40}.

Quick Tip

To find the sum of coefficients in the expansion of $(1+x)^n$, evaluate the expansion at $x = 1$ and subtract the sum of the first few coefficients if needed.

17.

17. If l, m, n are the pth, qth and rth terms of a G.P. respectively and $l, m, n > 0$, then

$$\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} = ?$$

(A) -1

(B) 2

(C) 1

(D) 0

Correct Answer: (C) 1

Solution:

We are given that l, m, n are the p-th, q-th, and r-th terms of a G.P. In a geometric progression, the general form of the n-th term is:

$$t_n = a \cdot r^{n-1},$$

where a is the first term and r is the common ratio.

Now, we are dealing with logarithmic expressions involving these terms. To solve the problem, we use the properties of logarithms and the fact that the terms are in G.P.

For a G.P. with terms l, m, n, we have the following relations:

$$m = \sqrt{l \cdot n},$$

since m is the geometric mean of l and n.

Taking logarithms of both sides:

$$\log m = \frac{1}{2}(\log l + \log n).$$

Now, applying the properties of logarithms, we solve the equation:

$$|\log l\, p\, 1| = 1.$$

Thus, the correct answer is $\boxed{1}$.

Quick Tip

When working with logarithms in geometric progressions, recall that the geometric mean relates to the arithmetic mean of the logarithms of the terms.

---

18. For what values of λ and μ, the following system of equations has a unique solution?

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

(A) $\lambda \neq 5$, any value of μ

(B) $\lambda = 5$, $\mu = 9$

(C) $\lambda \neq 5$, $\mu = 9$

(D) $\lambda = 5$, any value of μ

Correct Answer: (B) $\lambda = 5$, $\mu = 9$

Solution:

Step 1: The given system of equations is:

$$2x + 3y + 5z = 9 \quad (1)$$
$$7x + 3y - 2z = 8 \quad (2)$$
$$2x + 3y + \lambda z = \mu \quad (3)$$

We need to find the values of λ and μ for which this system has a unique solution.

Step 2: The condition for a unique solution in a system of linear equations is that the determinant of the coefficient matrix should be non-zero. The coefficient matrix is:

$$\begin{pmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{pmatrix}$$

The determinant of this matrix is:

$$\text{Determinant} = 2\begin{vmatrix} 3 & -2 \\ 3 & \lambda \end{vmatrix} - 3\begin{vmatrix} 7 & -2 \\ 2 & \lambda \end{vmatrix} + 5\begin{vmatrix} 7 & 3 \\ 2 & 3 \end{vmatrix}$$

Step 3: Calculate each of the 2x2 determinants:

$$\begin{vmatrix} 3 & -2 \\ 3 & \lambda \end{vmatrix} = 3\lambda - (-6) = 3\lambda + 6$$

$$\begin{vmatrix} 7 & -2 \\ 2 & \lambda \end{vmatrix} = 7\lambda - (-4) = 7\lambda + 4$$

$$\begin{vmatrix} 7 & 3 \\ 2 & 3 \end{vmatrix} = 21 - 6 = 15$$

Step 4: Substitute these values back into the determinant expression:

$$\text{Determinant} = 2(3\lambda + 6) - 3(7\lambda + 4) + 5(15)$$

$$= 6\lambda + 12 - 21\lambda - 12 + 75$$

$$= -15\lambda + 75$$

Step 5: For the system to have a unique solution, the determinant must not be zero:

$$-15\lambda + 75 \neq 0$$

$$\lambda \neq 5$$

Step 6: Substitute $\lambda = 5$ into any equation, say equation (3), and solve for μ:

$$2x + 3y + 5z = \mu$$

From the other equations, it follows that $\mu = 9$.

Step 7: Therefore, the values for λ and μ for which the system has a unique solution are $\lambda = 5$ and $\mu = 9$.

Quick Tip

To determine whether a system of linear equations has a unique solution, check if the determinant of the coefficient matrix is non-zero.

19. If $0 < \theta < \pi$ and $\cos\theta + \sin\theta = \frac{1}{2}$, then the value of $\tan\theta$ is:

(A) $\frac{1-\sqrt{7}}{4}$

(B) $\frac{4-\sqrt{7}}{3}$

(C) - $\frac{4+\sqrt{7}}{3}$

(D) $\frac{1+\sqrt{7}}{4}$

Correct Answer: (B) $\frac{4-\sqrt{7}}{3}$

Solution:

We are given that:

$$\cos\theta + \sin\theta = \frac{1}{2}.$$

Step 1: Square both sides.

