WBJEE 2024 Mathematics Question Paper with Solutions

Time Allowed : 180 minutes	Maximum Marks : 200	Total questions : 155

General Instructions

Read the following instructions very carefully and strictly follow them:

1. The test is of 3 hours duration.
2. The question paper consists of 155 questions. The maximum marks are 200.
3. There are three parts in the question paper consisting of Physics, Chemistry and Mathematics.
4. There are 75 questions in Mathematics and 40 each in Physics and Chemistry papers. 100 marks are allotted for Maths while 50 is allotted for Physics and Chemistry each, totaling 200 marks in both papers together.
5. There are three categories of WBJEE questions asked in the exam.
   (i) Category 1: 1 mark is awarded for choosing the correct option. 1/4th mark is deducted for an incorrect answer.
   (ii) Category 2: only 1 option is correct. 2 marks are awarded for each correct answer. ½ mark is deducted for an incorrect answer. (iii) Category 3: Category 3: more than 1 option is correct and all correct answers are to be chosen to receive 2 marks.

1. All values of a for which the inequality

$$\frac{1}{\sqrt{a}} \int_1^a \left(\frac{3}{2}\sqrt{x} + 1 - \frac{1}{\sqrt{x}} \right) dx < 4$$

is satisfied, lie in the interval.

(A) (1, 2)

(B) (0, 3)

(C) (0, 4)

(D) (1, 4)

Correct Answer: (C) (0, 4)

Solution:

1. Step 1: Write the integral in separate terms:

$$I(a) = \int_1^a \left(\frac{3}{2}\sqrt{x} + 1 - \frac{1}{\sqrt{x}} \right) dx$$

2. Step 2: Break it down into individual integrals:

$$I(a) = \int_1^a \frac{3}{2}\sqrt{x}\, dx + \int_1^a 1\, dx - \int_1^a \frac{1}{\sqrt{x}}\, dx$$

3. Step 3: Compute each of the integrals:

$$\int_1^a \frac{3}{2}\sqrt{x}\, dx = \left[\frac{3}{2} \cdot \frac{2}{3} x^{3/2} \right]_1^a = a^{3/2} - 1$$

$$\int_1^a 1\, dx = a - 1$$

$$\int_1^a \frac{1}{\sqrt{x}}\, dx = [2\sqrt{x}]_1^a = 2\sqrt{a} - 2$$

4. Step 4: Substitute the results into the inequality and simplify:

$$\frac{1}{\sqrt{a}} \left(a^{3/2} - 1 + a - 1 - 2\sqrt{a} + 2 \right) < 4$$

5. Step 5: Solve the inequality, which simplifies to $a \in (0, 4)$.

Quick Tip

To solve inequalities involving integrals, break the integral into simpler terms and simplify the resulting expressions for easier evaluation.

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2. For any integer n,

$$\int_0^{\pi} e^{\cos^2 x} \cdot \cos^3(2n+1)x \, dx \text{ has the value.}$$

(A) π

(B) 1

(C) 0

(D) $\frac{3\pi}{2}$

Correct Answer: (C) 0

Solution:

1. Step 1: Recognize that the integrand contains an oscillatory term $\cos^3((2n+1)x)$, which is an odd function.
2. Step 2: The integral of an odd function over a symmetric interval results in zero.
3. Step 3: Therefore, the value of the integral is 0.

Quick Tip

When integrating an odd function over a symmetric interval, the result is always zero.

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3. Let f be a differential function with

$$\lim_{x\to\infty} f(x) = 0. \text{ If } y' + yf'(x) - f(x)f'(x) = 0, \quad \lim_{x\to\infty} y(x) = 0 \text{ then,}$$

(A) $y + 1 = e^{-f(x)} + f(x)$

(B) $y + 1 = e^{-f(x)} + f(x)$

(C) $y + 2 = e^{-f(x)} + f(x)$

(D) $y - 1 = e^{-f(x)} + f(x)$

Correct Answer: (B) $y + 1 = e^{-f(x)} + f(x)$

Solution:

1. Step 1: Start with the given differential equation:

$$y' + yf'(x) - f(x)f'(x) = 0$$

This is a first-order linear differential equation involving both y and $f(x)$.

2. Step 2: Rearrange the terms to isolate y':

$$y' = f'(x)\,(f(x) - y)$$

Now, we have an equation that describes how y changes with respect to x, in terms of the function $f(x)$ and its derivative.

3. Step 3: Notice that the form of the equation suggests that y might be related to $f(x)$ in some way. Since $f(x) \to 0$ as $x \to \infty$, we expect $y(x)$ to simplify as well, possibly to a constant.

