JEE Main 2023 25 Jan Shift 2 Mathematics Question Paper with Solutions

Time Allowed :60 minutes	Maximum Marks :100	Total questions :30

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 90 questions. The maximum marks are 300.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics having 30 questions in each part of equal weightage.
4. Each part (subject) has two sections.
   (i) Section-A: This section contains 20 multiple choice questions which have only one correct answer. Each question carries 4 marks for correct answer and –1 mark for wrong answer.
   (ii) Section-B: This section contains 10 questions. In Section-B, attempt any five questions out of 10. The answer to each of the questions is a numerical value. Each question carries 4 marks for correct answer and –1 mark for wrong answer. For Section-B, the answer should be rounded off to the nearest integer.

SECTION-A

1. Let the function $f(x) = 2x^3 + (2p-7)x^2 + 3(2p-9)x - 6$ have a maxima for some value of $x < 0$ and a minima for some value of $x > 0$. Then, the set of all values of p is:

(1) $\left(\frac{9}{2}, \infty\right)$

(2) $\left(0, \frac{9}{2}\right)$

(3) $\left(-\infty, \frac{9}{2}\right)$
(4) $\left(-\frac{9}{2}, \frac{9}{2}\right)$
Correct Answer: (3) $\left(-\infty, \frac{9}{2}\right)$.
Solution:

To find the values of p for which the function $f(x)$ has both a maximum at some $x < 0$ and a minimum at some $x > 0$, we begin by analyzing the first derivative of $f(x)$.

First, we compute the first derivative:

$$f'(x) = \frac{d}{dx}\left(2x^3 + (2p-7)x^2 + 3(2p-9)x - 6\right) = 6x^2 + 2(2p-7)x + 3(2p-9)$$

For $f(x)$ to have both a maximum and a minimum, the equation $f'(x) = 0$ must have two distinct real roots, one positive and one negative.

1. Discriminant Condition:

The discriminant D of the quadratic equation $6x^2 + 2(2p-7)x + 3(2p-9) = 0$ must be positive for two distinct real roots:

$$D = [2(2p-7)]^2 - 4 \cdot 6 \cdot 3(2p-9) = 4(2p-7)^2 - 72(2p-9) > 0$$

Simplifying:

$$4(4p^2 - 28p + 49) - 144p + 648 > 0$$

$16\text{p}^2 - 112p + 196 - 144p + 648 > 0$

$16\text{p}^2 - 256p + 844 > 0$

$4\text{p}^2 - 64p + 211 > 0$

Solving the inequality $4p^2 - 64p + 211 > 0$, we find the critical points and determine the intervals where the inequality holds.

2. Product of Roots Condition:

For the roots to have opposite signs, the product of the roots must be negative:

$$\text{Product of roots} = \frac{3(2p-9)}{6} = \frac{6p-27}{6} = p - \frac{9}{2} < 0$$

$\Rightarrow p < \frac{9}{2}$

By combining both conditions, the set of all values of p that satisfy both is:

$$p \in \left(-\infty, \frac{9}{2}\right)$$

Thus, the correct answer is option (3).

Quick Tip

When analyzing extrema of polynomial functions, always examine the first derivative for critical points and ensure the existence of necessary conditions (like distinct real roots with appropriate signs) to confirm the presence of maxima and minima.

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2. Let z be a complex number such that $\frac{|z-2i|}{|z+i|} = 2$, $z \neq -i$. Then z lies on the circle of radius 2 and centre:

(1) $(2, 0)$

(2) $(0, 0)$

(3) $(0, 2)$

(4) $(0, -2)$

Correct Answer: (4) $(0, -2)$.

Solution:

We are given the condition:

$$\frac{|z-2i|}{|z+i|} = 2$$

Let $z = x + yi$, where x and y are real numbers. Substituting this into the equation:

$$\frac{\sqrt{x^2 + (y-2)^2}}{\sqrt{x^2 + (y+1)^2}} = 2$$

Squaring both sides to eliminate the square roots:

$$\frac{x^2 + (y-2)^2}{x^2 + (y+1)^2} = 4$$

Cross-multiply:

$$x^2 + (y-2)^2 = 4\left(x^2 + (y+1)^2\right)$$

Expanding both sides:

$$x^2 + y^2 - 4y + 4 = 4x^2 + 4y^2 + 8y + 4$$

Move all terms to one side:

$$x^2 + y^2 - 4y + 4 - 4x^2 - 4y^2 - 8y - 4 = 0$$

-3x$^2 - 3y^2 - 12y = 0$

3x$^2 + 3y^2 + 12y = 0$

x$^2 + y^2 + 4y = 0$ Next, complete the square for the y-terms to express this in standard circle form:

$$x^2 + y^2 + 4y = 0$$

x$^2 + (y^2 + 4y + 4) = 4$

x$^2 + (y + 2)^2 = 4$ This represents a circle with center $(0, -2)$ and radius 2.

Thus, the correct answer is option (4).

Quick Tip

When working with complex numbers and geometric loci, converting the given conditions into Cartesian coordinates and completing the square can help in identifying the geometric figure and its properties.

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3. If the function

$$f(x) = \begin{cases} (1 + |\cos x|)^{\lambda/|\cos x|}, & 0 < x < \dfrac{\pi}{2} \\ \mu, & x = \dfrac{\pi}{2} \\ e^{\frac{\cot 6x}{\cot 4x}}, & \dfrac{\pi}{2} < x < \pi \end{cases}$$

is continuous at $x = \frac{\pi}{2}$**, then** $9\lambda + 6\log_e \mu + \mu^6 - e^{6\lambda}$ **is equal to:**

(1) 11

(2) 8

(3) $2e^4 + 8$

(4) 10

Correct Answer: (4) 10.

Solution:

To ensure that the function $f(x)$ is continuous at $x = \frac{\pi}{2}$, the left-hand and right-hand limits as x approaches $\frac{\pi}{2}$ must both be equal to $f\left(\frac{\pi}{2}\right) = \mu$.

1. Left-Hand Limit ($x \to \frac{\pi}{2}^-$):

$$\lim_{x \to \frac{\pi}{2}^-} (1 + |\cos x|)^{\frac{\lambda}{|\cos x|}} = \lim_{x \to \frac{\pi}{2}^-} (1 + \cos x)^{\frac{\lambda}{\cos x}}$$

As $x \to \frac{\pi}{2}^-$, $\cos x \to 0^+$, so:

$$(1 + \cos x)^{\frac{\lambda}{\cos x}} \approx e^{\lambda}$$

Hence,

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = e^{\lambda}$$

2. Right-Hand Limit ($x \to \frac{\pi}{2}^+$):

$$\lim_{x \to \frac{\pi}{2}^+} e^{\frac{\cot 6x}{\cot 4x}} = e^{\lim_{x \to \frac{\pi}{2}^+} \frac{\cot 6x}{\cot 4x}}$$

Simplifying the exponent:

$$\frac{\cot 6x}{\cot 4x} = \frac{\frac{\cos 6x}{\sin 6x}}{\frac{\cos 4x}{\sin 4x}} = \frac{\cos 6x \cdot \sin 4x}{\cos 4x \cdot \sin 6x}$$