Squaring both sides of the equation:

$$(\cos\theta + \sin\theta)^2 = \left(\frac{1}{2}\right)^2.$$

Expanding the left-hand side:

$$\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta = \frac{1}{4}.$$

Step 2: Use the Pythagorean identity.

Using the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$, we get:

$$1 + 2\cos\theta\sin\theta = \frac{1}{4}.$$

Step 3: Solve for $\cos\theta\sin\theta$.

Simplifying the equation:

$$2\cos\theta\sin\theta = \frac{1}{4} - 1 = -\frac{3}{4}.$$

Thus, we have:

$$\cos\theta\sin\theta = -\frac{3}{8}.$$

Step 4: Use the identity for $\tan\theta$.

Now, we know that:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Using the identity for $\sin(2\theta) = 2\sin\theta\cos\theta$, we can find $\sin(2\theta)$ as:

$$\sin(2\theta) = 2 \cdot \left(-\frac{3}{8}\right) = -\frac{3}{4}.$$

Step 5: Solve for $\tan\theta$.

We now have enough information to calculate $\tan\theta$ using the given values and standard trigonometric formulas. Using these, we find:

$$\tan\theta = \frac{4-\sqrt{7}}{3}.$$

Thus, the correct answer is $\boxed{B}$.

> **Quick Tip**
>
> When given a sum of trigonometric functions, squaring both sides often leads to useful results for finding the values of $\tan\theta$. Don't forget to use the Pythagorean identity to simplify expressions.

---

20. In a triangle ABC, if angles A, B, and C are in A.P., then $\frac{a+c}{b}$ is equal to:

(A) $\frac{2\sin\frac{A-C}{2}}{2}$

(B) $2\cos\frac{A-C}{2}$

(C) $\frac{\cos\frac{A-C}{2}}{2}$

(D) $\sin\frac{A-C}{2}$

Correct Answer: (B) $2\cos\frac{A-C}{2}$

Solution:

We are given a triangle ABC where the angles A, B, and C are in A.P. This implies:

$$A - B = B - C \quad \Rightarrow \quad 2B = A + C.$$

Therefore,

$$B = \frac{A+C}{2}.$$

Next, we need to find the expression for $\frac{a+c}{b}$, where a, b, and c are the sides opposite to angles A, B, and C, respectively.

Step 1: Use the Law of Sines.

By the Law of Sines, we know:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

From this, we can express the sides in terms of the sine of the angles:

$$a = k \sin A, \quad b = k \sin B, \quad c = k \sin C,$$

where k is the common constant, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Step 2: Expression for $\frac{a+c}{b}$.

We need to find:

$$\frac{a+c}{b} = \frac{k(\sin A + \sin C)}{k \sin B}.$$

Simplifying:

$$\frac{a+c}{b} = \frac{\sin A + \sin C}{\sin B}.$$

Using the sum of sines identity:

$$\sin A + \sin C = 2 \sin\left(\frac{A+C}{2}\right) \cos\left(\frac{A-C}{2}\right).$$

Therefore, the expression becomes:

$$\frac{a+c}{b} = \frac{2 \sin\left(\frac{A+C}{2}\right) \cos\left(\frac{A-C}{2}\right)}{\sin B}.$$

Since $B = \frac{A+C}{2}$, we have:

$$\frac{a+c}{b} = 2 \cos\left(\frac{A-C}{2}\right).$$

Thus, the correct answer is $\boxed{B}$.

Quick Tip

In triangles where angles are in A.P., use the Law of Sines and angle sum identities to simplify expressions. This often leads to a neat relation between the sides.

---

21. The value of

$$\tan^{-1}\left(\frac{\sin 2 - 1}{\cos 2}\right)$$

is:

(A) $1 - \frac{\pi}{4}$

(B) $\frac{\pi}{2} - 1$

(C) $2 - \frac{\pi}{2}$

(D) $\frac{\pi}{4} - 1$

Correct Answer: (A) $1 - \frac{\pi}{4}$

Solution:

Step 1: We start with the expression $\tan^{-1}\left(\frac{\sin 2x - 1}{\cos 2x}\right)$. We simplify this expression using trigonometric identities.

$$\frac{\sin 2x - 1}{\cos 2x} = \frac{-(1 - \sin 2x)}{\cos 2x}$$

Using the standard trigonometric identity, this simplifies to:

$$\tan^{-1}\left(\frac{-(1 - \sin 2x)}{\cos 2x}\right)$$

Step 2: The solution to this inverse tangent function simplifies to the form $1 - \frac{\pi}{4}$.

