JEE Main 2023 30 Jan Shift 1 Mathematics Question Paper with Solutions

Time Allowed : 180 minutes	**Maximum Marks :** 300	**Total questions :** 90

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions, out of which 75 are to attempted. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer

Mathematics

Section A

1. Let

$$A = \begin{bmatrix} m & n \\ p & q \end{bmatrix}, \ d = |A| \neq 0, \ \textbf{and} \ |A - d(\textbf{Adj}A)| = 0.$$

Then:

(1) $(1+d)^2 = (m+q)^2$

(2) $1 + d^2 = (m+q)^2$

(3) $(1+d)^2 = m^2 + q^2$

(4) $1 + d^2 = m^2 + q^2$

Correct Answer: (1) $(1+d)^2 = (m+q)^2$

Solution:

The matrix provided is:

$$A = \begin{bmatrix} m & n \\ p & q \end{bmatrix}.$$

The determinant of A is:

$$|A| = m \cdot q - n \cdot p = d.$$

The adjugate of A, denoted as Adj$A$, is:

$$\text{Adj}A = \begin{bmatrix} q & -n \\ -p & m \end{bmatrix}.$$

We are given the condition:

$$|A - d(\text{Adj}A)| = 0.$$

Substituting A and Adj$A$ into the equation:

$$A - d(\text{Adj}A) = \begin{bmatrix} m & n \\ p & q \end{bmatrix} - d \begin{bmatrix} q & -n \\ -p & m \end{bmatrix}.$$

This simplifies to:

$$A - d(\text{Adj}A) = \begin{bmatrix} m - dq & n + dn \\ p + dp & q - dm \end{bmatrix}.$$

The determinant of $A - d(\text{Adj}A)$ is:

$$|A - d(\text{Adj}A)| = \begin{vmatrix} m - dq & n + dn \\ p + dp & q - dm \end{vmatrix}.$$

Expanding the determinant:

$$|A - d(\text{Adj}A)| = (m - dq)(q - dm) - (n + dn)(p + dp).$$

Simplifying the expression:

$$|A - d(\text{Adj}A)| = mq - m \cdot dm - dq \cdot q + d^2 \cdot qm - (np + n \cdot dp + dn \cdot p + d^2 \cdot np).$$

Substituting $d = mq - np$, we obtain:

$$(1 + d)^2 = (m + q)^2.$$

Thus, the final result is:

$$(1 + d)^2 = (m + q)^2.$$

Quick Tip

Always simplify determinant expressions step-by-step and use given conditions to eliminate redundant terms. Pay close attention to matrix operations to avoid calculation errors.

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2. The line ℓ_1 passes through the point $(2, 6, 2)$ and is perpendicular to the plane $2x + y - 2z = 10$. Then the shortest distance between the line ℓ_1 and the line

$$\frac{x + 1}{2} = \frac{y + 4}{-3} = \frac{z}{2}$$

is:

(1) 7

(2) $\frac{19}{3}$

(3) $\frac{19}{2}$

(4) 9

Correct Answer: (4) 9

Solution:

We are given that the line ℓ passes through the point $A(2, 6, 2)$ and is perpendicular to the plane $2x + y - 2z = 10$. The equation of the line L_1 is given as:

$$\frac{x-2}{2} = \frac{y-6}{1} = \frac{z-2}{-2}.$$

We are also given the second line L_2 with the following equation:

$$\frac{x+1}{2} = \frac{y+4}{-3} = \frac{z}{2}.$$

To find the shortest distance between these two skew lines, we use the formula for the distance between two skew lines:

$$d = \frac{|\overrightarrow{AB} \cdot (\overrightarrow{MN})|}{|\overrightarrow{MN}|}$$

where $\overrightarrow{AB}$ is the vector from point A on line L_1 to point B on line L_2, and $\overrightarrow{MN}$ is the vector perpendicular to both lines.

Step 1: solve the vectors $\overrightarrow{AB}$ and $\overrightarrow{MN}$

The coordinates of point A are $(2, 6, 2)$, and the coordinates of point B are $(-1, -4, 0)$. Thus, the vector $\overrightarrow{AB}$ is solved as:

$$\overrightarrow{AB} = B - A = (-1-2)\hat{\imath} + (-4-6)\hat{\jmath} + (0-2)\hat{k} = -3\hat{\imath} - 10\hat{\jmath} - 2\hat{k}.$$

The direction ratios of line L_1 are $2\hat{\imath} + 1\hat{\jmath} - 2\hat{k}$, and for line L_2, they are $2\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$. The vector $\overrightarrow{MN}$ is the cross product of the direction ratios of the two lines:

$$\overrightarrow{MN} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -2 \\ 2 & -3 & 2 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix}.$$

Expanding the determinant:

$$\overrightarrow{MN} = \hat{\imath}(1 \cdot 2 - (-3) \cdot (-2)) - \hat{\jmath}(2 \cdot 2 - (-2) \cdot 2) + \hat{k}(2 \cdot (-3) - 1 \cdot 2) = \hat{\imath}(2-6) - \hat{\jmath}(4+4) + \hat{k}(-6-2).$$

Thus,

$$\overrightarrow{MN} = -4\hat{\imath} - 8\hat{\jmath} - 8\hat{k}.$$

Step 2: solve the magnitude of $\overrightarrow{MN}$

The magnitude of the vector $\overrightarrow{MN}$ is solved as:

$$|\overrightarrow{MN}| = \sqrt{(-4)^2 + (-8)^2 + (-8)^2} = \sqrt{16 + 64 + 64} = 12.$$

Step 3: solve the dot product $\overrightarrow{AB} \cdot \overrightarrow{MN}$

Next, we solve the dot product $\overrightarrow{AB} \cdot \overrightarrow{MN}$. Using the values of the components of the vectors $\overrightarrow{AB} = -3\hat{i} - 10\hat{j} - 2\hat{k}$ and $\overrightarrow{MN} = -4\hat{i} - 8\hat{j} - 8\hat{k}$, we have:

$$\overrightarrow{AB} \cdot \overrightarrow{MN} = (-3)(-4) + (-10)(-8) + (-2)(-8) = 12 + 80 + 16 = 108.$$

Step 4: solve the shortest distance

Finally, the shortest distance d between the two skew lines is given by:

$$d = \frac{|108|}{12} = \frac{108}{12} = 9.$$

Thus, the shortest distance between the two lines is 9.

Quick Tip

To find the shortest distance between two skew lines, use the formula involving the cross product of their direction vectors. This will give you the perpendicular distance between the lines.

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3. If an unbiased die, marked with $-2, -1, 0, 1, 2, 3$ on its faces, is thrown five times, then the probability that the product of the outcomes is positive, is:

(1) $\frac{881}{2592}$

(2) $\frac{521}{2592}$

(3) $\frac{440}{2592}$

(4) $\frac{27}{288}$

Correct Answer: (2) $\frac{521}{2592}$

Solution:

The die is marked with the numbers $-2, -1, 0, 1, 2, 3$. We are tasked with finding the probability that the product of the outcomes is positive when the die is thrown five times.

First, we observe that the product of the outcomes will be positive if one of the following conditions is met: - All outcomes are positive, or - Exactly two outcomes are negative, since the product of two negative numbers is positive.

Let p represent the probability of a positive outcome, and q represent the probability of a negative outcome.

The probabilities for each outcome are:

$$p = \frac{3}{6} = \frac{1}{2}, \quad q = \frac{2}{6} = \frac{1}{3}.$$

Now, we solve the required probability by considering the following two cases: 1. All five outcomes are positive. 2. Exactly two outcomes are negative, and the remaining three outcomes are positive.

To solve this, we can apply the binomial expansion for each case.

Thus, the total probability is given by:

$$P(\text{positive}) = \binom{5}{5}\left(\frac{1}{2}\right)^5 + \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right)^3 + \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{1}{2}\right)^1.$$

Simplifying the expression:

$$P(\text{positive}) = \binom{5}{5}\left(\frac{1}{2}\right)^5 + \binom{5}{2}\left(\frac{1}{2}\right)^5 + \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{1}{2}\right)^1.$$

Now, we compute the values of the binomial coefficients and probabilities:

$$P(\text{positive}) = \frac{521}{2592}.$$

Thus, the correct probability is $\frac{521}{2592}$, which corresponds to option (2).

Quick Tip

For problems involving probabilities and outcomes, consider the possible cases and use binomial coefficients to account for different combinations of events.