4. Step 4: Assume that as $x \to \infty$, the behavior of the functions becomes simpler. Specifically, since $\lim_{x\to\infty} f(x) = 0$, we hypothesize that $y(x)$ may behave similarly to an exponential function that decays as $f(x)$ does.

5. Step 5: Assume a solution for $y(x)$ in the form:

$$y + 1 = e^{-f(x)} + f(x)$$

This form satisfies the differential equation and the condition that as $x \to \infty$, $y(x) \to 0$ because $f(x) \to 0$.

6. Step 6: Substitute this proposed solution into the original differential equation to verify that it satisfies the equation. If both sides of the equation balance, then our assumption is correct.

$$y' + yf'(x) - f(x)f'(x) = 0$$

After substituting $y = e^{-f(x)} + f(x) - 1$ into this equation, we find that it holds true, confirming the correctness of our solution.

7. Step 7: Therefore, the correct relationship is:

$$y + 1 = e^{-f(x)} + f(x)$$

Quick Tip

To solve such differential equations, look for terms that approach zero as x increases. The limits often simplify the function form.

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4. If $xy' + y - e^x = 0$, $y(a) = b$, then

$$\lim_{x \to 1} y(x) \text{ is}$$

(A) $e + 2ab - e^a$

(B) $e^2 + ab - e^{-a}$

(C) $e - ab + e^a$

(D) $e + ab - e^a$, $y' = \frac{dy}{dx}$

Correct Answer: (D) $e + ab - e^a$ **Solution:**

1. Step 1: Start with the given differential equation:

$$xy' + y - e^x = 0$$

Rearranging it:

$$xy' + y = e^x$$

This is a linear first-order differential equation.

2. Step 2: Multiply the equation by the integrating factor $\mu(x)$, which is $\mu(x) = x$.

$$x \cdot y' + x \cdot y = x \cdot e^x$$

This simplifies to:

$$\frac{d}{dx}(xy) = xe^x$$

3. Step 3: Integrate both sides of the equation with respect to x:

$$\int \frac{d}{dx}(xy)\,dx = \int xe^x\,dx$$

The right-hand side can be integrated by parts:

$$\int xe^x\,dx = xe^x - e^x + C$$

Thus, we have:

$$xy = xe^x - e^x + C$$

4. Step 4: Solve for y:

$$y = e^x - \frac{e^x}{x} + \frac{C}{x}$$

5. Step 5: Use the initial condition $y(a) = b$ to find C:

$$b = e^a - \frac{e^a}{a} + \frac{C}{a}$$

Solving for C:

$$C = a(b - e^a + \frac{e^a}{a})$$

6. Step 6: Substitute the value of C into the equation for y:

$$y = e^x - \frac{e^x}{x} + \frac{a(b - e^a + \frac{e^a}{a})}{x}$$

7. Step 7: Finally, take the limit as $x \to 1$:

$$\lim_{x \to 1} y(x) = e + ab - e^a$$

Answer: (D) $e + ab - e^a$

> **Quick Tip**
>
> For first-order linear differential equations, use an integrating factor to simplify and solve the equation.

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5. The area bounded by the curves $x = 4 - y^2$ and the Y-axis is:

(A) 16 sq. unit

(B) $\frac{32}{3}$ sq. unit

(C) $\frac{16}{3}$ sq. unit

(D) 32 sq. unit

Correct Answer: (B) $\frac{32}{3}$ sq. unit

Solution:

1. Step 1: The given equation is $x = 4 - y^2$. The area is bounded by the curve and the Y-axis. We first find the limits of integration by setting $x = 0$:

$$0 = 4 - y^2 \quad \Rightarrow \quad y^2 = 4 \quad \Rightarrow \quad y = \pm 2$$

2. Step 2: The area is the integral of $x = 4 - y^2$ from $y = -2$ to $y = 2$:

$$\text{Area} = \int_{-2}^{2} (4 - y^2)\, dy$$

3. Step 3: Compute the integral:

$$\text{Area} = \left[4y - \frac{y^3}{3}\right]_{-2}^{2}$$

4. Step 4: Substituting the limits:

$$\text{Area} = \left(4(2) - \frac{(2)^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right) = 16 - \frac{16}{3}$$

$$\text{Area} = \frac{48}{3} - \frac{16}{3} = \frac{32}{3}$$

Quick Tip

To compute areas bounded by curves, set the integral limits based on where the curve intersects the axes, then evaluate the integral.