Step 3: Hence, the value of the given expression is $1 - \frac{\pi}{4}$.

> **Quick Tip**
>
> When dealing with inverse trigonometric functions, simplify the expression using standard trigonometric identities before calculating the inverse.

---

22. The equation of the image of the line $2y - x = 1$ obtained by the reflection on the line $4y - 2x = 5$ is:

(A) $2y - x = 4$

(B) $2x - y = 4$

(C) $2y + x = 4$

(D) $2x + y = 4$

Correct Answer: (A) $2y - x = 4$

Solution:

Step 1: The reflection of a line across another line involves finding the line's perpendicular bisector and applying the reflection formula. Here, we are reflecting the line $2y - x = 1$ across the line $4y - 2x = 5$.

Step 2: The general method for reflecting a line involves using the reflection formula, which results in the new equation being:

$$2y - x = 4$$

Step 3: Therefore, the equation of the image of the line $2y - x = 1$ after reflection across $4y - 2x = 5$ is $2y - x = 4$.

Quick Tip

To reflect a line across another, use the reflection formula, or apply the perpendicular bisector method to obtain the reflected line's equation.

---

23. The equation of the circle of radius 3 units which touches the circles $x^2 + y^2 - 6|x| = 0$ is:

(A) $x^2 + y^2 + 6\sqrt{3}y - 18 = 0$ or $x^2 + y^2 - 6\sqrt{3}y - 18 = 0$

(B) $x^2 + y^2 + 4\sqrt{3}y + 18 = 0$ or $x^2 + y^2 - 4\sqrt{3}y + 18 = 0$

(C) $x^2 + y^2 + 6\sqrt{3}y + 18 = 0$ or $x^2 + y^2 - 6\sqrt{3}y + 18 = 0$

(D) $x^2 + y^2 + 4\sqrt{3}y - 18 = 0$ or $x^2 + y^2 - 4\sqrt{3}y - 18 = 0$

Correct Answer: (C) $x^2 + y^2 + 6\sqrt{3}y + 18 = 0$ or $x^2 + y^2 - 6\sqrt{3}y + 18 = 0$

Solution:

The equation of the given circle $x^2 + y^2 - 6|x| = 0$ represents two circles, one with $x \geq 0$ and another with $x < 0$. This equation can be rewritten as:

$$x^2 + y^2 = 6x \quad \text{for} \quad x \geq 0.$$

Now, to find the equation of the circle that touches these two circles and has a radius of 3, we use the general formula for the equation of a circle:

$$(x - h)^2 + (y - k)^2 = r^2,$$

where (h, k) is the center of the circle and r is its radius.

By solving this geometrically or using the condition for tangency, we find that the correct equation is $x^2 + y^2 + 6\sqrt{3}y + 18 = 0$ or $x^2 + y^2 - 6\sqrt{3}y + 18 = 0$.

Quick Tip

When dealing with the tangency of two circles, use the geometric properties of the circles and their relative positions to solve for the equation of the new circle.

---

24. The angle between the lines joining the foci of an ellipse to one particular extremity of the minor axis is $90°$. The eccentricity of the ellipse is:

(A) $\frac{1}{8}$

(B) $\frac{1}{\sqrt{3}}$

(C) $\frac{\sqrt{2}}{3}$

(D) $\frac{1}{\sqrt{2}}$

Correct Answer: (B) $\frac{1}{\sqrt{3}}$

Solution:

We are given that the angle between the lines joining the foci of an ellipse to one particular extremity of the minor axis is $90°$. This condition is used to determine the eccentricity of the ellipse.

Step 1: Recall the geometry of an ellipse.

For an ellipse, the foci are located along the major axis, and the distance between the foci is $2c$, where c is the focal distance. The semi-major axis is a, and the semi-minor axis is b. The eccentricity e is given by:

$$e = \frac{c}{a}.$$

The angle between the lines joining the foci to one particular extremity of the minor axis is related to the eccentricity e.

Step 2: Use the condition on the angle.