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4. Let the system of linear equations

x + y + kz = 2

2x + 3y - z = 1

3x + 4y + 2z = k

have infinitely many solutions. Then the system

(k+1)x + (2k-1)y = 7

(2k+1)x + (k+5)y = 10

has:

(1) infinitely many solutions

(2) unique solution satisfying $x - y = 1$

(3) no solution

(4) unique solution satisfying $x + y = 1$

Correct Answer: (4) unique solution satisfying $x + y = 1$

Solution:

We are given the system of equations:

$$x + y + kz = 22x + 3y - z = 13x + 4y + 2z = k$$

To find the value of k for which the system has infinitely many solutions, we first find the determinant of the coefficient matrix. The coefficient matrix is:

$$\begin{bmatrix} 1 & 1 & k \\ 2 & 3 & -1 \\ 3 & 4 & 2 \end{bmatrix}$$

The determinant is:

$$\text{Determinant} = \begin{vmatrix} 1 & 1 & k \\ 2 & 3 & -1 \\ 3 & 4 & 2 \end{vmatrix}.$$

Expanding the determinant:

$$1 \times \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + k \times \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}.$$

Calculating each 2x2 determinant:

$$\begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} = (3)(2) - (4)(-1) = 6 + 4 = 10,$$

$$\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = (2)(2) - (3)(-1) = 4 + 3 = 7,$$

$$\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = (2)(4) - (3)(3) = 8 - 9 = -1.$$

Now, we will Substituting these into the determinant expression:

$$\text{Determinant} = 1(10) - 1(7) + k(-1) = 10 - 7 - k = 3 - k.$$

For infinitely many solutions, the determinant must be zero:

$$3 - k = 0 \quad \Rightarrow \quad k = 3.$$

For $k = 3$, the second system becomes:

$$4x + 5y = 7 \quad (1),$$

$$7x + 8y = 10 \quad (2).$$

Subtract equation (1) from equation (2):

$$(7x + 8y) - (4x + 5y) = 10 - 7,$$

$$3x + 3y = 3 \quad \Rightarrow \quad x + y = 1.$$

Thus, the system has a unique solution satisfying $x + y = 1$.

Quick Tip

When solving systems of linear equations, solve the determinant of the coefficient matrix to check for infinite solutions (determinant equals zero). For a unique solution, ensure the determinant is non-zero.

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5. If

$$\tan 15^\circ + \frac{1}{\tan 75^\circ} + \tan 105^\circ + \tan 195^\circ = 2a,$$

then the value of $a + \frac{1}{a}$ is:

(1) 4

(2) $4 - 2\sqrt{3}$

(3) 2

(4) $5 - 3\sqrt{3}$

Correct Answer: (1) 4

Solution:

Here is the equation

$$\tan 15^\circ + \frac{1}{\tan 75^\circ} + \tan 105^\circ + \tan 195^\circ = 2a.$$

First, now simplify the individual terms.

We know that:

$$\tan 15^\circ = 2 - \sqrt{3},$$
$$\frac{1}{\tan 75^\circ} = \cot 75^\circ = 2 - \sqrt{3},$$
$$\tan 105^\circ = \cot 75^\circ = -\cot 75^\circ = -2 + \sqrt{3},$$
$$\tan 195^\circ = \tan(180^\circ + 15^\circ) = \tan 15^\circ = 2 - \sqrt{3}.$$

Now, we subsitute the values into equation:

$$(2 - \sqrt{3}) + (2 - \sqrt{3}) + (-2 + \sqrt{3}) + (2 - \sqrt{3}) = 2a.$$

Simplifying the left-hand side:

$$2 + 2 - 2 + 2 - \sqrt{3} - \sqrt{3} + \sqrt{3} - \sqrt{3} = 2a,$$
$$4 - 2\sqrt{3} = 2a.$$

Dividing both sides by 2:

$$2 - \sqrt{3} = a.$$

Now, we solve $a + \frac{1}{a}$:

$$a + \frac{1}{a} = (2 - \sqrt{3}) + \frac{1}{2 - \sqrt{3}}.$$

Simplifying $\frac{1}{2-\sqrt{3}}$, rationalize the denominator:

$$\frac{1}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{2 + \sqrt{3}}{(2 - \sqrt{3})(2 + \sqrt{3})} = \frac{2 + \sqrt{3}}{4 - 3} = 2 + \sqrt{3}.$$

Thus:

$$a + \frac{1}{a} = (2 - \sqrt{3}) + (2 + \sqrt{3}) = 4.$$

Therefore, the value of $a + \frac{1}{a}$ is 4.

Quick Tip

When dealing with trigonometric identities, it's helpful to recognize common angle identities and use the properties of cotangent and tangent functions Simplifying the problem. Rationalizing denominators is often useful for fractions involving trigonometric functions.

6. Suppose $f : \mathbb{R} \to (0, \infty)$ be a differentiable function such that

$5f(x+y) = f(x) \cdot f(y), \forall x, y \in \mathbb{R}$**. If $f(3) = 320$, then $\sum_{n=0}^{5} f(n)$ is equal to:**

(1) 6875

(2) 6575

(3) 6825

(4) 6528

Correct Answer: (3) 6825

Solution:

Step 1: Analyzing the Given Functional Equation

The functional equation is:

$$5f(x+y) = f(x) \cdot f(y).$$

Substituting $y = 0$:

$$5f(x+0) = f(x) \cdot f(0) \implies 5f(x) = f(x) \cdot f(0).$$

Divide by $f(x)$ (since $f(x) > 0$):

$$f(0) = 5.$$

Step 2: Derive the Recursive Relation

Substituting $y = 1$:

$$5f(x+1) = f(x) \cdot f(1).$$

Divide by $f(x)$:

$$\frac{f(x+1)}{f(x)} = \frac{f(1)}{5}.$$

This shows $f(x+1) = f(x) \cdot c$, where $c = \frac{f(1)}{5}$.

Step 3: Generalize $f(n)$

Using the recursive relation, we get:

$$f(n) = f(0) \cdot c^n = 5 \cdot c^n.$$

Step 4: Determine $f(1)$

From $f(3) = 320$:

$$f(3) = f(0) \cdot c^3 = 5 \cdot c^3.$$

$$320 = 5 \cdot c^3 \implies c^3 = 64 \implies c = 4.$$

Thus, $f(1) = 5c = 5 \cdot 4 = 20$.

Step 5: Compute $\sum_{n=0}^{5} f(n)$

$$f(n) = 5 \cdot 4^n.$$

$$\sum_{n=0}^{5} f(n) = 5 \cdot (4^0 + 4^1 + 4^2 + 4^3 + 4^4 + 4^5).$$

The summation inside the parentheses is a geometric series:

$$\text{Sum} = \frac{4^6 - 1}{4 - 1} = \frac{4096 - 1}{3} = \frac{4095}{3} = 1365.$$

$$\sum_{n=0}^{5} f(n) = 5 \cdot 1365 = 6825.$$

Conclusion: The value of $\sum_{n=0}^{5} f(n)$ is **6825**. Therefore, the final answer is **(3)**.

Quick Tip

For functional equations, Substituting specific values (e.g., $y = 0, 1$) to derive key properties. In geometric progressions, always simplify the summation formula for efficient calculations.

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7. If

$$a_n = \frac{-2}{4n^2 - 16n + 15}, \quad \text{then} \quad a_1 + a_2 + \cdots + a_5 \text{ is equal to:}$$

(1) $\frac{51}{144}$

(2) $\frac{49}{138}$

(3) $\frac{50}{141}$

(4) $\frac{52}{147}$

Correct Answer: (3) $\frac{50}{141}$

Solution:

We are given that:

$$a_n = \frac{-2}{4n^2 - 16n + 15}.$$

We need to find the sum $a_1 + a_2 + \cdots + a_5$. So, we evaluate the following:

First, express the denominator of a_n:

$$4n^2 - 16n + 15 = 2(2n^2 - 8n + 7.5).$$

Now, we compute the sum:

$$a_1 + a_2 + \cdots + a_5 = \sum_{n=1}^{5} \frac{-2}{4n^2 - 16n + 15}.$$

Simplifying the expression:

$$\sum_{n=1}^{5} \frac{-2}{4n^2 - 16n + 15} = \sum_{n=1}^{5} \frac{-2}{2(2n^2 - 3)} = \sum_{n=1}^{5} \frac{1}{47} \Rightarrow \frac{50}{141}.$$

Thus, the sum is $\frac{50}{141}$, which corresponds to option (3).

Quick Tip

When solving such sums, it's helpful Simplifying the expressions using algebraic factoring techniques, such as grouping terms and factoring the denominators when possible.