As $x \to \frac{\pi}{2}^+$:

$$6x \to 3\pi \Rightarrow \cos 6x = \cos 3\pi = -1, \quad \sin 6x = \sin 3\pi = 0$$

$$4x \to 2\pi \Rightarrow \cos 4x = \cos 2\pi = 1, \quad \sin 4x = \sin 2\pi = 0$$

Applying L'Hôpital's Rule to the indeterminate form:

$$\lim_{x \to \frac{\pi}{2}^+} \frac{\cot 6x}{\cot 4x} = \lim_{x \to \frac{\pi}{2}^+} \frac{-\csc^2 6x \cdot 6}{-\csc^2 4x \cdot 4} = \lim_{x \to \frac{\pi}{2}^+} \frac{6\sin^2 4x}{4\sin^2 6x} = \frac{6}{4} \cdot \left(\frac{\sin 4x}{\sin 6x}\right)^2 = \frac{3}{2} \cdot \left(\frac{2}{3}\right)^2 = \frac{3}{2} \cdot \frac{4}{9} = \frac{2}{3}$$

Therefore,

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = e^{\frac{2}{3}}$$

3. Continuity Condition:

$$e^{\lambda} = \mu = e^{\frac{2}{3}}$$

Thus,

$$\lambda = \frac{2}{3}, \quad \mu = e^{\frac{2}{3}}$$

4. Evaluating the Expression:

$$9\lambda + 6\ln \mu + \mu^6 - e^{6\lambda} = 9\left(\frac{2}{3}\right) + 6\ln\left(e^{\frac{2}{3}}\right) + \left(e^{\frac{2}{3}}\right)^6 - e^{6 \cdot \frac{2}{3}} = 6 + 6 \cdot \frac{2}{3} + e^4 - e^4 = 6 + 4 + 0 = 10$$

Thus, the correct answer is option (4).

Quick Tip

When ensuring the continuity of piecewise functions, equate the left-hand and right-hand limits to the function's value at the point of interest. Solving these equations helps determine the required parameters for continuity.

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4. Let $f(x) = 2x^n + \lambda$, where $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$. Given that $f(4) = 133$ and $f(5) = 255$, Then the sum of all the positive integer divisors of $f(3) - f(2)$?

(1) 61

(2) 60

(3) 58

(4) 59

Correct Answer: (2) 60.

Solution:

We are given the following equations:

$$f(4) = 2 \cdot 4^n + \lambda = 133 \quad (1)$$

$$f(5) = 2 \cdot 5^n + \lambda = 255 \quad (2)$$

By subtracting equation (1) from equation (2), we get:

$$2 \cdot 5^n + \lambda - (2 \cdot 4^n + \lambda) = 255 - 133$$

$$2(5^n - 4^n) = 122$$

$$5^n - 4^n = 61$$

Now, let's test integer values of n:

$$n = 3 : \quad 5^3 - 4^3 = 125 - 64 = 61$$

Thus, $n = 3$. Substituting this value back into equation (1) to find λ:

$$2 \cdot 4^3 + \lambda = 133$$

$2 \cdot 64 + \lambda = 133$

$128 + \lambda = 133$

$\lambda = 5$ Next, we compute $f(3) - f(2)$:

$$f(3) = 2 \cdot 3^3 + 5 = 2 \cdot 27 + 5 = 54 + 5 = 59$$

$$f(2) = 2 \cdot 2^3 + 5 = 2 \cdot 8 + 5 = 16 + 5 = 21$$

$$f(3) - f(2) = 59 - 21 = 38$$

The positive integer divisors of 38 are:

$$1, 2, 19, 38$$

The sum of the divisors is:

$$1 + 2 + 19 + 38 = 60$$

Therefore, the correct answer is option (2).

Quick Tip

When working with functions defined at specific points, set up a system of equations using the given values. Solving this system helps determine the unknown parameters, which can then be used to evaluate other expressions involving the function.

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5. If the four points, whose position vectors are $3\hat{i} - 4\hat{j} + 2\hat{k}$, $\hat{i} + 2\hat{j} - \hat{k}$, $-2\hat{i} - \hat{j} + 3\hat{k}$, and $5\hat{i} - 2\alpha\hat{j} + 4\hat{k}$ are coplanar, then α is equal to:

(1) $\frac{73}{17}$

(2) $-\frac{107}{17}$

(3) $-\frac{73}{17}$

(4) $\frac{107}{17}$

Correct Answer: (1) $\frac{73}{17}$.

Solution:

The volume of the tetrahedron created by four points must be zero in order for them to be coplanar. This corresponds to $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ having a scalar triple product of zero.

1. Find Vectors:

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (\hat{i} + 2\hat{j} - \hat{k}) - (3\hat{i} - 4\hat{j} + 2\hat{k}) = -2\hat{i} + 6\hat{j} - 3\hat{k}$$

$$\overrightarrow{AC} = \hat{c} - \hat{a} = (-2\hat{i} - \hat{j} + 3\hat{k}) - (3\hat{i} - 4\hat{j} + 2\hat{k}) = -5\hat{i} + 3\hat{j} + \hat{k}$$

$$\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = (5\hat{i} - 2\alpha\hat{j} + 4\hat{k}) - (3\hat{i} - 4\hat{j} + 2\hat{k}) = 2\hat{i} - (2\alpha - 4)\hat{j} + 2\hat{k}$$

2. Compute the Scalar Triple Product:

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = 0$$

First, find $\overrightarrow{AC} \times \overrightarrow{AD}$:

$$\overrightarrow{AC} \times \overrightarrow{AD} =$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 3 & 1 \\ 2 & -(2\alpha - 4) & 2 \end{vmatrix} =$$

$$= \hat{i}\,(3 \cdot 2 - 1 \cdot (-(2\alpha - 4))) - \hat{j}\,(-5 \cdot 2 - 1 \cdot 2) + \hat{k}\,(-5 \cdot (-(2\alpha - 4)) - 3 \cdot 2)$$

$$= \hat{i}(6 + 2\alpha - 4) - \hat{j}(-10 - 2) + \hat{k}(10\alpha - 20 - 6)$$

$$= \hat{i}(2\alpha + 2) - \hat{j}(-12) + \hat{k}(10a - 26)$$

$$= (2\alpha + 2)\hat{i} + 12\hat{j} + (10\alpha - 26)\hat{k}$$

Now, compute the dot product with $\overrightarrow{AB}$:

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = (-2) \cdot (2a+2) + 6 \cdot 12 + (-3) \cdot (10\alpha - 26) = -4\alpha - 4 + 72 - 30\alpha + 78 = (-34\alpha) + 146$$

Setting the scalar triple product to zero:

$$-34\alpha + 146 = 0$$

-34α = -146

a = $\frac{146}{34}$ = $\frac{73}{17}$

Thus, $\alpha = \dfrac{73}{17}$.

Quick Tip

To determine if points are coplanar, applying the scalar triple product of vectors formed by these points. If the scalar triple product is zero, the points lie on the same plane.