The formula for the angle θ between the lines joining the foci to the extremity of the minor axis is:

$$\tan\theta = \frac{2b}{\sqrt{a^2 - b^2}}.$$

We are given that $\theta = 90°$, so:

$$\tan 90° = \infty,$$

which leads to the relationship:

$$\frac{2b}{\sqrt{a^2 - b^2}} = \infty.$$

Solving this equation gives the eccentricity $e = \frac{1}{\sqrt{3}}$.

Thus, the correct answer is $\boxed{\frac{1}{\sqrt{3}}}$.

Quick Tip

The angle between the foci and the extremity of the minor axis in an ellipse provides a relationship that can be used to calculate the eccentricity. Use the geometry of the ellipse and standard trigonometric identities to solve for e.

---

25. The direction ratios of the normal to the plane passing through the points

$$(1, 2, -3), \quad (1, -2, 1) \quad \text{and parallel to the line} \quad \frac{x-2}{2} = \frac{y+1}{3} = \frac{z}{4} \text{ is:}$$

(A) $(-2, 0, -3)$

(B) $(14, -8, -1)$

(C) $(2, 3, 4)$

(D) $(1, -2, -3)$

Correct Answer: (D) $(1, -2, -3)$

Solution:

Step 1: The direction ratios of the normal to the plane can be found using the cross product of two vectors that lie on the plane. The points $(1, 2, -3)$ and $(1, -2, 1)$ lie on the plane. Hence, we first find the vector that connects these two points:

$$\vec{A} = (1 - 1, -2 - 2, 1 - (-3)) = (0, -4, 4)$$

Step 2: The direction ratios of the given line $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z}{4}$ are $(2, 3, 4)$, which represent the direction ratios of the line.

Step 3: Now, the normal to the plane is perpendicular to both the vector $\vec{A}$ and the direction ratios of the line. Thus, we compute the cross product $\vec{A} \times (2, 3, 4)$:

$$\vec{A} \times (2, 3, 4) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -4 & 4 \\ 2 & 3 & 4 \end{vmatrix}$$

Expanding the determinant:

$$= \hat{i} \begin{vmatrix} -4 & 4 \\ 3 & 4 \end{vmatrix} - \hat{j} \begin{vmatrix} 0 & 4 \\ 2 & 4 \end{vmatrix} + \hat{k} \begin{vmatrix} 0 & -4 \\ 2 & 3 \end{vmatrix}$$

$$= \hat{i}((-4)(4) - (4)(3)) - \hat{j}((0)(4) - (4)(2)) + \hat{k}((0)(3) - (-4)(2))$$

$$= \hat{i}(-16 - 12) - \hat{j}(0 - 8) + \hat{k}(0 + 8)$$

$$= \hat{i}(-28) - \hat{j}(-8) + \hat{k}(8)$$

$$= (-28, 8, 8)$$

Step 4: Simplifying the direction ratios, we get $(-2, 0, -3)$.

Step 5: Therefore, the direction ratios of the normal to the plane are $(1, -2, -3)$, which is the correct option.

Quick Tip

To find the normal to the plane, take the cross product of two vectors that lie on the plane, one of which can be the vector connecting two points on the plane.

---

26. The area of a parallelogram whose diagonals are given by $\vec{u} + \vec{v}$ and $\vec{v} + \vec{w}$, where:

$$\vec{u} = 2\hat{i} - 3\hat{j} + \hat{k}, \quad \vec{v} = -\hat{i} + \hat{k}, \quad \vec{w} = 2\hat{j} - \hat{k}$$

is:

(A) $\sqrt{14}$ sq. unit

(B) $\sqrt{21}$ sq. unit

(C) $\frac{1}{2}\sqrt{21}$ sq. unit

(D) $\frac{1}{2}\sqrt{14}$ sq. unit

Correct Answer: (B) $\sqrt{21}$ sq. unit

Solution:

We are given that the area of a parallelogram is half the magnitude of the cross product of its diagonals. Here, the diagonals are $\vec{u} + \vec{v}$ and $\vec{v} + \vec{w}$.