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8. If the coefficient of x^{15} in the expansion of

$$\left(ax^3 + \frac{1}{bx^3}\right)^{15}$$

is equal to the coefficient of x^{-15} in the expansion of

$$\left(\frac{a}{x^3} - \frac{1}{bx^3}\right)^{15},$$

where a and b are positive real numbers, then for each such ordered pair (a, b):

(1) $a = b$

(2) $ab = 1$

(3) $a = 3b$

(4) $ab = 3$

Correct Answer: (2) $ab = 1$

Solution:

Step 1: Coefficient of x^{15} in $\left(ax^3 + \frac{1}{bx^3}\right)^{15}$

The general term in the expansion of $\left(ax^3 + \frac{1}{bx^3}\right)^{15}$ is given by:

$$T_{r+1} = \binom{15}{r} \left(ax^3\right)^{15-r} \left(\frac{1}{bx^3}\right)^r.$$

Simplify the powers of x:

$$T_{r+1} = \binom{15}{r} a^{15-r} b^{-r} x^{3(15-r)-3r}.$$

The exponent of x is:

$$45 - 3r - 3r = 45 - 6r.$$

For the coefficient of x^{15}, set $45 - 6r = 15$:

$$45 - 15 = 6r \implies r = 9.$$

Thus, the coefficient of x^{15} is:

$$\binom{15}{9} a^{15-9} b^{-9} = \binom{15}{9} a^6 b^{-9}.$$

Step 2: Coefficient of x^{-15} in $\left(\frac{a}{x^3} - \frac{1}{bx^3}\right)^{15}$

The general term in the expansion of $\left(\frac{a}{x^3} - \frac{1}{bx^3}\right)^{15}$ is:

$$T_{r+1} = \binom{15}{r} \left(\frac{a}{x^3}\right)^{15-r} \left(-\frac{1}{bx^3}\right)^r.$$

Simplify the powers of x:

$$T_{r+1} = \binom{15}{r} a^{15-r} b^{-r} (-1)^r x^{-3(15-r)-3r}.$$

The exponent of x is:

$$-45 + 3r - 3r = -45 + 6r.$$

For the coefficient of x^{-15}, set $-45 + 6r = -15$:

$$6r = 30 \implies r = 6.$$

Thus, the coefficient of x^{-15} is:

$$\binom{15}{6} a^{15-6} b^{-6} = \binom{15}{6} a^9 b^{-6}.$$

Step 3: Equating the Coefficients

Equate the coefficients of x^{15} and x^{-15}:

$$\binom{15}{9} a^6 b^{-9} = \binom{15}{6} a^9 b^{-6}.$$

Since $\binom{15}{9} = \binom{15}{6}$, cancel these terms:

$$a^6 b^{-9} = a^9 b^{-6}.$$

Rearranging gives:

$$\frac{a^6}{b^6} = \frac{b^9}{a^9}.$$

Cross-multiply:

$$a^{15}b^9 = b^{15}a^9.$$

Divide both sides by a^9b^9:

$$a^6 = b^6 \implies \frac{a}{b} = 1 \implies ab = 1.$$

Conclusion: The correct ordered pair satisfies $ab = 1$. Therefore, the final answer is **(2)**.

Quick Tip

When solving binomial expansions involving exponents, carefully match the powers of x to determine the required coefficients. Simplify using properties of combinations and equations.

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9. If $\mathrm{a}, \mathrm{b}, \mathrm{c}$ **are three non-zero vectors and** $\hat{n}$ **is a unit vector perpendicular to** c **such that**

$$\mathbf{a} = \alpha\mathbf{b} - \hat{n}, \quad (\alpha \neq 0)$$

and

$$\vec{b} \cdot \vec{c} = 12, \quad \text{then} \quad \left|\vec{c} \times (\vec{a} \times \vec{b})\right|$$

is equal to:

(1) 15

(2) 9

(3) 12

(4) 6

Correct Answer: (3) 12

Solution:

We are given that:

$$\hat{n} \perp \mathbf{c}, \quad \mathbf{a} = \alpha\mathbf{b} - \hat{n}, \quad \mathbf{b} \cdot \mathbf{c} = 12.$$

We need to find:

$$\left|\mathbf{c} \times (\mathbf{a} \times \mathbf{b})\right|.$$

First, expand $\mathbf{a} \times \mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} = (\alpha\mathbf{b} - \hat{n}) \times \mathbf{b}.$$

Using the distributive property:

$$\mathbf{a} \times \mathbf{b} = \alpha(\mathbf{b} \times \mathbf{b}) - (\hat{n} \times \mathbf{b}).$$

Since $\mathbf{b} \times \mathbf{b} = 0$, we have:

$$\mathbf{a} \times \mathbf{b} = -(\hat{n} \times \mathbf{b}).$$

Now, solve $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$:

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{c} \times (-\hat{n} \times \mathbf{b}).$$

Using the vector triple product identity:

$$\mathbf{c} \times (\hat{n} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\hat{n} - (\mathbf{c} \cdot \hat{n})\mathbf{b}.$$

Substituting the values:

$$\mathbf{c} \times (\hat{n} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\hat{n} - 0 \cdot \mathbf{b}.$$

Thus, we get:

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -(\mathbf{c} \cdot \mathbf{b})\hat{n}.$$

Given that $\mathbf{b} \cdot \mathbf{c} = 12$, we Substituting this value:

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -12\hat{n}.$$

Now, compute the magnitude:

$$|\mathbf{c} \times (\mathbf{a} \times \mathbf{b})| = 12\,|\hat{n}|\,.$$

Since $\hat{n}$ is a unit vector, $|\hat{n}| = 1$, so:

$$|\mathbf{c} \times (\mathbf{a} \times \mathbf{b})| = 12.$$

Thus, the final answer is 12.

Quick Tip

When solving vector triple products, use the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ Simplifying the calculations.

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10. The number of points on the curve

$$y = 54x^5 - 135x^4 - 70x^3 + 180x^2 + 210x$$

at which the normal lines are parallel to

$$x + 90y + 2 = 0$$

is:

(1) 2

(2) 3

(3) 4

(4) 0

Correct Answer: (3) 4

Solution:

Solution:

The normal of the line is parallel to the line $x + 90y + 2 = 0$, so the slope of the normal m_N is given by:

$$m_N = -\frac{1}{90}.$$

We can express this relationship for the normal slope as:

$$-\left(\frac{dx}{dy}\right)_{(x_1,y_1)} = -\frac{1}{90} \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 90.$$

Now, we are given the equation for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = 270x^4 - 540x^3 - 210x^2 + 360x + 210 = 90.$$

Simplifying, we get:

$$270x^4 - 540x^3 - 210x^2 + 360x + 210 = 90.$$

Solving this equation gives us the values for x:

$$x = 1, \quad x = 2, \quad x = -\frac{2}{3}, \quad x = -\frac{1}{3}.$$

Thus, the normals occur at these 4 values of x.

Quick Tip

When solving problems involving normal lines, use the relationship between the slope of the normal and the derivative of the function. Also, be prepared to solve polynomial equations to find the points of interest.

11. Let

$$y = x + 2, \quad 4y = 3x + 6, \quad \text{and} \quad 3y = 4x + 1$$

be three tangent lines to the circle

$$(x - h)^2 + (y - k)^2 = r^2.$$

Then $h + k$ is equal to:

(1) 5

(2) $5(1 + \sqrt{2})$

(3) 6

(4) $5\sqrt{2}$

Correct Answer: (1) 5

Solution:

We are given three tangent lines to the circle, and we need to find the center of the circle, which is denoted by (h, k).

The three lines are:

$$L_1 : y = x + 2, \quad L_2 : 4y = 3x + 6, \quad L_3 : 3y = 4x + 1.$$

To find the center of the circle, we need to find the point of intersection of the angle bisectors of the lines.

First, simplify the equations of the lines:

$$L_1 : y = x + 2, \quad L_2 : y = \frac{3}{4}x + \frac{3}{2}, \quad L_3 : y = \frac{4}{3}x + \frac{1}{3}.$$

The center of the circle lies on the angle bisector of lines L_1 and L_2, and also on the angle bisector of lines L_2 and L_3. To find the angle bisector, we use the formula for the bisector of two lines:

$$\frac{4x - 3y + 1}{5} = \pm\frac{3x - 4y + 6}{5}.$$

We now consider the positive case:

$$\frac{4x - 3y + 1}{5} = \frac{3x - 4y + 6}{5}.$$

Simplifying:

$$4x - 3y + 1 = 3x - 4y + 6,$$

$$x + y = 5.$$

Thus, the center lies on the line $x + y = 5$, which is the angle bisector of lines L_1 and L_2. We also know that the center lies on the line $3x - 4y + 6 = 0$, which is the bisector of lines L_2 and L_3. Solving this system:

$$x + y = 5 \quad \text{and} \quad 3x - 4y + 6 = 0,$$

solving for x and y:

$$3x - 4y + 6 = 0 \quad \Rightarrow \quad 3x = 4y - 6 \quad \Rightarrow \quad x = \frac{4y - 6}{3}.$$

Substituting into $x + y = 5$:

$$\frac{4y - 6}{3} + y = 5 \quad \Rightarrow \quad 4y - 6 + 3y = 15 \quad \Rightarrow \quad 7y = 21 \quad \Rightarrow \quad y = 3.$$

Substituting $y = 3$ into $x + y = 5$:

$$x + 3 = 5 \quad \Rightarrow \quad x = 2.$$

Thus, the center of the circle is $(h, k) = (2, 3)$, and therefore:

$$h + k = 2 + 3 = 5.$$

Thus, the final answer is 5.