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6. Let

$A = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} /\sqrt{10}$ **and** $B = \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}$**, where** $i = \sqrt{-1}$**. If** $M = A^T BA$**,**

, then the inverse of the matrix $AM^{2023}A^T$ is:

(1) $\begin{bmatrix} 1 & -2023i \\ 0 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 0 \\ -2023i & 1 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 0 \\ 2023i & 1 \end{bmatrix}$

(4) $\begin{bmatrix} 1 & 2023i \\ 0 & 1 \end{bmatrix}$

Correct Answer: (4).

Solution:

We begin by calculating M as $M = A^T BA$. The adjoint (conjugate transpose) of A, A^T, and the product $A^T BA$ lead to a specific form of M. Assuming M in its simplified form:

$$M = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}$$

Next, to compute M^{2023}, we observe the pattern of powers of M:

$$M^2 = \begin{bmatrix} 1 & 2i \\ 0 & 1 \end{bmatrix}, \quad M^3 = \begin{bmatrix} 1 & 3i \\ 0 & 1 \end{bmatrix}, \dots, M^n = \begin{bmatrix} 1 & ni \\ 0 & 1 \end{bmatrix}$$

Thus, we find:

$$M^{2023} = \begin{bmatrix} 1 & 2023i \\ 0 & 1 \end{bmatrix}$$

Now, to find the inverse of $AM^{2023}A^T$, we use the known forms of A, M^{2023}, and A^T. The computation yields:

$$AM^{2023}A = \begin{bmatrix} 1 & 2023i \\ 0 & 1 \end{bmatrix}$$

Therefore, the inverse is:

$$\begin{bmatrix} 1 & -2023i \\ 0 & 1 \end{bmatrix}$$

Thus, the correct answer is option (4).

Quick Tip

Matrix exponentiation can simplify significantly for special matrices, such as upper triangular matrices with ones on the diagonal. Using the properties of specific matrix forms can make computations much easier.

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7. Let $\triangle$ and $\bigtriangledown \in [\wedge, \vee]$ be such that the expression $(p \to q) \wedge (p \bigtriangledown q)$ is a tautology. Then:

(1) $\triangle = \wedge, \bigtriangledown = \vee$

(2) $\triangle = \vee, \bigtriangledown = \wedge$

(3) $\triangle = \vee, \bigtriangledown = \vee$

(4) $\triangle = \wedge, \bigtriangledown = \wedge$

Correct Answer: (3).

Solution:

For the given expression to be a tautology, every possible valuation of p and q must make the expression true. Since $p \to q$ is logically equivalent to $\neg p \vee q$, we can simplify the expression as follows:

$$(\neg p \vee q) \wedge (p \vee q)$$

Applying the distributive laws:

$$(\neg p \wedge p) \vee (\neg p \wedge q) \vee (p \wedge q) \vee (q \wedge q)$$

Next, simplifying further, knowing that $\neg p \wedge p$ is always false:

$$(\neg p \wedge q) \vee (p \wedge q) \vee q = q$$

Therefore, for the expression to be a tautology, it must always evaluate to true, which occurs when $\wedge$ and $\vee$ are defined such that any expression involving these operators always results in a true outcome.

Quick Tip

Using logical equivalences and truth table analysis is essential for verifying the properties of logical expressions. These tools help determine the conditions under which expressions hold as tautologies or contradictions.

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8. The number of numbers, strictly between 5000 and 10000, that can be formed using the digits 1, 3, 5, 7, 9 without repetition, is:

(1) 6

(2) 12

(3) 120

(4) 72

Correct Answer: (4) 72.

Solution:

The numbers must be four-digit numbers starting with 5, 7, or 9, as they need to fall between 5000 and 10000. Additionally, the digits used must be 1, 3, 5, 7, and 9, with no repetition.

There are three choices for the first digit (5, 7, or 9), four remaining choices for the second digit (excluding the first digit), three choices for the third digit, and two choices for the fourth digit:

$$3 \times 4 \times 3 \times 2 = 72$$

Therefore, there are 72 such numbers.

Quick Tip

When calculating permutations for digit-based problems, it's helpful to consider the constraints on the most significant digit, particularly when constructing numbers within a specified range.

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9. The number of functions $f : \{1, 2, 3, 4\} \to \{a \in Z : |a| \leq 8\}$ **satisfying** $f(n) + 1/nf(n+1)$ **=1, for** $\forall n \in \{1, 2, 3\}$ **is:**

(1) 3

(2) 4

(3) 1

(4) 2

Correct Answer: (4) 2.

Solution:

To satisfy the condition $f(n) + 1$ being divisible by n, the function values at each n must meet the following requirements:

$$f : \{1, 2, 3, 4\} \to \{a \in \mathbb{Z} : |a| \leq 8\}$$

$$f(n) + \frac{1}{n}f(n+1) = 1, \quad \forall n \in \{1, 2, 3\}$$

$$f(n+1) \text{ must be divisible by } n$$

$$f(4) \Rightarrow -6, -3, 0, 3, 6$$

$$f(3) \Rightarrow -8, -6, -4, -2, 0, 2, 4, 6, 8$$

$$f(2) \Rightarrow -8, \ldots\ldots\ldots\ldots\ldots, 8$$

$$f(1) \Rightarrow -8, \ldots\ldots\ldots\ldots\ldots, 8$$

$$\frac{f(4)}{3} \text{ must be odd since } f(3) \text{ should be even}$$

$\therefore$ therefore 2 solutions possible.

Quick Tip

When counting functions with divisibility conditions, consider modular constraints for each function value and combine the possibilities, ensuring to account for any restrictions on distinctness or overlaps where needed.

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10. The equations of two sides of a variable triangle are $x = 0$ and $y = 3$, and its third side is a tangent to the parabola $y^2 = 6x$. The locus of its circumcentre is:

(1) $4y^2 - 18y - 3x - 18 = 0$

(2) $4y^2 + 18y + 3x + 18 = 0$

(3) $4y^2 - 18y + 3x + 18 = 0$

(4) $4y^2 - 18y - 3x + 18 = 0$

Correct Answer: (3).

Solution:

To find the locus of the circumcenter, we analyze the properties of the triangle and its relationship with the parabola. The triangle's sides, aligned with the y-axis and the line $y = 3$, form a right angle at the origin. The third side, which is tangent to the parabola, imposes a specific geometric condition. This condition requires determining the tangent line's slope and its intersection points.

Starting with the derivative of the parabola $y^2 = 6x$, we have:

$$y^2 = 6x$$

and

$$y^2 = 4ax$$

Equating the two expressions gives:

$$4a = 6 \Rightarrow a = \frac{3}{2}$$

Next, the equation of the tangent line is:

$$y = mx + \frac{3}{2m}$$

(m $\neq 0$) Using the circumcenter formulas, we can calculate the coordinates h and k as:

$$h = \frac{6m - 3}{4m^2},$$

k = 6m + 3$\frac{}{4m}$ Substituting into the equation:

$$3h = 2(-2k^2 + 9k - 9)$$

This leads to:

$$4y^2 - 18y + 3x + 18 = 0$$

Solving this equation for the coordinates of the point of tangency and applying the circumcenter formulae yields the locus as a line equation.

Quick Tip

The locus of a circumcenter in geometric configurations involving conics can be derived by combining geometric insights with calculus (for tangency conditions) and coordinate geometry (for circumcenter locations).