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6. $f(x) = \cos x - 1 + \frac{x^2}{2!}, x \in \mathbb{R}$

Then $f(x)$ is:

(A) decreasing function

(B) increasing function

(C) neither increasing nor decreasing

(D) constant for $x > 0$

Correct Answer: (C) neither increasing nor decreasing

Solution:

1. Step 1: The given function is:

$$f(x) = \cos x - 1 + \frac{x^2}{2!}$$

2. Step 2: Find the derivative of $f(x)$ to determine if it is increasing or decreasing:

$$f'(x) = -\sin x + x$$

3. Step 3: The derivative $f'(x) = -\sin x + x$ depends on the value of x. For large values of x, x dominates, making $f'(x) > 0$, which indicates that $f(x)$ is increasing for large x.

4. Step 4: Since $f'(x)$ is not always positive or negative, the function is neither always increasing nor decreasing.

Quick Tip

When analyzing the behavior of a function, examine the sign of its derivative. If the derivative is not always positive or negative, the function may neither increase nor decrease uniformly.

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7. Let $y = f(x)$ be any curve on the X-Y plane and P be a point on the curve. Let C be a fixed point not on the curve. The length PC is either a maximum or a minimum. Then:

(A) PC is perpendicular to the tangent at P

(B) PC is parallel to the tangent at P

(C) PC meets the tangent at an angle of 45°

(D) PC meets the tangent at an angle of 60°

Correct Answer: (A) PC is perpendicular to the tangent at P

Solution:

1. Step 1: The length PC is the distance from the fixed point C to the point P on the curve. We are told that PC is either a maximum or a minimum.

2. Step 2: For the length PC to be at a maximum or minimum, the line connecting P and C must be perpendicular to the tangent at P.

3. Step 3: The condition for the minimum or maximum distance is that the vector $\overrightarrow{PC}$ is perpendicular to the tangent line at P.

Answer: (A) PC is perpendicular to the tangent at P

Quick Tip

When dealing with optimization problems involving distances and curves, the distance is maximized or minimized when the vector from the point of interest is either parallel or perpendicular to the tangent.

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8. If a particle moves in a straight line according to the law $x = a\sin(\sqrt{t} + b)$, then the particle will come to rest at two points whose distance is:

(A) a

(B) $\frac{a}{2}$

(C) $2a$

(D) $4a$

Correct Answer: (C) $2a$

Solution:

The particle's position is given by:

$$x = a\sin(\sqrt{t} + b).$$

Step 1: Velocity of the particle. The velocity v is the derivative of x with respect to time t:

$$v = \frac{dx}{dt}.$$

Differentiate x:

$$v = a\cos(\sqrt{t} + b) \cdot \frac{d}{dt}(\sqrt{t} + b).$$

Since $\frac{d}{dt}(\sqrt{t} + b) = \frac{1}{2\sqrt{t}}$, the velocity becomes:

$$v = a\cos(\sqrt{t} + b) \cdot \frac{1}{2\sqrt{t}}.$$

Step 2: Condition for the particle to come to rest. The particle comes to rest when $v = 0$, i.e., when:

$$\cos(\sqrt{t} + b) = 0.$$

The general solution for $\cos(\theta) = 0$ is:

$$\sqrt{t} + b = \frac{\pi}{2} + n\pi \quad \text{for integers } n.$$

From this, we solve for t:

$$\sqrt{t} = \frac{\pi}{2} + n\pi - b.$$

Let the particle come to rest at two consecutive points corresponding to $n = k$ and $n = k + 1$. The values of $\sqrt{t}$ at these points are:

$$\sqrt{t_1} = \frac{\pi}{2} + k\pi - b, \quad \sqrt{t_2} = \frac{\pi}{2} + (k+1)\pi - b.$$

Step 3: Distance between rest points. At the rest points, the position x is:

$$x_1 = a\sin(\sqrt{t_1} + b), \quad x_2 = a\sin(\sqrt{t_2} + b).$$

Substitute $\sqrt{t_1} + b = \frac{\pi}{2} + k\pi$ and $\sqrt{t_2} + b = \frac{\pi}{2} + (k+1)\pi$:

$$x_1 = a\sin\left(\frac{\pi}{2} + k\pi\right), \quad x_2 = a\sin\left(\frac{\pi}{2} + (k+1)\pi\right).$$

Using the property of sine:

$$\sin\left(\frac{\pi}{2} + k\pi\right) = (-1)^k, \quad \sin\left(\frac{\pi}{2} + (k+1)\pi\right) = (-1)^{k+1}.$$

Thus:

$$x_1 = a(-1)^k, \quad x_2 = a(-1)^{k+1}.$$

The distance between the two points is:

$$\text{Distance} = |x_2 - x_1| = |a(-1)^{k+1} - a(-1)^k|.$$

Since $(-1)^{k+1} - (-1)^k = -2(-1)^k$, the absolute value gives:

$$\text{Distance} = 2a.$$

Conclusion: The particle comes to rest at two points whose distance is:

$$\boxed{2a}.$$

Quick Tip

When dealing with trigonometric functions in particle motion problems, set the function equal to zero to find the points of rest.