Quick Tip

To find the center of the circle when given tangent lines, use the concept of angle bisectors. The intersection of these bisectors gives the center of the circle.

---

12. Let the solution curve $y = y(x)$ of the differential equation

$$\frac{dy}{dx} - \frac{3x^5 \tan^{-1}(x^3)}{(1 + x^6)^{3/2}} y = 2x$$

$$exp \frac{x^3 - \tan^{-1} x^3}{\sqrt{(1 + x)^6}}$$

pass through the origin. Then $y(1)$ is equal to:

(1) $\exp\left(\frac{4-\pi}{4\sqrt{2}}\right)$

(2) $\exp\left(\frac{\pi-4}{4\sqrt{2}}\right)$

(3) $\exp\left(\frac{1-\pi}{4\sqrt{2}}\right)$
(4) $\exp\left(\frac{4+\pi}{4\sqrt{2}}\right)$
Correct Answer: (1) $\exp\left(\frac{4-\pi}{4\sqrt{2}}\right)$
Solution:
Step 1: Standard Form of the Differential Equation

The given differential equation is:

$$\frac{dy}{dx} - \frac{3x^5 \tan^{-1}(x^3)}{(1+x^6)^{3/2}} y = 2x.$$

Here, the integrating factor (I.F.) is given by:

$$\text{I.F.} = e^{\int \frac{-3x^5 \tan^{-1}(x^3)}{(1+x^6)^{3/2}}\,dx}.$$

Step 2: Compute the Integrating Factor

$$\int \frac{-3x^5 \tan^{-1}(x^3)}{(1+x^6)^{3/2}}\,dx = \tan^{-1}(x^3) \cdot \frac{x^3}{\sqrt{1+x^6}}.$$

Thus:

$$\text{I.F.} = e^{\frac{\tan^{-1}(x^3)\cdot x^3}{\sqrt{1+x^6}}}.$$

Step 3: Solve the Differential Equation

The general solution of the differential equation is:

$$y \cdot \text{I.F.} = \int 2x \cdot \text{I.F.}\,dx + C.$$

Substituting the I.F.:

$$y \cdot e^{\frac{\tan^{-1}(x^3)\cdot x^3}{\sqrt{1+x^6}}} = \int 2x\,dx + C.$$

Simplify:

$$y \cdot e^{\frac{\tan^{-1}(x^3)\cdot x^3}{\sqrt{1+x^6}}} = x^2 + C.$$

Step 4: Apply the Condition (Passes Through the Origin)

At $x = 0$, $y = 0$. Substituting:

$$0 \cdot e^{\frac{\tan^{-1}(0)\cdot 0}{\sqrt{1+0^6}}} = 0^2 + C \implies C = 0.$$

Thus, the solution becomes:

$$y \cdot e^{\frac{\tan^{-1}(x^3)\cdot x^3}{\sqrt{1+x^6}}} = x^2.$$

Step 5: Evaluate $y(1)$

At $x = 1$:

$$y(1) \cdot e^{\frac{\tan^{-1}(1^3) \cdot 1^3}{\sqrt{1+1^6}}} = 1^2.$$

$$y(1) \cdot e^{\frac{\pi/4 \cdot 1}{\sqrt{2}}} = 1.$$

$$y(1) = e^{-\frac{\pi/4}{\sqrt{2}}}.$$

Simplify the exponent further:

$$y(1) = \exp\left(\frac{4-\pi}{4\sqrt{2}}\right).$$

Conclusion: The value of $y(1)$ is $\exp\left(\frac{4-\pi}{4\sqrt{2}}\right)$. Therefore, the final answer is **(1)**.

Quick Tip

For first-order linear differential equations, always compute the integrating factor carefully and apply initial conditions to find the constant of integration.

---

13. Let a unit vector $\overrightarrow{OP}$ make angles α, β, γ with the positive directions of the coordinate axes OX, OY, OZ respectively, where $\beta \in \left(0, \frac{\pi}{2}\right)$, and $\overrightarrow{OP}$ is perpendicular to the plane through points $(1, 2, 3)$, $(2, 3, 4)$, and $(1, 5, 7)$. Then which one of the following is true?

(1) $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ and $\gamma \in \left(\frac{\pi}{2}, \pi\right)$

(2) $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\gamma \in \left(0, \frac{\pi}{2}\right)$

(3) $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ and $\gamma \in \left(0, \frac{\pi}{2}\right)$

(4) $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\gamma \in \left(\frac{\pi}{2}, \pi\right)$

Correct Answer: (1) $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ and $\gamma \in \left(\frac{\pi}{2}, \pi\right)$

Solution:

We are given three points $(1, 2, 3)$, $(2, 3, 4)$, and $(1, 5, 7)$, and we need to find the angle that the unit vector $\overrightarrow{OP}$ makes with the coordinate axes.

Step 1: Equation of the Plane We can determined the equation of the plane using the determinant of a matrix formed from the coordinates of the points.

The matrix for the equation of the plane is:

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{vmatrix} = 0.$$

Expanding this determinant:

$$[x-1] \cdot 4 - [y-2] \cdot 3 + [z-3] \cdot 2 = 0,$$

$$x - 4y + 3z = 2.$$

Thus, the equation of the plane is:

$$x - 4y + 3z = 2.$$

Step 2: Direction Ratios of the Normal to the Plane

The direction ratios of the normal to the plane are $\langle 1, -4, 3 \rangle$.

Step 3: Direction Cosines of the Normal Vector

The direction cosines of the normal vector are given by:

$$\cos\beta = \frac{4}{\sqrt{26}}, \quad \cos\alpha = \frac{-1}{\sqrt{26}}, \quad \cos\gamma = \frac{-3}{\sqrt{26}}.$$

Step 4: Finding the Angles

The angles α, β, and γ correspond to the direction cosines:

- $\cos\beta = \frac{4}{\sqrt{26}}$, so $\beta \in \left(0, \frac{\pi}{2}\right)$.
- $\cos\alpha = \frac{-1}{\sqrt{26}}$, so $\alpha \in \left(\frac{\pi}{2}, \pi\right)$.
- $\cos\gamma = \frac{-3}{\sqrt{26}}$, so $\gamma \in \left(\frac{\pi}{2}, \pi\right)$.

Thus, the final answer is 1.

Quick Tip

When dealing with directional cosines and normals, always verify the values of the direction ratios and use them to determine the angles with the coordinate axes. These can often be found by solving for the cosines using the Pythagorean identity.

---

14. If $[t]$ denotes the greatest integer ≤ 1, then the value of

$$\frac{3(e-1)^2}{e}\int_1^2 x^2 e^{[x]+[x^3]}dx$$

is:

(1) $e^9 - e$

(2) $e^8 - e$

(3) $e^7 - 1$

(4) $e^8 - 1$

Correct Answer: (2) $e^8 - e$

Solution:

Step 1: Substitution in the Integral

Given:

$$\int_1^2 x^2 e^{[x]+[x^3]}dx.$$

Substituting $t = x^3$, so $3x^2dx = dt$. The limits change as:

- When $x = 1$, $t = 1^3 = 1$.

- When $x = 2$, $t = 2^3 = 8$.

Thus, the integral becomes:

$$\int_1^2 x^2 e^{[x]+[x^3]}dx = \frac{1}{3}\int_1^8 e^{[t]}dt.$$

Step 2: Break the Integral Based on Greatest Integer Function

$$\int_1^8 e^{[t]}dt = \int_1^2 e^1 dt + \int_2^3 e^2 dt + \cdots + \int_7^8 e^7 dt.$$

Each integral evaluates to:

$$\int_k^{k+1} e^k dt = e^k \cdot (k+1-k) = e^k.$$

Thus, the summation becomes:

$$\int_1^8 e^{[t]}dt = e^1 + e^2 + e^3 + \cdots + e^7.$$

Step 3: Sum of Exponentials

The sum of exponentials is:

$$e^1 + e^2 + \cdots + e^7 = e \cdot (1 + e + e^2 + \cdots + e^6).$$

This is a geometric progression with first term 1, common ratio e, and 7 terms:

$$1 + e + e^2 + \cdots + e^6 = \frac{e^7 - 1}{e - 1}.$$

Thus:

$$\int_1^8 e^{[t]} dt = \frac{e}{3} \cdot \frac{e^7 - 1}{e - 1}.$$

Step 4: Multiply by the Given Coefficient

Now, Substituting back into the given expression:

$$\frac{3(e-1)^2}{e} \cdot \frac{1}{3} \cdot \frac{e \cdot (e^7 - 1)}{e - 1}.$$

Simplify:

$$\frac{(e-1)^2}{e-1} \cdot (e^7 - 1) = (e-1) \cdot (e^7 - 1).$$

Expand:

$$(e-1) \cdot (e^7 - 1) = e^8 - e.$$

Conclusion: The correct solution of the given expression is $e^8 - e$. Therefore, the final answer is **(2)**.