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11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \log_{\sqrt{m}}\left(\sqrt{2}(\sin x - \cos x) + m - 2\right)$, for some m, such that the range of f is [0, 2]. Then the value of m is:

(1) 5

(2) 3

(3) 2

(4) 4

Correct Answer: (1) 5.

Solution:

To explain the solution for the given problem in detail, let's expand each step:

Step 1: Understanding the inequality

The given inequality is:

$$-\sqrt{2} \leq \sin x - \cos x \leq \sqrt{2}$$

This inequality defines the range of the expression $\sin x - \cos x$. The maximum and minimum values of $\sin x - \cos x$ are derived from the trigonometric identity:

$$\sin x - \cos x = \sqrt{2} \sin \left(x - \frac{\pi}{4} \right)$$

Since $\sin(x - \frac{\pi}{4})$ is bounded between -1 and 1, the range of $\sin x - \cos x$ is:

$$-\sqrt{2} \leq \sin x - \cos x \leq \sqrt{2}.$$

Step 2: Scaling the inequality

Multiplying the entire inequality by $\sqrt{2}$, we get:

$$-\sqrt{2} \cdot \sqrt{2} \leq \sqrt{2}(\sin x - \cos x) \leq \sqrt{2} \cdot \sqrt{2}$$

$$-2 \leq \sqrt{2}(\sin x - \cos x) \leq 2$$

Here, scaling by $\sqrt{2}$ linearly stretches the bounds of $\sin x - \cos x$ by a factor of $\sqrt{2}$.

Step 3: Substitution

Let:

$$\sqrt{2}(\sin x - \cos x) = k$$

Then the inequality becomes:

$$-2 \leq k \leq 2 \quad \ldots (i)$$

Step 4: Analyzing the function $f(x)$

The function is given as:

$$f(x) = \log_{\sqrt{m}}(k + m - 2)$$

We are provided with the condition:

$$0 \leq f(x) \leq 2$$

Substituting $f(x)$ into the inequality:

$$0 \leq \log_{\sqrt{m}}(k + m - 2) \leq 2$$

The logarithmic inequality implies:

$$1 \leq k + m - 2 \leq m$$

Rearranging the terms:

$$-m + 3 \leq k \leq 2 \quad \ldots (ii)$$

Step 5: Combining inequalities

From equations (i) and (ii):

$$-2 \leq k \leq 2 \quad \text{(from (i))}$$

$$-m + 3 \leq k \leq 2 \quad \text{(from (ii))}$$

For consistency between the two inequalities, the lower bound of k gives:

$$-m + 3 = -2$$

Solving for m:

$$m = 5$$

Final Answer:

$$\boxed{m = 5}$$

Quick Tip

When dealing with logarithmic functions, always ensure that the argument stays positive and within the expected range by adjusting conditions or re-evaluating the parameter constraints.

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12. Let A, B, C be 3×3 matrices such that A is symmetric and B and C are skew-symmetric. Consider the statements:

(S1) $A^{13}B^{26} - B^{26}A^{13}$ **is symmetric.**

(S2) $A^{26}C^{13} - C^{13}A^{26}$ **is symmetric. Then:**

(1) Only S2 is true

(2) Only S1 is true

(3) Both S1 and S2 are false

(4) Both S1 and S2 are true

Correct Answer: (1) Only S2 is true.

Solution:

Given, $A^T = A$, $B^T = -B$, $C^T = -C$

Let $M = A^{13}B^{26} - B^{26}A^{13}$

Then, $M^T = (A^{13}B^{26} - B^{26}A^{13})^T$

$= (A^{13}B^{26})^T - (B^{26}A^{13})^T$

$= (B^T)^{26}(A^T)^{13} - (A^T)^{13}(B^T)^{26}$

$= B^{26}A^{13} - A^{13}B^{26} = -M$

Hence, M is skew-symmetric.

Let, $N = A^{26}C^{13} - C^{13}A^{26}$

Then, $N^T = (A^{26}C^{13})^T - (C^{13}A^{26})^T$

$= (C^{13})^T(A^{26})^T - (A^{26})^T(C^{13})^T$

$= (C^T)^{13}(A^T)^{26} - (A^T)^{26}(C^T)^{13}$

$= (-C)^{13}A^{26} - A^{26}(-C)^{13}$

$= -C^{13}A^{26} + A^{26}C^{13}$

$= A^{26}C^{13} - C^{13}A^{26} = N$

Hence, N is symmetric.

$\therefore$ Only S2 is true.

Quick Tip

Understanding the properties of symmetric and skew-symmetric matrices, as well as how they interact under matrix operations, is essential for solving problems involving matrix expressions and their symmetries.

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13. Let y = $y(t)$ **be a solution of the differential equation** $\frac{dy}{dt} + \alpha y = \gamma e^{-\beta t}$**, where** $\alpha, \beta, \gamma > 0$ **. If** $\lim_{t\to\infty} y(t)$**, then:**

(1) Is 0

(2) does not exist

(3) Is 1

(4) Is -1

Correct Answer: (1) Is 0.

Solution:

The given differential equation is:

$$\frac{dy}{dt} + \alpha y = \gamma e^{-\beta t},$$

where α, β, γ are constants. We are given that:

$$\lim_{t\to\infty} y(t) = 0.$$

Step 1: we write the equation

Factoring y from the right-hand side:

$$\frac{dy}{dt} = y\left(e^{-\beta t} - \alpha\right).$$

Dividing through by y (assuming $y \neq 0$):

$$\frac{1}{y}\frac{dy}{dt} = e^{-\beta t} - \alpha.$$

Step 2: Solve the differential equation

Rewriting:

$$\frac{dy}{y} = \left(e^{-\beta t} - \alpha\right) dt.$$

Integrating both sides:

$$\int \frac{1}{y}\,dy = \int \left(e^{-\beta t} - \alpha\right) dt.$$

The left-hand side integrates to:

$$\ln|y| = \int e^{-\beta t}\,dt - \alpha t + C,$$

where C is the constant of integration.

For the first term on the right-hand side:

$$\int e^{-\beta t}\,dt = -\frac{1}{\beta}e^{-\beta t}.$$

Thus:

$$\ln|y| = -\frac{1}{\beta}e^{-\beta t} - \alpha t + C.$$

Exponentiating both sides:

$$y = e^{-\frac{1}{\beta}e^{-\beta t} - \alpha t + C}.$$

Simplify the exponent:

$$y = e^{C} \cdot e^{-\frac{1}{\beta}e^{-\beta t}} \cdot e^{-\alpha t}.$$

Let $e^{C} = K$ (a constant). Then:

$$y = K \cdot e^{-\frac{1}{\beta}e^{-\beta t}} \cdot e^{-\alpha t}.$$

Step 3: Analyze $y(t)$ as $t \to \infty$

1. As $t \to \infty$, $e^{-\beta t} \to 0$. Therefore, the term $e^{-\frac{1}{\beta}e^{-\beta t}} \to e^{0} = 1$.

2. The dominant term in $y(t)$ is $e^{-\alpha t}$, which goes to 0 as $t \to \infty$, provided $\alpha > 0$.

Thus:

$$\lim_{t\to\infty} y(t) = 0.$$

Final Answer: The solution satisfies the condition $\lim_{t\to\infty} y(t) = 0$, and the correct option is: 1 (0).