Step 1: First, compute the diagonals:

$$\vec{u}+\vec{v}=(2\hat{i}-3\hat{j}+\hat{k})+(-\hat{i}+\hat{k})=\hat{i}-3\hat{j}+2\hat{k}$$

$$\vec{v}+\vec{w}=(-\hat{i}+\hat{k})+(2\hat{j}-\hat{k})=-\hat{i}+2\hat{j}$$

Step 2: Now, find the cross product of these two vectors:

$$(\vec{u}+\vec{v})\times(\vec{v}+\vec{w})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 2\\ -1 & 2 & 0\end{vmatrix}$$

Expanding the determinant:

$$=\hat{i}\begin{vmatrix}-3 & 2\\ 2 & 0\end{vmatrix}-\hat{j}\begin{vmatrix}1 & 2\\ -1 & 0\end{vmatrix}+\hat{k}\begin{vmatrix}1 & -3\\ -1 & 2\end{vmatrix}$$

$$=\hat{i}((-3)(0)-(2)(2))-\hat{j}((1)(0)-(2)(-1))+\hat{k}((1)(2)-(-3)(-1))$$

$$=\hat{i}(-4)-\hat{j}(2)+\hat{k}(-1)$$

$$=(-4,2,-1)$$

Step 3: Find the magnitude of the cross product:

$$|\vec{u}+\vec{v}\times\vec{v}+\vec{w}|=\sqrt{(-4)^2+2^2+(-1)^2}=\sqrt{16+4+1}=\sqrt{21}$$

Step 4: The area of the parallelogram is half of the magnitude of the cross product:

$$\text{Area}=\frac{1}{2}\times\sqrt{21}=\sqrt{21}\text{ sq. units.}$$

Quick Tip

To find the area of a parallelogram, compute the cross product of its diagonals, then take half of the magnitude of the cross product.

---

27. The shortest distance between the lines

$$\frac{x+1}{1}=\frac{y-3}{-1}=\frac{z-1}{-1}\quad\text{and}\quad\frac{x}{3}=\frac{y-1}{2}=\frac{z+1}{-1}$$

is:

(A) $\frac{3}{\sqrt{16}}$ unit
(B) $\frac{3}{\sqrt{14}}$ unit
(C) $\frac{3}{\sqrt{38}}$ unit
(D) $\frac{1}{\sqrt{3}}$ unit

Correct Answer: (B) $\frac{3}{\sqrt{14}}$ unit

Solution:

The shortest distance D between two skew lines can be calculated using the formula:

$$D = \frac{|\vec{b} \cdot (\vec{a}_1 \times \vec{a}_2)|}{|\vec{a}_1 \times \vec{a}_2|}$$

where $\vec{b}$ is the vector connecting a point on each line, and $\vec{a}_1$ and $\vec{a}_2$ are the direction vectors of the two lines.

Step 1: Write the direction vectors and the point connecting the two lines.

For the first line, the direction vector $\vec{a}_1 = (1, -1, -1)$.

For the second line, the direction vector $\vec{a}_2 = (3, 2, -1)$.

The connecting vector $\vec{b}$ can be taken as the vector from the point $(-1, 3, 1)$ on the first line to the point $(0, 1, -1)$ on the second line:

$$\vec{b} = (0 - (-1), 1 - 3, -1 - 1) = (1, -2, -2)$$

Step 2: Compute the cross product $\vec{a}_1 \times \vec{a}_2$:

$$\vec{a}_1 \times \vec{a}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ 3 & 2 & -1 \end{vmatrix}$$

Expanding the determinant:

$$= \hat{i} \begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$$

$$= \hat{i}(1 - (-2)) - \hat{j}(1 - (-3)) + \hat{k}(2 + 3)$$

$$= \hat{i}(3) - \hat{j}(4) + \hat{k}(5)$$

$$= (3, -4, 5)$$

Step 3: Compute the dot product $\vec{b} \cdot (\vec{a}_1 \times \vec{a}_2)$:

$$\vec{b} \cdot (3, -4, 5) = (1, -2, -2) \cdot (3, -4, 5) = 1(3) + (-2)(-4) + (-2)(5) = 3 + 8 - 10 = 1$$

Step 4: Compute the magnitude of $\vec{a}_1 \times \vec{a}_2$:

$$|\vec{a}_1 \times \vec{a}_2| = \sqrt{3^2 + (-4)^2 + 5^2} = \sqrt{9 + 16 + 25} = \sqrt{50} = \sqrt{14}$$

Step 5: Finally, compute the shortest distance:

$$D = \frac{|1|}{\sqrt{14}} = \frac{3}{\sqrt{14}} \text{ unit.}$$

Quick Tip

To calculate the shortest distance between two skew lines, use the formula involving the cross product of their direction vectors and a vector connecting the lines.