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9. A unit vector in XY-plane making an angle 45° with $\hat{i} + \hat{j}$ and an angle 60° with $3\hat{i} - 4\hat{j}$ is:

(A) $\frac{13}{14}\hat{i} + \frac{1}{14}\hat{j}$

(B) $\frac{1}{14}\hat{i} + \frac{13}{14}\hat{j}$

(C) $\frac{13}{14}\hat{i} - \frac{1}{14}\hat{j}$

(D) $\frac{1}{14}\hat{i} - \frac{13}{14}\hat{j}$

Correct Answer: (A) $\frac{13}{14}\hat{i} + \frac{1}{14}\hat{j}$

Solution:

1. Step 1: The unit vector $\mathbf{v}$ in XY-plane making an angle of 45° with $\hat{i} + \hat{j}$ is given by:

$$\mathbf{v} = \hat{i}\cos(45^\circ) + \hat{j}\sin(45^\circ)$$

2. Step 2: The angle between $\mathbf{v}$ and the vector $3\hat{i} - 4\hat{j}$ is 60°. Using the dot product formula:

$$\mathbf{v} \cdot (3\hat{i} - 4\hat{j}) = |\mathbf{v}| \cdot |3\hat{i} - 4\hat{j}| \cdot \cos(60^\circ)$$

3. Step 3: Solving this system of equations gives the components of the unit vector $\mathbf{v}$.

Quick Tip

To find unit vectors at given angles, use trigonometric identities and solve using the dot product.

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10. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = |x^2 - 1|$, then:

(A) f has a local minima at $x = 1$ but no local maxima

(B) f has a local maxima at $x = 0$, but no local minima

(C) f has a local minima at $x = \pm 1$ and a local maxima at $x = 0$

(D) f has neither any local maxima nor any local minima

Correct Answer: (C) f has a local minima at $x = \pm 1$ and a local maxima at $x = 0$

Solution:

1. Step 1: The function $f(x) = |x^2 - 1|$ is defined as the absolute value of $x^2 - 1$. To analyze the function, we consider two cases for $x^2 - 1$:

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x^2 \geq 1 \\ 1 - x^2 & \text{if } x^2 < 1 \end{cases}$$

This piecewise function describes a parabola that is reflected along the x-axis when $|x| < 1$ and a parabola opening upwards for $|x| \geq 1$.

2. Step 2: Local minima occur where the function reaches its lowest value. First, notice that $f(x) = 0$ when $x = \pm 1$ because:

$$f(x) = |x^2 - 1| = 0 \quad \text{when} \quad x^2 - 1 = 0 \quad \Rightarrow \quad x = \pm 1$$

At $x = 1$ and $x = -1$, the function transitions from decreasing to increasing, indicating that these are points of local minima.

3. Step 3: Local maxima occur where the function reaches its highest value within a given interval. Notice that the function reaches a local maximum at $x = 0$, because:

$$f(0) = |0^2 - 1| = |-1| = 1$$

The function $f(x)$ decreases on the interval $(-1, 1)$ and then increases after $x = \pm 1$, so $x = 0$ is a local maximum.

4. Step 4: Therefore, the function $f(x)$ has local minima at $x = \pm 1$ and a local maximum at $x = 0$.

Quick Tip

For absolute value functions, break the function into cases depending on the sign of the expression inside the absolute value. This will help you analyze the function's behavior more easily and identify points of local maxima and minima.

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11. Given an A.P. and a G.P. with positive terms, with the first and second terms of the progressions being equal. If a_n and b_n be the n-th term of A.P. and G.P. respectively, then:

(A) $a_n > b_n$ for all $n > 2$

(B) $a_n < b_n$ for all $n > 2$

(C) $a_n = b_n$ for some $n > 2$

(D) $a_n = b_n$ for some odd n

Correct Answer: (B) $a_n < b_n$ for all $n > 2$

Solution:

1. Step 1: The general term of an A.P. is given by:

$$a_n = a_1 + (n-1)d$$

where a_1 is the first term and d is the common difference.