Quick Tip

When solving first-order linear differential equations, the behavior of the solution at infinity often depends crucially on the coefficients' values and their relation to each other.

14. $\sum_{k=0}^{6} {}^{(51-k)}C_3$**: is equal to**

(1) ${}^{51}C_4 - {}^{45}C_4$

(2) ${}^{51}C_3 - {}^{45}C_3$

(3) ${}^{52}C_4 - {}^{45}C_4$

(4) ${}^{52}C_3 - {}^{45}C_3$

Correct Answer: (3) ${}^{52}C_4 - {}^{45}C_4$.

Solution:

The given summation is:

$$\sum_{k=0}^{6} {}^{(51-k)}C_3.$$

Step 1: we write the summation

This summation can be expanded as:

$$\sum_{k=0}^{6} {}^{(51-k)}C_3 = {}^{51}C_3 + {}^{50}C_3 + \cdots + {}^{45}C_3.$$

This is a finite summation of combinations.

Step 2: applying the telescoping property of combinations

We utilize the following identity for summation of combinations:

$$\sum_{r=a}^{b} {}^{r}C_p = {}^{(b+1)}C_{(p+1)} - {}^{a}C_{(p+1)}.$$

Here, let $a = 45$, $b = 51$, and $p = 3$. Substituting these values:

$$\sum_{k=0}^{6} {}^{(51-k)}C_3 = {}^{52}C_4 - {}^{45}C_4.$$

Step 3: Verify the answer

From the above calculation, we see that:

$$\sum_{k=0}^{6} {}^{(51-k)}C_3 = {}^{52}C_4 - {}^{45}C_4.$$

This matches option (3).

Final Answer:

$$\boxed{{}^{52}C_4 - {}^{45}C_4}$$

Quick Tip

Summation of combinations often follows telescoping identities. These can simplify complex expressions into compact forms.

15. The shortest distance between the lines $x + 1 = 2y = -12z$ and $x = y + 2 = 6z - 6$ is:

(1) 2

(2) 3

(3) $\frac{5}{2}$

(4) $\frac{3}{2}$

Correct Answer: (1) 2.

Solution:

The problem involves finding the shortest distance between two skew lines in three-dimensional space. The parametric equations of the two lines are given as:

$$\frac{x+1}{1} = \frac{y}{1/2} = \frac{z}{-1/12} \quad \text{and} \quad \frac{x}{1} = \frac{y+2}{1} = \frac{z-1}{1/6}$$

$$= \text{Shortest distance} = \frac{|(\vec{b} - \vec{a}) \cdot (\vec{p} \times \vec{q})|}{|\vec{p} \times \vec{q}|}$$

$$\text{S.D.} = \frac{(-\hat{i} + 2\hat{j} - \hat{k}) \cdot (\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|}$$

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \frac{1}{2} & -\frac{1}{12} \\ 1 & 1 & \frac{1}{6} \end{vmatrix} = \frac{1}{6}\hat{i} - \frac{1}{4}\hat{j} + \frac{1}{2}\hat{k} \quad \text{or} \quad 2\hat{i} - 3\hat{j} + 6\hat{k}$$

$$\text{S.D.} = \frac{(-\hat{i} + 2\hat{j} - \hat{k}) \cdot (2\hat{i} - 3\hat{j} + 6\hat{k})}{\sqrt{2^2 + 3^2 + 6^2}} = \left| \frac{-14}{7} \right| = 2$$

To compute the shortest distance between these two lines, we applying the standard formula, The shortest distance between the two skew lines is: S.D. = 2.

Quick Tip

Ensure precision in using vector operations and calculating distances in geometry problems, especially when dealing with the cross product and norms, to avoid miscalculations.

16. Let N be the sum of the numbers appeared when two fair dice are rolled and let the probability that $N-2, \sqrt{3N}, N+2$ are in geometric progression be $\frac{k}{48}$. Then the value of k is:

(1) 2

(2) 4

(3) 16

(4) 8

Correct Answer: (2) 4.

Solution:

For $N-2, \sqrt{3N}, N+2$ to be in geometric progression, the condition:

$$\sqrt{3N}^2 = (N-2)(N+2)$$

$$3N = N^2 - 4 \quad \Rightarrow \quad N^2 - 3N - 4 = 0$$

Solving this quadratic equation, we find:

$$N = 4 \quad \text{(since } N \text{ must be a possible sum of two dice)}$$

The favorable outcomes for $N = 4$ when two dice are rolled are $(1,3), (2,2), (3,1)$, totaling 3 outcomes. Since there are $6 \times 6 = 36$ total outcomes:

$$P(A) = \frac{3}{36} = \frac{1}{12}$$

Thus, $k = 4$ as $\frac{1}{12} = \frac{k}{48}$.

Quick Tip

In problems involving dice and probabilities, check for the total number of outcomes and ensure the conditions for geometric sequences or other properties are correctly applied.

---

17. The integral $16 \int_1^2 \frac{dx}{x^3(x^2+2)^2}$ is equal to:

(1) $\frac{11}{6} + \log_e 4$

(2) $\frac{11}{12} + \log_e 4$

(3) $\frac{11}{12} - \log_e 4$

(4) $\frac{11}{6} - \log_e 4$

Correct Answer: (4) $\frac{11}{6} - \log_e 4$.

Solution:

The given integral is:

$$I = 16 \int_1^2 \frac{dx}{x^3(x^2+2)^2}$$

Rewriting:

$$= 16 \int_1^2 \frac{dx}{x^3 x^4 \left(1 + \frac{2}{x^2}\right)^2}$$

$$\text{Let, } 1 + \frac{2}{x^2} = t \implies -\frac{4}{x^3}dx = dt$$

Simplify:

$$I = -4 \int_3^4 \frac{dt}{(t-1)^2 t^2}$$

$$I = -4 \int_3^4 \left(\frac{2}{t-1} - \frac{2}{t} + \frac{1}{t^2}\right) dt$$

Now integrate:

$$I = -1 \left[t - 2\ln|t| - \frac{1}{t}\right]_3^4$$

Applying the limits:

$$= -1 \left[\left(4 - 2\ln 4 - \frac{1}{4}\right) - \left(3 - 2\ln 3 - \frac{1}{3}\right)\right]$$

Simplify:

$$= -1 \left[2\ln 2 - \frac{11}{6}\right]$$

$$= \frac{11}{6} - \ln 4$$

Final Answer:

$$\frac{11}{6} - \ln 4$$

Quick Tip

In integral calculus, substitution can simplify the integral into a more manageable form, especially when dealing with rational functions. Partial fractions decompose complex fractions into simpler parts that are easier to integrate.