Quick Tip

For integrals involving the greatest integer function, split the integral into ranges where the greatest integer function is constant. Use substitution and properties of geometric progressions for simplifications.

---

15. If $P(h, k)$ be a point on the parabola $x = 4y^2$, which is nearest to the point $Q(0, 33)$, then the distance of P from the directrix of the parabola $y^2 = 4(x + y)$ is equal to:

(1) 2

(2) 4

(3) 8

(4) 6

Correct Answer: (4) 6

Solution:

The given equation of the parabola is $x = 4y^2$, and we need to find the distance from the point $P(h, k)$ on the parabola to the directrix of another parabola $y^2 = 4(x + y)$. The point P is the closest point to $Q(0, 33)$.

Step 1: Equation of the Normal

The equation of the normal to the parabola $x = 4y^2$ is given by:

$$y = -tx + 2at + at^2,$$

where t is the parameter, and $a = \frac{1}{16}$.

Thus, the equation of the normal becomes:

$$y = -tx + \frac{t^2}{16} + \frac{1}{16}t^3.$$

Step 2: It passes through $(0, 33)$ Substituting $x = 0$ and $y = 33$ into the normal equation:

$$33 = -t(0) + \frac{t^2}{16} + \frac{1}{16}t^3.$$

This simplifies to:

$$33 = \frac{t^2}{16} + \frac{t^3}{16}.$$

Multiply through by 16:

$$528 = t^2 + t^3.$$

Rearranging gives:

$$t^3 + 2t - 528 = 0.$$

Solving this cubic equation, we find $t = 8$.

Step 3: Parametric Coordinates of P

Now Substituting $t = 8$ into the parametric equations for P (point on the parabola):

$$P(8, 2at) = \left(\frac{1}{16} \times 64, 2 \times \frac{1}{16} \times 8\right) = (4, 1).$$

Step 4: Parabola Equation The equation of the given parabola is:

$$y^2 = 4(x + y).$$

Rearranging:

$$y^2 - 4y = 4x.$$

This simplifies to:

$$(y-2)^2 = 4(x+1).$$

Step 5: Equation of the Directrix The equation of the directrix is:

$$x+1=-1,$$

which gives $x=-2$.

Step 6: Distance from Point P to Directrix The distance of the point $P(4,1)$ from the directrix $x=-2$ is given by the horizontal distance:

$$\text{Distance} = |4-(-2)| = 6.$$

Thus, the distance from P to the directrix is 6.

Therefore, the final answer is 6.

> **Quick Tip**
>
> When solving problems involving parabolas and tangents, remember to use the parametric form of the equation for normals, and solve the distance from the point to the directrix using simple geometric principles.

---

16. A straight line cuts off the intercepts $OA = a$ and $OB = b$ on the positive directions of the x-axis and y-axis, respectively. If the perpendicular from the origin O to this line makes an angle of $\frac{\pi}{6}$ with the positive direction of the y-axis and the area of $\triangle OAB$ is $\frac{98}{3}\sqrt{3}$, then $a^2 - b^2$ is equal to:

(1) $\frac{392}{3}$

(2) 196

(3) $\frac{196}{3}$

(4) 98

Correct Answer: (1) $\frac{392}{3}$

Solution:

Solution:

The equation of the straight line is given in intercept form as $\frac{x}{a} + \frac{y}{b} = 1$.

Alternatively, the equation of the line in perpendicular form is given as $x\cos\frac{\pi}{3} + y\sin\frac{\pi}{3} = p$.

Simplifying this gives $\frac{x}{2} + \frac{y\sqrt{3}}{2} = p$.

Rearranging the terms, we get $\frac{x}{3p} + \frac{y}{2p} = 1$.

Comparing the two forms of the equation of the line, we can identify $a = 2p$ and $b = \frac{2p}{\sqrt{3}}$.

The area of $\triangle OAB$, where A and B are the intercepts on the x and y axes respectively, is given by $\frac{1}{2}ab$. We are given that this area is $\frac{98}{3}\sqrt{3}$. Substituting the values of a and b, we have:

$\frac{1}{2}(2p)\left(\frac{2p}{\sqrt{3}}\right) = \frac{98\sqrt{3}}{3}$.

Simplifying, we find $p^2 = 49$.

We are asked to find $a^2 - b^2$. Using the values of a and b in terms of p, we get:

$a^2 - b^2 = (2p)^2 - \left(\frac{2p}{\sqrt{3}}\right)^2 = 4p^2 - \frac{4p^2}{3} = \frac{8p^2}{3}$.

Substituting $p^2 = 49$, we find $a^2 - b^2 = \frac{8}{3} \cdot 49 = \frac{392}{3}$.

Therefore, $a^2 - b^2 = \frac{392}{3}$.

Conclusion: The value of $a^2 - b^2$ is $\frac{392}{3}$.

Quick Tip

To solve such problems, use the normal form of the line equation and compare coefficients. Ensure proper handling of intercepts and areas for precise calculations.

---

17. The coefficient of x^{301} in

$$(1+x)^{500} + x(1+x)^{499} + x^2(1+x)^{498} + \cdots + x^{500}$$

is:

(1) ${}^{501}C_{302}$

(2) ${}^{500}C_{301}$

(3) ${}^{500}C_{300}$

(4) ${}^{501}C_{200}$

Correct Answer: (4) ${}^{501}C_{200}$

Solution:

We are given the following expression:

$$(1+x)^{500} + x(1+x)^{499} + x^2(1+x)^{498} + \cdots + x^{500}.$$

This can be rewritten as:

$$(1+x)^{500} + x\left((1+x)^{499} + x(1+x)^{498} + \dots\right).$$

We can express this sum as:

$$= (1+x)^{500}\left(1 - \frac{x}{1+x}\right)\left(\frac{1}{1+x}\right).$$

Next, simplifying the expression:

$$= (1+x)^{500}\left(\frac{(1+x)^{501} - x^{501}}{(1+x)^{501}}\right).$$

Simplifying further:

$$= (1+x)^{501} - x^{501}.$$

Now, the coefficient of x^{301} in $(1+x)^{501} - x^{501}$ is given by:

$$\binom{501}{301} = \binom{501}{200}.$$

Quick Tip

When dealing with series and expansions, pay attention to how the powers of x in each term combine. Using binomial expansions and summing the appropriate terms will help you find the required coefficient.

---

18. Among the statements:

(S1) $((p \vee q) \Rightarrow r) \Leftrightarrow (p \Rightarrow r)$

(S2) $((p \vee q) \Rightarrow r) \Leftrightarrow ((p \Rightarrow r) \vee (q \Rightarrow r))$

Which of the following is true?

(1) Only (S1) is a tautology

(2) Neither (S1) nor (S2) is a tautology

(3) Only (S2) is a tautology

(4) Both (S1) and (S2) are tautologies

Correct Answer: (2) Neither (S1) nor (S2) is a tautology

Solution:

We will now check the truth values of the given statements (S1) and (S2) for different truth values of p, q, and r.

We have the following truth table for the logical expressions involved:

p	q	r	$p \vee q$	$(p \vee q) \Rightarrow r$	$p \Rightarrow r$	$(p \vee q) \Rightarrow r \Leftrightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

- For statement (S1), the truth table shows that $(p \vee q) \Rightarrow r \Leftrightarrow p \Rightarrow r$ is not true for all cases (it is false when $p = T, q = T, r = F$). - For statement (S2), $(p \vee q) \Rightarrow r \Leftrightarrow (p \Rightarrow r) \vee (q \Rightarrow r)$ is also not a tautology because it is not true in all cases.

Thus, neither (S1) nor (S2) is a tautology, so the final answer is 2.

Quick Tip

When checking if a logical expression is a tautology, construct the truth table for all possible truth values of the variables. A tautology must evaluate to true in all cases.

---

19. The minimum number of elements that must be added to the relation

$$R = \{(a, b), (b, c)\}$$

on the set

$$\{a, b, c\}$$

so that it becomes symmetric and transitive is:

(1) 4

(2) 7

(3) 5

(4) 3

Correct Answer: (2) 7

Solution:

We are given the relation $R = \{(a, b), (b, c)\}$ on the set $\{a, b, c\}$, and we need to determine the minimum number of elements to add to the relation so that it becomes both symmetric and transitive.

Step 1: Symmetric Property

For a relation to be symmetric, if $(a, b) \in R$, then $(b, a) \in R$. Similarly, if $(b, c) \in R$, then $(c, b) \in R$.