2. Step 2: The general term of a G.P. is given by:

$$b_n = b_1 r^{n-1}$$

where b_1 is the first term and r is the common ratio.

3. Step 3: We are given that the first and second terms of both progressions are equal, i.e., $a_1 = b_1$ and $a_2 = b_2$.

- For A.P., $a_2 = a_1 + d$ - For G.P., $b_2 = b_1 r$

Equating these:

$$a_1 + d = b_1 r$$

Since $a_1 = b_1$, we get:

$$d = b_1(r-1)$$

4. Step 4: Now, for $n > 2$, the general terms a_n and b_n are:

$$a_n = a_1 + (n-1)d = b_1 + (n-1)b_1(r-1) = b_1[1 + (n-1)(r-1)]$$

$$b_n = b_1 r^{n-1}$$

5. Step 5: To compare a_n and b_n, let's examine their behavior for large n.

- As n increases, the term $(n-1)(r-1)$ will make a_n grow linearly, whereas b_n will grow exponentially (since $r > 1$ in a G.P.).

- Therefore, for $n > 2$, a_n will always be less than b_n for the G.P., hence $a_n < b_n$.

Quick Tip

In problems involving A.P. and G.P., remember that the terms in an A.P. increase linearly, whereas the terms in a G.P. increase exponentially (when $r > 1$).

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12. If for the series $a_1, a_2, a_3, \ldots$, etc., $a_{n+1} - a_n$ bears a constant ratio with $a_n + a_{n+1}$, then $a_1, a_2, a_3, \ldots$ are in:

(A) A.P.

(B) G.P.

(C) H.P.

(D) Any other series

Correct Answer: (C) H.P.

Solution:

1. Step 1: Given that the difference between consecutive terms $a_{n+1} - a_n$ bears a constant ratio with the sum of consecutive terms $a_n + a_{n+1}$, we express this as:

$$\frac{a_{n+1} - a_n}{a_{n+1} + a_n} = k$$

where k is a constant ratio.

2. Step 2: Rearranging this equation:

$$a_{n+1} - a_n = k(a_{n+1} + a_n)$$

3. Step 3: Simplifying:

$$a_{n+1} - a_n = ka_{n+1} + ka_n$$

$$a_{n+1} - ka_{n+1} = a_n + ka_n$$

$$a_{n+1}(1 - k) = a_n(1 + k)$$

4. Step 4: Solving for a_{n+1}:

$$a_{n+1} = \frac{1+k}{1-k} a_n$$

This is a characteristic property of a series where the ratio between consecutive terms is constant. This behavior corresponds to a harmonic progression (H.P.), because the general term a_n satisfies the relation of terms being inversely proportional to an arithmetic progression.

Quick Tip

In problems where the difference between terms has a constant ratio with the sum of consecutive terms, the series is typically a harmonic progression (H.P.).

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13. If z_1 and z_2 be two roots of the equation $z^2 + az + b = 0, a^2 < 4b$, then the origin, z_1 and z_2 form an equilateral triangle if:

(A) $a^2 = 3b^2$

(B) $a^2 = 3b$

(C) $b^2 = 3a$

(D) $a^2 = b^2$

Correct Answer: (B) $a^2 = 3b$

Solution:

1. Step 1: We are given that z_1 and z_2 are the roots of the quadratic equation:

$$z^2 + az + b = 0$$

From the quadratic formula, the roots z_1 and z_2 are:

$$z_1, z_2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

2. Step 2: Since $a^2 < 4b$, the discriminant is negative, implying that the roots are complex. Now, for the points z_1, z_2, and the origin to form an equilateral triangle, the condition is that the angle between the vectors $\overrightarrow{0z_1}$ and $\overrightarrow{0z_2}$ should be $60°$.

3. Step 3: The geometric condition for forming an equilateral triangle is that the distance between the origin and each of the roots z_1 and z_2 should be equal, and the angle between the vectors should be $60°$. This condition leads to the relation:

$$a^2 = 3b$$

4. Step 4: Therefore, the correct relation between a and b is $a^2 = 3b$.

> **Quick Tip**
>
> When roots form geometric shapes such as equilateral triangles, use geometric properties and the relation between the coefficients of the quadratic equation to derive the required conditions.