18. Let T and C respectively be the transverse and conjugate axes of the hyperbola $16x^2 - y^2 + 64x + 4y + 44 = 0$. Then the area of the region above the parabola $x^2 = y + 4$, below the transverse axis T and on the right of the conjugate axis C is:

(1) $4\sqrt{6} + \frac{44}{3}$

(2) $4\sqrt{6} + \frac{28}{3}$

(3) $4\sqrt{6} - \frac{44}{3}$

(4) $4\sqrt{6} - \frac{28}{3}$

Correct Answer: (2) $4\sqrt{6} + \frac{28}{3}$.

Solution:

First, find the integration bounds and fix the hyperbola equation before solving for the area. Completing the square and putting the terms in standard form are necessary steps in transforming the hyperbola and determining its axis. The intended area is obtained by combining the parabola's constraints with integration across the designated region. Finding intersections and integrating the appropriate function over the designated limits are part of the computations.

$$16(x^2 + 4x) - (y^2 - 4y) + 44 = 0$$

$$16(x + 2)^2 - 64 - (y - 2)^2 + 4 + 44 = 0$$

$$16(x + 2)^2 - (y - 2)^2 = 16$$

$$\frac{(x + 2)^2}{1} - \frac{(y - 2)^2}{16} = 1$$

$$A = \int_{-2}^{\sqrt{6}} (2 - (x^2 - 4))dx$$

$$A = \left[6\sqrt{6} - \frac{6\sqrt{6}}{3}\right] - \left(-12 + \frac{8}{3}\right)$$

$$\int_{-2}^{\sqrt{6}} (6 - x^2)dx = \left[6x - \frac{x^3}{3}\right]_{-2}^{\sqrt{6}}$$

$$A = \frac{12\sqrt{6}}{3} + \frac{28}{3}$$

$$A = 4\sqrt{6} + \frac{28}{3}$$

Quick Tip

For geometry problems involving area calculations, accurately plotting the region and determining bounds for integration are crucial. Transforming equations to standard forms can simplify understanding the geometric configuration.

---

19. Let $\vec{a} = -\hat{i} - \hat{j} + \hat{k}$, $\vec{a} \cdot \vec{b} = 1$ and $\vec{a} \times \vec{b} = \hat{i} - \hat{j}$. Then $\vec{a} - 6\vec{b}$ is equal to:

(1) $3(\hat{i} - \hat{j} - \hat{k})$

(2) $3(\hat{i} + \hat{j} + \hat{k})$

(3) $3(\hat{i} - \hat{j} + \hat{k})$

(4) $3(\hat{i} + \hat{j} - \hat{k})$

Correct Answer: (2) $3(\hat{i} + \hat{j} + \hat{k})$.

Solution:

The dot product and cross product information are used to solve for $\vec{b}$ given the vector equations and attributes. After $\vec{b}$ is computed, $\vec{a} - 6\vec{b}$ is determined by scalar multiplication and direct subtraction. The approach entails manipulating vector components algebraically and confirming that each choice corresponds to the computed outcome.

Given:

$$\vec{a} \times \vec{b} = \hat{i} - \hat{j}$$

Taking the cross product with $\vec{a}$, we get:

$$\vec{a} \times (\vec{a} \times \vec{b}) = \vec{a} \times (\hat{i} - \hat{j})$$

Using the vector triple product identity $\vec{a} \times (\vec{b} \times \mathbf{c}) = (\vec{a} \cdot \mathbf{c})\vec{b} - (\vec{a} \cdot \vec{b})\mathbf{c}$, we obtain:

$$\vec{a} \times (\vec{a} \times \vec{b}) = (\vec{a} \cdot \hat{j})\hat{i} - (\vec{a} \cdot \hat{i})\hat{j}$$

Assuming:

$$\vec{a} \cdot \hat{i} = \vec{a}_i, \quad \vec{a} \cdot \hat{j} = \vec{a}_j$$

Thus:

$$\vec{a} \times (\vec{a} \times \vec{b}) = \vec{a}_j\hat{i} - \vec{a}_i\hat{j}$$

Let's express a in terms of its components:

$$\vec{a} = \vec{a}_i\hat{i} + \vec{a}_j\hat{j} + \vec{a}_k\hat{k}$$

Using the results above:

$$\vec{a} \times (\hat{i} - \hat{j}) = (\vec{a}_j \hat{i} - \vec{a}_i \hat{j}) + 2\vec{a}_k \hat{k}$$

This simplifies to:

$$\vec{a} - 3\vec{b} = \hat{i} + \hat{j} + 2\hat{k}$$

$$2\vec{a} - 6\vec{b} = 2\hat{i} + 2\hat{j} + 4\hat{k}$$

$$\vec{a} - 6\vec{b} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

Quick Tip

In vector problems, use properties of dot products and cross products to find unknown vectors. These properties can simplify the equations and lead directly to the solution.

---

20. The foot of the perpendicular from the point $(2, 0, 5)$ on the line $\frac{x+1}{2} = \frac{y-1}{5} = \frac{z+1}{-1}$ is (α, β, γ). Then, which of the following is NOT correct?

(1) $\frac{\alpha\beta}{\gamma} = \frac{4}{15}$

(2) $\frac{\alpha}{\beta} = -8$

(3) $\frac{\beta}{\gamma} = -5$

(4) $\frac{\gamma}{\alpha} = \frac{5}{8}$

Correct Answer: (3) $\frac{\beta}{\gamma} = -\frac{5}{8}$.

Solution:

To find the foot of the perpendicular, apply the formula for the point-line distance and the projection of a point onto a line using vector operations. Calculate $\mathbf{PA} \cdot \mathbf{b} = 0$ where $\mathbf{P}$ is the point on the line closest to $\mathbf{A}$ and $\mathbf{b}$ is the direction vector of the line. This provides the specific coordinates α, β, γ of the point $\mathbf{P}$, and subsequently, the ratios of these coordinates are compared against the options to identify the incorrect statement.

Given the line equation:

$$\frac{x+1}{2} = \frac{y-1}{5} = \frac{z+1}{-1} \quad \text{(let)}$$

and the point $A(2, 0, 5)$, we want to find the foot of the perpendicular from A to the line.

Let the foot of the perpendicular be $P(\alpha, \beta, \gamma)$. The position vector of P can be described as:

$$P = (2\lambda - 1, 5\lambda + 1, -\lambda - 1)$$

The vector $\overrightarrow{PA}$ is given by:

$$\overrightarrow{PA} = (3 - 2\lambda)\hat{i} - (5\lambda + 1)\hat{j} + (6 + \lambda)\hat{k}$$

The direction ratios of the line are:

$$\mathbf{b} = 2\hat{i} + 5\hat{j} - \hat{k}$$

For P to be the foot of the perpendicular, $\overrightarrow{PA}$ must be perpendicular to $\mathbf{b}$, hence:

$$\overrightarrow{PA} \cdot \mathbf{b} = 0$$

$$(3 - 2\lambda) \cdot 2 - (5\lambda + 1) \cdot 5 + (6 + \lambda) \cdot (-1) = 0$$

$$6 - 4\lambda - 25\lambda - 5 - 6 - \lambda = 0 \quad \Rightarrow \quad -30\lambda - 5 = 0 \quad \Rightarrow \quad \lambda = -\frac{1}{6}$$

Substitute $\lambda = -\frac{1}{6}$ back into the coordinates of P:

$$\alpha = 2\left(-\frac{1}{6}\right) - 1 = -4/3$$

$$\beta = 5\left(-\frac{1}{6}\right) + 1 = -\frac{5}{6} + 1 = \frac{1}{6}$$

$$\gamma = -\left(-\frac{1}{6}\right) - 1 = \frac{1}{6} - 1 = -\frac{5}{6}$$

Thus, the foot of the perpendicular from A to the line is at:

$$P\left(\frac{-4}{3}, \frac{1}{6}, -\frac{5}{6}\right)$$

Quick Tip

Using vector projection and dot product formulas effectively can solve problems involving distances and perpendiculars in three-dimensional geometry efficiently.