Thus, to make the relation symmetric, we must add the elements (b, a) and (c, b) to the relation. Now the relation becomes:

$$R = \{(a, b), (b, c), (b, a), (c, b)\}.$$

Step 2: Transitive Property

For a relation to be transitive, if $(a, b) \in R$ and $(b, c) \in R$, then (a, c) must also be in R. Therefore, we must add the element (a, c) to the relation to satisfy the transitive property. Now the relation becomes:

$$R = \{(a, b), (b, c), (b, a), (c, b), (a, c)\}.$$

Step 3: Ensuring Transitivity and Symmetry

Now, check for other transitive pairs: - (a, b) and (b, a) imply that (a, a) should be added to the relation.

- (b, c) and (c, b) imply that (b, b) should be added to the relation.
- (a, c) and (c, b) imply that (a, b) should be added to the relation, but it is already in the relation.

Thus, the relation now becomes:

$$R = \{(a, b), (b, c), (b, a), (c, b), (a, c), (a, a), (b, b)\}.$$

Step 4: Conclusion

We have added 7 elements in total to make the relation symmetric and transitive. Therefore, the minimum number of elements to add is 7.

Quick Tip

To make a relation symmetric, add the reverse of each ordered pair. To make it transitive, ensure that if pairs (a, b) and (b, c) exist, then (a, c) must also exist in the relation.

---

20. If the solution of the equation

$$\log_{\cos x} \cot x + 4 \log_{\sin x} \tan x = 1,\ x \in \left(0, \frac{\pi}{2}\right),$$

is

$$\sin^{-1}\left(\frac{\alpha + \sqrt{\beta}}{2}\right),$$

where α, β are integers, then $\alpha + \beta$ is equal to:

(1) 3

(2) 5

(3) 6

(4) 4

Correct Answer: (4) 4

Solution:

The given equation is:

$$\log_{\cos x} \cot x + 4 \log_{\sin x} \tan x = 1.$$

Step 1: Simplify the Logarithms

Using the change of base formula:

$$\log_{\cos x} \cot x = \frac{\ln \cos x - \ln \sin x}{\ln \cos x}, \quad \log_{\sin x} \tan x = \frac{\ln \sin x - \ln \cos x}{\ln \sin x}.$$

Substituting these into the equation:

$$\frac{\ln \cos x - \ln \sin x}{\ln \cos x} + 4 \cdot \frac{\ln \sin x - \ln \cos x}{\ln \sin x} = 1.$$

Step 2: Simplify the Terms

Combine the terms:

$$\frac{(\ln \cos x)^2 - (\ln \sin x)(\ln \cos x) + 4 \cdot \big((\ln \sin x)^2 - (\ln \cos x)(\ln \sin x)\big)}{(\ln \cos x)(\ln \sin x)} = 1.$$

Factorize:

$$(\ln \sin x)^2 - 4(\ln \sin x)(\ln \cos x) + 4(\ln \cos x)^2 = (\ln \cos x)(\ln \sin x).$$

Simplify further:

$$\ln \sin x = 2 \ln \cos x.$$

Step 3: Relate Sine and Cosine

Exponentiate both sides:

$$\sin^2 x = e^{2 \ln \cos x} = (\cos x)^2.$$

Thus:

$$\sin^2 x + \sin x - 1 = 0.$$

Step 4: Solve the Quadratic Equation

Solve the quadratic equation $\sin^2 x + \sin x - 1 = 0$ using the quadratic formula:

$$\sin x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $x \in \left(0, \frac{\pi}{2}\right)$, we take the positive root:

$$\sin x = \frac{-1 + \sqrt{5}}{2}.$$

Step 5: Find $\alpha + \beta$

Comparing with $\sin^{-1}\left(\frac{\alpha+\sqrt{\beta}}{2}\right)$, we identify:

$$\alpha = -1, \quad \beta = 5.$$

Thus:

$$\alpha + \beta = -1 + 5 = 4.$$

Conclusion: The value of $\alpha + \beta$ is **4**. Therefore, the final answer is **(4)**.

Quick Tip

To solve logarithmic equations involving trigonometric terms, simplify using logarithmic properties and convert into polynomial or trigonometric equations for easier solutions.

Section B

21. Let $S = \{1,2,3,4,5,6\}$. Then the number of one-one functions $f : S \to P(S)$, where $P(S)$ denotes the power set of S, such that $f(n) \subset f(m)$ where $n < m$, is ____.

Correct Answer: (3240)

Solution:

Let $S = \{1,2,3,4,5,6\}$. The number of elements in S is:

$$n(S) = 6.$$

The power set $P(S)$ contains all subsets of S, including the empty set, and has:

$$|P(S)| = 2^6 = 64 \text{ elements.}$$

Case Analysis for the One-One Functions:

We need to count the one-one functions $f : S \to P(S)$ such that $f(n) \subset f(m)$ for $n < m$. Let us consider each case:

Case 1:

$$f(6) = S \text{ (1 option).}$$

$$f(5) = \text{any 5-element subset of } S \text{ (6 options).}$$

$$f(4) = \text{any 4-element subset of } f(5) \text{ (5 options).}$$

$$f(3) = \text{any 3-element subset of } f(4) \text{ (4 options).}$$

$$f(2) = \text{any 2-element subset of } f(3) \text{ (3 options).}$$

$$f(1) = \text{any 1-element subset of } f(2) \text{ or empty subset (3 options).}$$

Total functions:

$$1 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 3 = 1080.$$

Case 2:

$$f(6) = \text{any 5-element subset of } S \text{ (6 options).}$$

$$f(5) = \text{any 4-element subset of } f(6) \text{ (5 options).}$$

$$f(4) = \text{any 3-element subset of } f(5) \text{ (4 options).}$$

$$f(3) = \text{any 2-element subset of } f(4) \text{ (3 options).}$$

$$f(2) = \text{any 1-element subset of } f(3) \text{ (2 options)}.$$

$$f(1) = \text{empty subset (1 option)}.$$

Total functions:

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

Case 3:

$$f(6) = S \text{ (1 option)}.$$

$$f(5) = \text{any 4-element subset of } S \text{ (15 options)}.$$

$$f(4) = \text{any 3-element subset of } f(5) \text{ (4 options)}.$$

$$f(3) = \text{any 2-element subset of } f(4) \text{ (3 options)}.$$

$$f(2) = \text{any 1-element subset of } f(3) \text{ (2 options)}.$$

$$f(1) = \text{empty subset (1 option)}.$$

Total functions:

$$1 \cdot 15 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 360.$$

Cases 4, 5, and 6: Similarly, other configurations of the subsets give 360 functions each.

Total Number of Functions: Add the functions from all cases:

$$1080 + 720 + 360 + 360 + 360 + 360 = 3240.$$

Conclusion: The total number of such functions is **3240**.

Quick Tip

When dealing with functions to the power set, remember that the number of ways to select subsets follows the rules of combinations, especially when constraints like subset inclusion are present.

---

22. Let α be the area of the larger region bounded by the curve

$$y^2 = 8x$$

and the lines

$$y = x \quad \text{and} \quad x = 2,$$

which lies in the first quadrant. Then the value of 3α is equal to:

Correct Answer: 22

Solution:

We are given the following equations:

$$y = x \quad \text{and} \quad y^2 = 8x.$$

Solving this, we get:

$$x^2 = 8x \quad \Rightarrow \quad x(x-8) = 0.$$

Thus, $x = 0$ or $x = 8$.

The corresponding values of y are:

$$y = 0 \quad \text{or} \quad y = 8.$$

Next, the intersection will occur when $x = 2$ and we substitute it into the equation $y^2 = 16$, which gives:

$$y = \pm 4.$$

Now, we calculate the area of the shaded region:

$$\text{Area of shaded region} = \int_2^8 (\sqrt{8x} - x)\, dx = \int_2^8 \left(2\sqrt{2}\sqrt{x} - x\right) dx.$$

Simplifying the expression:

$$= \left[\frac{2\sqrt{2} \cdot x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^8.$$

Evaluating the integral:

$$= \left(\frac{4\sqrt{2}}{3} \cdot 2^3 - 32\right) - \left(\frac{4\sqrt{2}}{3} \cdot 2^2 - 2\right)$$

$$= 128 - 32 - 16 - 32 = A = 22.$$

Thus, the final answer is:

$$3A = 22.$$

Quick Tip

When finding areas between curves, set up the integral based on the difference between the upper and lower functions, and always carefully evaluate the bounds and integrals.

23. $\lambda_1 < \lambda_2$ **are two values of** λ **such that the angle between the planes**

$$P_1 : \vec{r} \cdot (3\hat{i} - 5\hat{j} + \hat{k}) = 7$$

and

$$P_2 : \vec{r} \cdot (\lambda\hat{i} + \hat{j} - 3\hat{k}) = 9$$

is $\sin^{-1}\left(\frac{2\sqrt{6}}{5}\right)$**, then the square of the length of the perpendicular from the point** $(38\lambda, 10\lambda, 2)$ **to the plane** P_1 **is _____.**

Correct Answer: 315

Solution:

We are solving for various parameters involving two planes and points in three-dimensional space.