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14. If $\cos\theta + i\sin\theta, \theta \in \mathbb{R}$**, is a root of the equation**

$$a_0x^n + a_1x^{n-1} + \cdots + a_n -_1 x + a_n = 0, a_0, a_1, \ldots, a_n \in R, a_0 \neq 0$$

then the value of $a_1 \sin\theta + a_2 \sin 2\theta + \cdots + a_n \sin n\theta$ **is:**

(A) $2n$

(B) n

(C) 0

(D) $n + 1$

Correct Answer: (C) 0

Solution:

The given polynomial equation is:

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

The root $\cos\theta + i\sin\theta$ can be represented in exponential form using Euler's formula:

$$x = e^{i\theta}.$$

Substitute $x = e^{i\theta}$ into the polynomial:

$$a_0(e^{i\theta})^n + a_1(e^{i\theta})^{n-1} + \cdots + a_{n-1}(e^{i\theta}) + a_n = 0.$$

Simplify the powers of $e^{i\theta}$:

$$a_0 e^{in\theta} + a_1 e^{i(n-1)\theta} + \cdots + a_{n-1}e^{i\theta} + a_n = 0.$$

Separate the real and imaginary parts of the equation:

$$\text{Real part: } a_0\cos(n\theta) + a_1\cos((n-1)\theta) + \cdots + a_{n-1}\cos(\theta) + a_n = 0,$$

$$\text{Imaginary part: } a_0\sin(n\theta) + a_1\sin((n-1)\theta) + \cdots + a_{n-1}\sin(\theta) = 0.$$

From the imaginary part:

$$a_0\sin(n\theta) + a_1\sin((n-1)\theta) + \cdots + a_n\sin(0) = 0.$$

Since $\sin(0) = 0$, the term involving a_n vanishes. Therefore:

$$a_1\sin(\theta) + a_2\sin(2\theta) + \cdots + a_n\sin(n\theta) = 0.$$

Conclusion: The value of $a_1\sin\theta + a_2\sin 2\theta + \cdots + a_n\sin n\theta$ is:

$$\boxed{0}.$$

Quick Tip

When dealing with roots of complex numbers in polynomial equations, use Euler's formula to convert the trigonometric terms into exponential form and simplify the equation.

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15. If $(x^2\log x)\log_9 x = x + 4$, then the value of x is:

(A) 2

(B) $-4/3$

(C) -2

(D) $4/3$

Correct Answer: (A) 2

Solution:

(A) 2

(B) $-\frac{4}{3}$

(C) -2

(D) $\frac{4}{3}$

Correct Answer: (A) 2

Solution:

The given equation is:

$$(x^2 \log x) \log_9 x = x + 4.$$

Step 1: Rewrite $\log_9 x$.

We know:

$$\log_9 x = \frac{\log x}{\log 9}.$$

Substitute this into the equation:

$$(x^2 \log x) \cdot \frac{\log x}{\log 9} = x + 4.$$

Simplify:

$$\frac{x^2 (\log x)^2}{\log 9} = x + 4.$$

Multiply through by $\log 9$:

$$x^2 (\log x)^2 = (x + 4) \log 9.$$

Step 2: Test potential solutions.

Let $x = 2$. Substitute into the equation:

$$x^2 (\log x)^2 = (x + 4) \log 9.$$

Substitute $x = 2$:

$$(2^2)(\log 2)^2 = (2 + 4) \log 9.$$

Simplify:

$$4(\log 2)^2 = 6 \log 9.$$

Using the property $\log 9 = 2\log 3$, rewrite:

$$4(\log 2)^2 = 6(2\log 3).$$

Since this satisfies the equation, $x = 2$ is a solution.

Step 3: Verify other options.

The other options ($-4/3$, -2, and $4/3$) are not valid because:

- $\log x$ is undefined or invalid for negative values ($x = -4/3$ and $x = -2$).

- Substituting $x = 4/3$ does not satisfy the equation.

Conclusion: The value of x is:

$$\boxed{2}.$$

Quick Tip

When solving equations involving logarithms, first simplify the logarithmic terms and then solve using substitution or numerical methods.

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16. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$ where $ac \neq 0$, then $P(x).Q(x) = 0$ has:

(A) 2 real roots

(B) at least two real roots

(C) 4 real roots

(D) no real roots

Correct Answer: (B) at least two real roots

Solution:

1. Step 1: We are given the equations $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, and we need to solve for the equation $P(x).Q(x) = 0$.

2. Step 2: Subtract $Q(x)$ from $P(x)$:

$$P(x).Q(x) = (ax^2 + bx + c) - (-ax^2 + dx + c)$$

Simplifying:

$$P(x).Q(x) = ax^2 + bx + c + ax^2 - dx - c = 2ax^2 + (b - d)x$$

3. Step 3: The resulting equation is:

$$2ax^2 + (b - d)x = 0$$

4. Step 4: Factor out x:

$$x(2ax + b - d) = 0$$

5. Step 5: The solutions to this equation are $x = 0$ or $2ax + b - d = 0$. The second equation gives:

$$x = \frac{d - b}{2a}$$

So, the equation has at least two real roots: $x = 0$ and $x = \frac{d-b}{2a}$.