---

21. For the two positive numbers a, b, if a, b and $\frac{1}{18}$ are in a geometric progression, while $\frac{1}{a}, 10, \frac{1}{b}$ are in an arithmetic progression, then $16a + 12b$ is equal to:

Correct Answer: 3.

Solution:

From the given conditions:

$$\frac{a}{18} = b^2 \quad (1).$$

Also, since $\frac{1}{a}, 10, \frac{1}{b}$ are in arithmetic progression:

$$\frac{1}{a} + \frac{1}{b} = 20.$$

Using equation (1) and simplifying:

$$a + b = 20ab.$$

Substituting $b = \sqrt{\frac{a}{18}}$,

$\Rightarrow 18b^2 + b = 360b^3$

$\Rightarrow 360b^3 - 18b^2 - b = 0$

$\Rightarrow 360b^2 - 18b - 1 = 0 \quad \{\because b \neq 0\}$

$\Rightarrow b = \frac{18 \pm \sqrt{324+1440}}{720}$

$\Rightarrow b = \frac{18+\sqrt{1764}}{720} \quad \{\because b > 0\}$

$\Rightarrow b = \frac{18+42}{720}$

$\Rightarrow b = \frac{60}{720}$

$\Rightarrow b = \frac{1}{12}$

we solve for a and b, eventually leading to:

$$a = \frac{1}{8}, \quad b = \frac{1}{12}.$$

Now compute $16a + 12b$:

$$16a + 12b = 16 \times \frac{1}{8} + 12 \times \frac{1}{12} = 2 + 1 = 3.$$

Quick Tip

In problems involving sequences, identify the relationship (e.g., arithmetic, geometric) and set up equations to simplify complex terms.

---

22. Points $P(-3,2), Q(9,10)$, and $R(\alpha,4)$ lie on a circle C with PR as its diameter. The tangents to C at Q and R intersect at point S. If S lies on the line $2x - ky = 1$, then k is equal to:

Correct Answer: 3.

Solution:

The slopes of PQ and QR satisfy:

$$m_{PQ} \cdot m_{QR} = -1.$$

From the given coordinates:

$$m_{PQ} = \frac{10-2}{9+3} = \frac{8}{12} = \frac{2}{3},$$
$$m_{QR} = \frac{10-4}{9-\alpha}.$$

Using $m_{PQ} \cdot m_{QR} = -1$:

$$\frac{2}{3} \cdot \frac{6}{9-\alpha} = -1 \implies \alpha = 13.$$

Next, calculate the equation of QS:

$$y - 10 = -\frac{4}{7}(x-9) \implies 4x + 7y = 106 \quad (1).$$

Similarly, the equation of RS is:

$$y - 4 = -8(x - 13) \implies 8x + y = 108 \quad (2).$$

Solve equations (1) and (2):

$$x = \frac{25}{2}, \quad y = 8.$$

Substituting into $2x - ky = 1$:

$$2 \cdot \frac{25}{2} - 8k = 1 \implies 25 - 8k = 1 \implies k = 3.$$

Quick Tip

When solving geometry problems involving tangents and intersections, carefully compute slopes and solve linear equations step-by-step.

---

23. Let $a \in \mathbb{R}$ and let α, β be the roots of the equation $x^2 + 60^{\frac{1}{4}}x + a = 0$. If $\alpha^4 + \beta^4 = -30$, then the product of all possible values of a is:

Correct Answer: 45.

Solution:

From the quadratic equation:

$$\alpha + \beta = -60^{\frac{1}{4}}, \quad \alpha\beta = a.$$

Using the condition $\alpha^4 + \beta^4 = -30$:

$$(\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2 = -30.$$

Substitute $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$:

$$\left((-60^{\frac{1}{4}})^2 - 2a\right)^2 - 2a^2 = -30.$$

Simplify and solve for a, leading to:

$$2a^2 - 4.60^{\frac{1}{2}}a + 90 = 0.$$

The product of all possible a is:

$$\boxed{45}.$$

Quick Tip

When solving polynomial problems, utilize root properties such as sums and products of roots effectively to derive key equations.

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24. Suppose Anil's mother wants to give 5 whole fruits to Anil from a basket of 7 red apples, 5 white apples, and 8 oranges. If in the selected 5 fruits, at least 2 oranges, at least one red apple, and at least one white apple must be given, then the number of ways Anil's mother can offer 5 fruits to Anil is:

Correct Answer: 6860.

Solution:

Possible selections are:

(1) 2 oranges, 1 red apple, 2 white apples.

(2) 2 oranges, 2 red apples, 1 white apple.

(3) 3 oranges, 1 red apple, 1 white apple.

The total number of ways for each case:

$$\Rightarrow^8 C_2\ {}^7C_1\ {}^5C_2 +^8 C_2\ {}^7C_2\ {}^5C_1 +^8 C_3\ {}^7C_1\ {}^5C_1$$

Adding them:

$$1960 + 2940 + 1960 = 6860.$$

Final Answer:

$$\boxed{6860}.$$

Quick Tip

When solving combinatorics problems, break them into cases and calculate each case using basic counting principles.

25. If m and n respectively are the numbers of positive and negative values of θ in the interval $[-\pi, \pi]$ that satisfy the equation $\cos 2\theta \cdot \cos \frac{\theta}{2} = \cos 3\theta \cdot \cos \frac{9\theta}{2}$, then mn is equal to:

Correct Answer: 25.

Solution:

The given equation is:

$$\cos 2\theta \cdot \cos \frac{\theta}{2} = \cos 3\theta \cdot \cos \frac{9\theta}{2}.$$

Simplify using trigonometric identities:

$\Rightarrow 2\cos 2\theta \cdot \cos \frac{\theta}{2} = 2\cos \frac{9\theta}{2} \cdot \cos 3\theta$

$\Rightarrow \cos \frac{5\theta}{2} + \cos \frac{3\theta}{2} = \cos \frac{15\theta}{2} + \cos \frac{3\theta}{2}$

$\Rightarrow \cos \frac{15\theta}{2} = \cos \frac{5\theta}{2}$

$\Rightarrow \frac{15\theta}{2} = 2k\pi \pm \frac{5\theta}{2}$

$5\theta = 2k\pi \quad \text{or} \quad 10\theta = 2k\pi$

$\theta = \frac{2k\pi}{5} \quad \theta = \frac{k\pi}{5}$

$\therefore \theta = \left\{-\pi, \frac{-4\pi}{5}, \frac{-3\pi}{5}, \frac{-2\pi}{5}, -\frac{\pi}{5}, 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi\right\}$

There are $m = 5$ positive and $n = 5$ negative values:

$$mn = 5 \cdot 5 = 25.$$

Final Answer:

$$\boxed{25}.$$

Quick Tip

For trigonometric equations, use symmetry and identities to simplify, and carefully consider the intervals of solutions.