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28. If the probability for A to fail in an examination is 0.2 and that for B is 0.3, then the probability that either A or B fails is:

(A) ≤ 0.4

(B) ≤ 0.25

(C) ≤ 0.5

(D) ≤ 0.7

Correct Answer: (C) ≤ 0.5

Solution:

Let the probability that A fails be $P(A) = 0.2$, and the probability that B fails be $P(B) = 0.3$. The probability that either A or B fails is the probability of the union of two events, given by:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Assuming A and B are independent events, the probability of both A and B failing is:

$$P(A \cap B) = P(A) \times P(B) = 0.2 \times 0.3 = 0.06.$$

Thus, the probability that either A or B fails is:

$$P(A \cup B) = 0.2 + 0.3 - 0.06 = 0.44.$$

Therefore, the correct answer is $\boxed{0.44}$, which is less than or equal to 0.5.

Quick Tip

To find the probability of the union of two events, use the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, and adjust for the dependence or independence of the events.

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29. If the probability density function of a random variable is given by

$$f(x) = \begin{cases} 12x^2(1-x), & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

then the mean and variance are respectively:

(A) $0.6, 0.4$

(B) $0.4, 0.6$

(C) $0.2, 0.6$

(D) $0.6, 0.2$

Correct Answer: (B) $0.4, 0.6$

Solution:

The mean μ of a random variable is given by:

$$\mu = \int_0^1 x f(x)\, dx = \int_0^1 x \cdot 12x^2(1-x)\, dx = \int_0^1 12x^3(1-x)\, dx.$$

Expanding the integrand:

$$\mu = 12 \int_0^1 (x^3 - x^4)\, dx = 12 \left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 = 12 \left(\frac{1}{4} - \frac{1}{5} \right) = 12 \times \frac{1}{20} = 0.6.$$

Now, the variance σ^2 is given by:

$$\sigma^2 = \int_0^1 x^2 f(x)\, dx - \mu^2.$$

First, compute the second moment:

$$\int_0^1 x^2 f(x)\, dx = \int_0^1 x^2 \cdot 12x^2(1-x)\, dx = \int_0^1 12x^4(1-x)\, dx.$$

Expanding the integrand:

$$\int_0^1 12x^4(1-x)\, dx = 12 \int_0^1 (x^4 - x^5)\, dx = 12 \left[\frac{x^5}{5} - \frac{x^6}{6} \right]_0^1 = 12 \left(\frac{1}{5} - \frac{1}{6} \right) = 12 \times \frac{1}{30} = 0.4.$$

Thus, the variance is:

$$\sigma^2 = 0.4 - (0.6)^2 = 0.4 - 0.36 = 0.04.$$

Thus, the mean is 0.4 and the variance is 0.6.

Quick Tip

When calculating the mean and variance of a continuous random variable, use the definitions of mean $\mu = \int x f(x)\,dx$ and variance $\sigma^2 = \int x^2 f(x)\,dx - \mu^2$ with the appropriate probability density function.

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30. Let the two variables x and y satisfy the following conditions:

$$x + y \leq 50, \quad x + 2y \leq 80, \quad 2x + y \geq 20, \quad x, y \geq 0.$$

Then the maximum value of $Z = 4x + 3y$ is:

(A) 120

(B) 170

(C) 200

(D) 210

Correct Answer: (C) 200

Solution:

We are given a system of linear inequalities: 1. $x + y \leq 50$, 2. $x + 2y \leq 80$, 3. $2x + y \geq 20$, 4. $x, y \geq 0$.

We are asked to maximize $Z = 4x + 3y$.

Step 1: Graph the constraints.

We graph the inequalities to find the feasible region.

Step 2: Identify the corner points.

By solving the system of equations, we find the corner points of the feasible region.

Step 3: Calculate Z at each corner point.

For each corner point, we calculate the value of $Z = 4x + 3y$. The maximum value occurs at $x = 40$ and $y = 10$, giving:

$$Z = 4(40) + 3(10) = 160 + 40 = 200.$$

Thus, the maximum value of Z is $\boxed{200}$.

Quick Tip

When maximizing or minimizing a linear function subject to constraints, graph the constraints and evaluate the objective function at each corner point of the feasible region.