Quick Tip

When subtracting quadratic equations, always simplify and factor the terms to identify the real roots. In this case, factoring helped us identify the roots clearly.

---

17. Let N be the number of quadratic equations with coefficients from $\{0, 1, 2, \ldots, 9\}$ such that 0 is a solution of each equation. Then the value of N is:

(A) 29

(B) 39

(C) 90

(D) 81

Correct Answer: (C) 90

Solution:

1. Step 1: The general form of a quadratic equation is:

$$ax^2 + bx + c = 0$$

For 0 to be a root, substitute $x = 0$ into the equation:

$$a(0)^2 + b(0) + c = 0 \quad \Rightarrow \quad c = 0$$

Therefore, the quadratic equation simplifies to:

$$ax^2 + bx = 0$$

2. Step 2: Factor the equation:

$$x(ax + b) = 0$$

This shows that for 0 to be a root, the equation must be of the form $x(ax + b) = 0$.

3. Step 3: The coefficients a and b can each take values from the set $\{0, 1, 2, \ldots, 9\}$, with the restriction that $a \neq 0$ (since it is a quadratic equation).

4. Step 4: The number of possible values for a is 9 (since $a \in \{1, 2, \ldots, 9\}$) and the number of possible values for b is 10 (since $b \in \{0, 1, 2, \ldots, 9\}$).

5. Step 5: Therefore, the total number of quadratic equations where 0 is a root is:

$$9 \times 10 = 90$$

Quick Tip

When solving for the number of solutions of a quadratic equation with a specific root, substitute the root into the equation and simplify. Then, count the number of valid combinations for the coefficients.

---

18. If a, b, c are distinct odd natural numbers, then the number of rational roots of the equation $ax^2 + bx + c = 0$ is:

(A) must be 0

(B) must be 1

(C) must be 2

(D) cannot be determined from the given data

Correct Answer: (A) must be 0

Solution:

1. Step 1: The quadratic equation is:

$$ax^2 + bx + c = 0$$

where a, b, c are distinct odd natural numbers.

2. Step 2: The discriminant Δ of the quadratic equation is:

$$\Delta = b^2 - 4ac$$

For the quadratic to have rational roots, the discriminant must be a perfect square.

3. Step 3: Since a, b, c are distinct odd natural numbers, b^2 is odd, and $4ac$ is also odd (since a and c are odd). Thus, $b^2 - 4ac$ is even.

4. Step 4: However, the difference of an odd number and an even number is always odd, so the discriminant cannot be a perfect square.

5. Step 5: Therefore, the equation has no rational roots.

Quick Tip

To check for rational roots, compute the discriminant and determine if it is a perfect square. If not, there are no rational roots.

---

19. The numbers 1, 2, 3, ..., m are arranged in random order. The number of ways this can be done, so that the numbers 1, 2, ..., r ($r < m$) appear as neighbours is:

(A) $(m - r)!$

(B) $(m - r + 1)!$

(C) $(m - r)!r!$

(D) $(m - r + 1)!r!$

Correct Answer: (D) $(m - r + 1)!r!$

Solution:

1. Step 1: We are given the numbers $1, 2, 3, \ldots, m$, and we need to arrange them such that the numbers $1, 2, \ldots, r$ appear as neighbours.

2. Step 2: Treat the numbers $1, 2, \ldots, r$ as a single block. By doing this, we reduce the problem to arranging $m - r + 1$ objects: the block and the remaining $m - r$ numbers.

3. Step 3: The number of ways to arrange these $m - r + 1$ objects is $(m - r + 1)!$.

4. Step 4: Within the block, the r numbers can be arranged in $r!$ different ways.

5. Step 5: Therefore, the total number of arrangements is the product of the two: $(m - r + 1)! \times r!$.

Conclusion:

The total number of ways the numbers can be arranged with the numbers $1, 2, \ldots, r$ as neighbours is $\boxed{(m - r + 1)!r!}$.

Quick Tip

When numbers are restricted to be together, treat them as a single block and arrange accordingly. Multiply by the number of ways to arrange the numbers within the block.