---

26. If
$\int_{\frac{1}{3}}^{3} |\log_e x|\, dx = \frac{m}{n} \log_e \left(\frac{n^2}{e}\right)$, **where m and n are coprime natural numbers, then** $m^2 + n^2 - 5$ **is equal to ____.**

Correct Answer: 20.

Solution:

The given integral is: $\int_{\frac{1}{3}}^{3} |\ln x|\, dx = \int_{\frac{1}{3}}^{1} (-\ln x)\, dx + \int_{1}^{3} (\ln x)\, dx$

$= -[x \ln x - x]_{\frac{1}{3}}^{1} + [x \ln x - x]_{1}^{3}$

$= -[-1 - (\frac{1}{3} \ln \frac{1}{3} - \frac{1}{3})] + [3 \ln 3 - 3 - (-1)]$

$= -[-1 - (\frac{1}{3} \ln \frac{1}{3} - \frac{1}{3})] + [3 \ln 3 - 3 - (-1)]$

$= [1 + \frac{1}{3} \ln \frac{1}{3} - \frac{1}{3}] + [3 \ln 3 - 2]$

$= [-\frac{2}{3} - \frac{1}{3} \ln \frac{1}{3}] + [3 \ln 3 - 2]$

$= -\frac{4}{3} + \frac{8}{3} \ln 3$

$= \frac{4}{3}(2 \ln 3 - 1)$

$= \frac{4}{3} \left(\ln \frac{9}{e}\right)$

Thus:

$$m = 4,\ n = 3 \quad \implies m^2 + n^2 - 5 = 16 + 9 - 5 = 20.$$

Quick Tip

When solving logarithmic integrals, split the integral over ranges for simplification and carefully calculate terms.

27. The remainder when $(2023)^{2023}$ **is divided by 35 is:**

Correct Answer: 7.

Solution:

Applying modular arithmetic:

$$(2023)^{2023} = (2030 - 7)^{2023}$$

$$= (35K - 7)^{2023}$$

Thus:

$$= ^{2023}C_0(35K)^{2023}(-7)^0 + ^{2023}C_1(35K)^{2022}(-7)^1 + \cdots + ^{2023}C_{2023}(-7)^{2023}$$

$$= 35N - 7^{2023}$$

Applying properties of modular arithmetic: -$7^{2023} = -7 \times 7^{2022} = -7 \times (7^2)^{1011}$

$$= -7 \times (50 - 1)^{1011}$$

Simplifying further using binomial theorem, and compute the remainder as:

$$= -7 \left(^{1011}C_0 50^{1011} - ^{1011}C_1 50^{1010} + \cdots + ^{1011}C_{1011} \right)$$

$$= -7(5\lambda - 1)$$

$$= -35\lambda + 7$$

when $(2023)^{2023}$ is divided by 35 remainder is 7

Quick Tip

Use modular arithmetic and binomial expansion to handle powers in modulo problems.

28. If the shortest distance between the line joining the points $(1,2,3)$ and $(2,3,4)$, and the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-2}{0}$ is α, then $28\alpha^2$ is equal to:

Correct Answer: 18.

Solution:

The shortest distance between two skew lines is:

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + \hat{j} + \hat{k}) \quad \vec{r} = \vec{a} + \lambda\vec{p}$

$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} - \hat{j}) \quad \vec{r} = \vec{b} + \mu\vec{q}$

$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix} = \hat{i} + 2\hat{j} - 3\hat{k}$

$d = \left| \frac{(\vec{b}-\vec{a})\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|} \right|$

$d = \left| \frac{(-3\hat{j}-\hat{k})\cdot(\hat{i}+2\hat{j}-3\hat{k})}{\sqrt{14}} \right|$

$= \left| \frac{-6+3}{\sqrt{14}} \right| = \frac{3}{\sqrt{14}}$

$\alpha = \frac{3}{\sqrt{14}}$

Now,

$28\alpha^2 = 28 \times \frac{9}{14} = 18$

Quick Tip

For shortest distance problems, use vector formulas and carefully compute cross products and dot products.

29. 25% of the population are smokers. A smoker has 27 times more chances to develop lung cancer than a non-smoker. If a person is diagnosed with lung cancer, and the probability that this person is a smoker is $\frac{k}{10}$, then the value of k is:

Correct Answer: 9.

Solution:

Applying Bayes' theorem:

$$P(E_1) = \frac{1}{4}, \quad P(E_2) = \frac{3}{4}, \quad P(E/E_1) = \frac{27}{28}, \quad P(E/E_2) = \frac{1}{28}.$$

The total probability is :

$$P(E) = P(E_1)P(E/E_1) + P(E_2)P(E/E_2).$$

Substituting the values and solve:

$$P(E_1/E) = \frac{\frac{1}{4} \cdot \frac{27}{28}}{\frac{1}{4} \times \frac{27}{28} + \frac{3}{4} \times \frac{1}{28}} = \frac{9}{10}.$$

Therefore:

$$k = 9.$$

Quick Tip

For probability problems, use Bayes' theorem to calculate posterior probabilities effectively.

30. A triangle is formed by the X-axis, Y-axis, and the line $3x + 4y = 60$. Then the number of points $P(a, b)$, where a is an integer and b is a multiple of a, which lie strictly inside the triangle, is:____

Correct Answer: 31.

Solution:

The intercepts of the line are $(20, 0)$ and $(0, 15)$. Calculating the points inside the triangle satisfying $b = ka$.

$$(1,1)(1,2) - (1,14) \Rightarrow 14 \text{ pts.}$$

$$\text{If } x = 2,\ y = \frac{27}{2} = 13.5$$

$$(2,2)(2,4)\dots(2,12) \Rightarrow 6 \text{ pts.}$$

$$\text{If } x = 3,\ y = \frac{51}{4} = 12.75$$

$$(3,3)(3,6)(3,12) \Rightarrow 4 \text{ pts.}$$

$$\text{If } x = 4,\ y = 12$$

$$(4,4)(4,8) \Rightarrow 2 \text{ pts.}$$

$$\text{If } x = 5,\ y = \frac{45}{4} = 11.25$$

$$(5,5)(5,10) \Rightarrow 2 \text{ pts.}$$

$$\text{If } x = 6,\ y = \frac{21}{2} = 10.5$$

$$(6,6) \Rightarrow 1 \text{ pt.}$$

$$\text{If } x = 7,\ y = \frac{39}{4} = 9.75$$

$$(7,7) \Rightarrow 1 \text{ pt.}$$

$$\text{If } x = 8,\ y = 9$$

$$(8,8) \Rightarrow 1 \text{ pt.}$$

$$\text{If } x = 9,\ y = \frac{33}{4} = 8.25$$

The total is:

$$31 \text{ points.}$$

Quick Tip

When solving geometry problems with constraints, carefully analyze integer solutions and relationships between coordinates.