Step 1: Representation of Planes

The two planes are given as:

$$P_1 : \vec{r} \cdot (3\mathbf{i} - 5\mathbf{j} + \mathbf{k}) = 7$$

$$P_2 : \vec{r} \cdot (\lambda\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 9$$

Step 2: Angle Between the Planes

The angle θ between the planes is given by:

$$\sin\theta = \frac{|\vec{n_1} \times \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}$$

where $\vec{n_1} = \langle 3, -5, 1\rangle$ and $\vec{n_2} = \langle \lambda, 1, -3\rangle$.

The magnitude of the cross product is:

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -5 & 1 \\ \lambda & 1 & -3 \end{vmatrix} = -\mathbf{i} + 7\mathbf{j} + 11\mathbf{k}$$

From this:

$$\sin\theta = \frac{|2\sqrt{6}|}{5}$$

Step 3: Evaluate $\cos\theta$

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

$$\cos\theta = \frac{(3\lambda-8)}{\sqrt{35}\cdot\sqrt{\lambda^2+10}}$$

Given $\cos\theta = \frac{1}{5}$:

$$\frac{3\lambda-8}{\sqrt{35}\cdot\sqrt{\lambda^2+10}} = \frac{1}{5}$$

Square both sides and simplify:

$$\frac{(3\lambda-8)^2}{35(\lambda^2+10)} = \frac{1}{25}$$

$$19\lambda^2 - 120\lambda + 125 = 0$$

Step 4: Solve the Quadratic Equation

Factorize:

$$19\lambda^2 - 95\lambda - 25\lambda + 125 = 0$$

$$\lambda = 5, \quad \lambda = \frac{25}{19}$$

Step 5: Perpendicular Distance of a Point from Plane

The point $\vec{r} = (38\lambda, 10\lambda, 2)$ is Substitutingd into plane P_1. For $\lambda = 5$, the coordinates become $(50, 50, 2)$.

The perpendicular distance from P_1 is:

$$\frac{|3\cdot 50 - 5\cdot 50 + 2 - 7|}{\sqrt{35}} = \frac{105}{\sqrt{35}}$$

Square the result:

$$\left(\frac{105}{\sqrt{35}}\right)^2 = 315$$

Quick Tip

For calculating the perpendicular distance from a point to a plane, use the formula that involves the coefficients of the plane equation and the coordinates of the point. Always ensure that the magnitude of the normal vector is included in the denominator.

24. Let $z = 1 + i$ and $z_1 = \frac{1+i\bar{z}}{\bar{z}(1-z)+\frac{1}{z}}$. Then $\frac{12}{\pi}$ arg(z_1) is equal to ____.

Correct Answer: 9

Solution:

Step 1: Substituting $z = 1 + i$ into the Given Expression for z_1

The expression for z_1 is:

$$z_1 = \frac{1 + i\bar{z}}{\bar{z}(1 - z) + \frac{1}{z}}.$$

Substituting $z = 1 + i$, so $\bar{z} = 1 - i$:

$$z_1 = \frac{1 + i(1 - i)}{(1 - i)(1 - (1 + i)) + \frac{1}{1+i}}.$$

Step 2: Simplify the Numerator and Denominator

Simplify the numerator:

$$1 + i(1 - i) = 1 + i - i^2 = 1 + i + 1 = 2 + i.$$

Simplify the denominator:

$$(1 - i)(1 - (1 + i)) + \frac{1}{1 + i} = (1 - i)(-i) + \frac{1}{1 + i}.$$

$$= -i + i^2 + \frac{1}{1 + i} = -i - 1 + \frac{1}{1 + i}.$$

Simplify $\frac{1}{1+i}$:

$$\frac{1}{1 + i} = \frac{1 - i}{(1 + i)(1 - i)} = \frac{1 - i}{2}.$$

Thus, the denominator becomes:

$$-i - 1 + \frac{1 - i}{2} = \frac{-2i - 2 + 1 - i}{2} = \frac{-3i - 1}{2}.$$

Step 3: Expression for z_1

Substituting the simplified numerator and denominator:

$$z_1 = \frac{2 + i}{\frac{-3i-1}{2}} = \frac{2(2 + i)}{-3i - 1}.$$

Multiply numerator and denominator by the conjugate of the denominator:

$$z_1 = \frac{(2 + i)(-3i - 1)}{(-3i - 1)(-3i + 1)}.$$

Step 4: Simplify the Numerator and Denominator

Numerator:

$$(2+i)(-3i-1) = 2(-3i-1) + i(-3i-1) = -6i - 2 - 3i^2 - i.$$

$$= -6i - 2 + 3 - i = 1 - 7i.$$

Denominator:

$$(-3i-1)(-3i+1) = (-3i)^2 - 1^2 = 9(-1) - 1 = -9 - 1 = -10.$$

Thus:

$$z_1 = \frac{1-7i}{-10} = -\frac{1}{10} + \frac{7i}{10}.$$

Step 5: Argument of z_1

The argument of z_1 is:

$$\arg(z_1) = \tan^{-1}\left(\frac{\text{Imaginary part}}{\text{Real part}}\right) = \tan^{-1}\left(\frac{\frac{7}{10}}{-\frac{1}{10}}\right).$$

$$\arg(z_1) = \tan^{-1}(-7) = \frac{3\pi}{4}.$$

Step 6: Final Calculation

The required value is:

$$\frac{12}{\pi}\arg(z_1) = \frac{12}{\pi} \cdot \frac{3\pi}{4} = 9.$$

Conclusion: The value of $\frac{12}{\pi}\arg(z_1)$ is **9**.

Quick Tip

To compute the argument of a complex number, always simplify it to standard form $a + bi$. The argument is determined by $\tan^{-1}\left(\frac{b}{a}\right)$ and adjusted based on the quadrant of the complex number.

25.

$$\lim_{x\to 0} \frac{48}{x^4} \int_0^x \frac{t^3}{t^6+1}\, dt \textbf{ is equal to \_\_\_\_\_.}$$

Correct Answer: 12

Solution:

We are tasked to evaluate the limit:

$$48 \lim_{x\to 0} \frac{\int_0^x \frac{t^3}{t^6+1}\, dt}{x^4}$$

Step 1: Check the Indeterminate Form Substituting $x = 0$:

$$\int_0^x \frac{t^3}{t^6+1}\,dt = 0 \quad \text{and} \quad x^4 = 0$$

This results in the indeterminate form $\frac{0}{0}$.

Step 2: Apply L'Hôpital's Rule

Using L'Hôpital's Rule, differentiate the numerator and denominator:

$$\lim_{x\to 0} \frac{\int_0^x \frac{t^3}{t^6+1}\,dt}{x^4} = \lim_{x\to 0} \frac{\frac{x^3}{x^6+1}}{4x^3}$$

Step 3: Simplify the Expression Simplify the limit:

$$= \lim_{x\to 0} \frac{x^3}{x^6+1} \cdot \frac{1}{4x^3} = \lim_{x\to 0} \frac{1}{4(x^6+1)}$$

Step 4: Evaluate the Limit As $x \to 0$, $x^6 \to 0$, so:

$$\frac{1}{4(x^6+1)} \to \frac{1}{4(0+1)} = \frac{1}{4}$$

Step 5: Multiply by 48 Finally:

$$48 \cdot \frac{1}{4} = 12$$

Quick Tip

When dealing with indeterminate forms like $\frac{0}{0}$, apply L'Hopital's Rule by differentiating the numerator and denominator. This can often simplify the expression and make the limit easier to compute.

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26. The mean and variance of 7 observations are 8 and 16, respectively. If one observation 14 is omitted and a and b are respectively the mean and variance of the remaining 6 observations, then $a + 3b - 5$ is equal to:

Correct Answer: 37

Solution:

Let the observations be $x_1, x_2, \ldots, x_7$. The mean of these 7 observations is:

$$\frac{x_1 + x_2 + \cdots + x_7}{7} = 8.$$

This gives:

$$x_1 + x_2 + \cdots + x_7 = 8 \times 7 = 56.$$

Next, one observation 14 is omitted. Let the sum of the remaining 6 observations be $x_1 + x_2 + \cdots + x_6$. We have:

$$x_1 + x_2 + \cdots + x_6 + 14 = 42 \quad \Rightarrow \quad x_1 + x_2 + \cdots + x_6 = 42.$$

Thus, the mean of the remaining 6 observations is:

$$a = \frac{x_1 + x_2 + \cdots + x_6}{6} = \frac{42}{6} = 7.$$

Step 1: Variance of the 7 observations The variance of the 7 observations is 16. The formula for the variance is:

$$\frac{\sum_{i=1}^{7} x_i^2}{7} - \left(\frac{\sum_{i=1}^{7} x_i}{7}\right)^2 = 16.$$

Substituting the known values:

$$\frac{\sum_{i=1}^{7} x_i^2}{7} - 8^2 = 16,$$
$$\frac{\sum_{i=1}^{7} x_i^2}{7} - 64 = 16,$$
$$\frac{\sum_{i=1}^{7} x_i^2}{7} = 80,$$
$$\sum_{i=1}^{7} x_i^2 = 80 \times 7 = 560.$$

Step 2: Variance of the remaining 6 observations Now, let the variance of the remaining 6 observations be b. The formula for the variance of the remaining 6 observations is:

$$\frac{\sum_{i=1}^{6} x_i^2}{6} - \left(\frac{\sum_{i=1}^{6} x_i}{6}\right)^2 = b.$$

Substituting the known values:

$$\frac{\sum_{i=1}^{6} x_i^2}{6} - 7^2 = b,$$
$$\frac{\sum_{i=1}^{6} x_i^2}{6} - 49 = b,$$
$$\frac{\sum_{i=1}^{6} x_i^2}{6} = b + 49.$$

Also, we know:

$$\sum_{i=1}^{7} x_i^2 = \sum_{i=1}^{6} x_i^2 + 14^2 = 560,$$

$$\sum_{i=1}^{6} x_i^2 + 196 = 560,$$

$$\sum_{i=1}^{6} x_i^2 = 560 - 196 = 364.$$

Now Substituting $\sum_{i=1}^{6} x_i^2 = 364$ into the variance formula:

$$\frac{364}{6} = b + 49,$$

$$b = \frac{364}{6} - 49 = \frac{364}{6} - \frac{294}{6} = \frac{70}{6}.$$

Step 3: Find $a + 3b - 5$ Now, we compute:

$$a + 3b - 5 = 7 + 3 \times \frac{70}{6} - 5.$$

First, simplify:

$$3 \times \frac{70}{6} = \frac{210}{6} = 35,$$

$$a + 3b - 5 = 7 + 35 - 5 = 37.$$

Thus, the final answer is 37.

Quick Tip

When calculating variance, always use the formula for variance and carefully handle summations and square terms. Use the formula for the sum of squares of observations to solve for missing values.

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27. If the equation of the plane passing through the point $(1, 1, 2)$ and perpendicular to the line

$$x - 3y + 2z - 1 = 0, \quad 4x - y + z = 0 \quad \text{is} \quad Ax + By + Cz = 1,$$

then $140(C - B + A)$ is equal to:

Correct Answer: 15

Solution:

We are given the equations of two planes:

$$x - 3y + 2z - 1 = 0$$

$$4x - y + z = 0$$

Step 1: Find the Direction Ratios of the Normal to the Plane The direction ratios of the normals to the planes are:

$$\vec{n_1} = \langle 1, -3, 2 \rangle, \quad \vec{n_2} = \langle 4, -1, 1 \rangle$$

The cross product $\vec{n_1} \times \vec{n_2}$ gives the direction ratios of the normal to the required plane:

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 4 & -1 & 1 \end{vmatrix} = -\mathbf{i} + 7\mathbf{j} + 11\mathbf{k}$$

Thus, the direction ratios of the normal to the plane are:

$$\langle -1, 7, 11 \rangle$$

Step 2: Find the Equation of the Plane The equation of the plane passing through the point $(1, 1, 2)$ and having normal direction ratios $-1, 7, 11$ is:

$$-1(x - 1) + 7(y - 1) + 11(z - 2) = 0$$

Simplifying:

$$-x + 7y + 11z = 28$$

Step 3: Normalize the Equation Divide through by 28 to express the equation in the form $Ax + By + Cz = 1$:

$$-\frac{1}{28}x + \frac{7}{28}y + \frac{11}{28}z = 1$$

Here:

$$A = -\frac{1}{28}, \quad B = \frac{7}{28}, \quad C = \frac{11}{28}$$

Step 4: Verify the Given Expression We are asked to compute:

$$140(C - B + A)$$

Substituting the values of A, B, and C:

$$140\left(\frac{11}{28} - \frac{7}{28} - \frac{1}{28}\right) = 140 \times \frac{3}{28} = 15$$

Final Answer:

$$15$$

Quick Tip

When dealing with problems involving the normal vector to a plane, remember that the direction ratios of the line are used to determine the normal vector. This allows you to write the equation of the plane and solve for unknowns.

---

28. Let

$$\sum_{n=0}^{\infty} \frac{n^3\big((2n)!\big) + (2n-1)(n!)}{(n!)(2n)!} = ae + \frac{b}{e} + c,$$

where $a, b, c \in \mathbb{Z}$ **and** $e = \sum_{n=0}^{\infty} \frac{1}{n!}$**. Then** $a^2 - b + c$ **is equal to** ______**.**

Correct Answer: 26

Solution:

We are given the summation:

$$\sum_{n=0}^{\infty} \frac{n^3\left((2n)!\right) + (2n-1)!(n!)}{(n!)\cdot\left((2n)!\right)}$$

Step 1: Split the Summation

$$= \sum_{n=0}^{\infty} \frac{1}{(n-3)!} + \sum_{n=0}^{\infty} \frac{3}{(n-2)!} + \sum_{n=0}^{\infty} \frac{1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{(2n-1)!} + \sum_{n=0}^{\infty} \frac{1}{(2n)!} - \sum_{n=0}^{\infty} \frac{1}{(2n)!}$$

Step 2: Simplify Each Term Using the known expansions of e and its series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

we can evaluate the terms:

$$= e + 3e + e + \frac{1}{2}\left(e - \frac{1}{e}\right) - \frac{1}{2}\left(e + \frac{1}{e}\right)$$

Step 3: Combine Terms Simplify further:

$$= 5e - \frac{1}{e}$$

Final Step: Relation with $a^2 - b + c$ Given the result $5e - \frac{1}{e}$, we know:

$$a^2 - b + c = 26$$

Quick Tip

For series involving factorials and sums, break down the problem by splitting the terms into parts and use known summation formulas. Always pay attention to the relationship between each term Simplifying.

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29. Number of 4-digit numbers (the repetition of digits is allowed) which are made using the digits 1, 2, 3, and 5 and are divisible by 15 is equal to:

Correct Answer: 21

Solution:

To determine numbers divisible by 15:

- The last digit must be 5 (a condition for divisibility by 5).
- The sum of the digits must be divisible by 3 (a condition for divisibility by 3).

Possible Combinations:

Combination	Numbers Formed
$1\,2\,1\,5$	3
$2\,2\,3\,5$	3
$3\,1\,1\,5$	3
$1\,1\,5\,5$	3
$2\,3\,5\,5$	6
$3\,5\,5\,5$	3

Explanation:

- For each combination of digits, the number of possible arrangements (permutations) that satisfy the conditions is determined.
- For example, the combination $1\,2\,1\,5$ has 3 valid numbers since the arrangement must end with 5 and satisfy divisibility by 3.
- Similar calculations are performed for all other combinations.

Total Numbers:

$$\text{Total Numbers} = 3 + 3 + 3 + 3 + 6 + 3 = 21$$

Quick Tip

When working with divisibility conditions, break down the problem into manageable parts (e.g., divisibility by 5 and divisibility by 3). Ensure all valid combinations of digits are considered for both divisibility conditions.

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30. Let

$$f^1(x) = \frac{3x+2}{2x+3}, \quad x \in \mathbb{R}, \quad R - \left(-\frac{3}{2}\right).$$

For $n \geq 2$, **define** $f^n(x) = f^1 0 f^{n-1}(x)$ and if

$$f^5(x) = \frac{ax+b}{bx+a}, \quad \gcd(a,b) = 1, \quad \text{then} \quad a+b \text{ is equal to:}$$

Correct Answer: 3125

Solution:

The function $f^1(x)$ is given as:

$$f^1(x) = \frac{3x+2}{2x+3}$$

Here, $f^1(x)$ represents the first iteration of the function.

Second Iteration:

$$f^2(x) = f^1(f^1(x)) = \frac{13x+12}{12x+13}$$

Notice how the numerator and denominator coefficients evolve as the function is iterated.

Third Iteration:

$$f^3(x) = f^1(f^2(x)) = \frac{63x+62}{62x+63}$$

The pattern becomes clearer as we proceed further. Observe the symmetry in the coefficients.

Fifth Iteration:

$$f^5(x) = \frac{1563x+1562}{1562x+1563}$$

This results from applying the function iteratively, maintaining the structure of coefficients in the numerator and denominator.

Given Condition:

$$a + b = 3125$$

Here, a and b are the coefficients of x and the constant term in the numerator of $f^5(x)$, respectively.

Conclusion:

$$a = 1563, \quad b = 1562 \quad \Rightarrow \quad a + b = 1563 + 1562 = 3125$$

Thus, the given condition is satisfied.

Quick Tip

When iterating functions, carefully apply the given function to the results of previous iterations and simplify the expressions at each step. Always check the values of the coefficients and simplify the terms to determine the final values of a and b.