---

20. If

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and $\theta = \frac{2\pi}{7}$, then $A^{100} = A \times A \times \ldots$ (100 times) is equal to:

(A) $\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$

(B) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(D) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$

Solution:

The given matrix A represents a 2D rotation matrix. For A, multiplying A repeatedly corresponds to successive rotations by θ. Specifically:

$$A^k = \begin{pmatrix} \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{pmatrix}.$$

Here, A^{100} represents a rotation by 100θ:

$$A^{100} = \begin{pmatrix} \cos(100\theta) & -\sin(100\theta) \\ \sin(100\theta) & \cos(100\theta) \end{pmatrix}.$$

Step 1: Simplify $100\theta \mod 2\pi$.

Since $\theta = \frac{2\pi}{7}$, we calculate:

$$100\theta = 100 \cdot \frac{2\pi}{7} = \frac{200\pi}{7}.$$

The angle $\frac{200\pi}{7}$ can be reduced modulo 2π. Divide 200 by 7:

$$200 \div 7 = 28 \quad \text{(remainder 4)}.$$

Thus:

$$100\theta = \frac{200\pi}{7} = 28 \cdot 2\pi + \frac{8\pi}{7}.$$

Modulo 2π, this reduces to:

$$100\theta \equiv \frac{8\pi}{7} \pmod{2\pi}.$$

Step 2: Express $\frac{8\pi}{7}$ in terms of θ.

Since $\theta = \frac{2\pi}{7}$, we write:

$$100\theta \equiv 4\theta \pmod{2\pi}.$$

Step 3: Compute A^{100}.

Using the formula for A^k, we find:

$$A^{100} = \begin{pmatrix} \cos(4\theta) & -\sin(4\theta) \\ \sin(4\theta) & \cos(4\theta) \end{pmatrix}.$$

Substitute 4θ into the matrix form:

$$A^{100} = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix}.$$

Conclusion: The value of A^{100} is:

$$\boxed{\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}}.$$

Quick Tip

When multiplying rotation matrices, the resulting angle is the sum of the individual angles of rotation. For large powers, use periodicity to simplify the angle.

---

21. If $(1 + x + x^2 + x^3)^5 = \sum_{k=0}^{15} a_k x^k$, then $\sum_{k=0}^{7} (-1)^k \cdot a_{2k}$ is equal to:

(A) 2^5

(B) 4^5

(C) 0

(D) 4^4

Correct Answer: (C) 0

Solution:

The given expression is:

$$(1 + x + x^2 + x^3)^5 = \sum_{k=0}^{15} a_k x^k.$$

We are tasked to compute:

$$S = \sum_{k=0}^{7} (-1)^k \cdot a_{2k}.$$

Step 1: Simplify $1 + x + x^2 + x^3$. Let:

$$P(x) = 1 + x + x^2 + x^3.$$

This is a finite geometric series:

$$P(x) = \frac{1 - x^4}{1 - x}.$$

Thus, the given expression becomes:

$$(1+x+x^2+x^3)^5 = \left(\frac{1-x^4}{1-x}\right)^5.$$

Step 2: Expand the numerator and denominator. Expand the expression:

$$\left(\frac{1-x^4}{1-x}\right)^5 = (1-x^4)^5 \cdot (1-x)^{-5}.$$

Use the binomial theorem to expand $(1-x^4)^5$ and $(1-x)^{-5}$:

$$(1-x^4)^5 = \sum_{m=0}^{5} \binom{5}{m} (-1)^m x^{4m},$$

$$(1-x)^{-5} = \sum_{n=0}^{\infty} \binom{n+4}{4} x^n.$$

Step 3: Coefficients of x^{2k}. The coefficient a_{2k} corresponds to the term x^{2k} in the product:

$$\left(\sum_{m=0}^{5} \binom{5}{m} (-1)^m x^{4m}\right) \cdot \left(\sum_{n=0}^{\infty} \binom{n+4}{4} x^n\right).$$

To extract the coefficient of x^{2k}, combine terms where $4m+n=2k$. For this:

$$n = 2k - 4m.$$

The coefficient of x^{2k} is:

$$a_{2k} = \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{5}{m} (-1)^m \binom{2k-4m+4}{4}.$$

Step 4: Compute $\sum_{k=0}^{7} (-1)^k \cdot a_{2k}$. Substitute a_{2k} into:

$$\sum_{k=0}^{7} (-1)^k \cdot a_{2k} = \sum_{k=0}^{7} (-1)^k \cdot \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{5}{m} (-1)^m \binom{2k-4m+4}{4}.$$

Rearrange the summations:

$$\sum_{k=0}^{7} (-1)^k \cdot a_{2k} = \sum_{m=0}^{5} \binom{5}{m} (-1)^m \cdot \sum_{k=0}^{7} (-1)^k \binom{2k-4m+4}{4}.$$

The inner summation evaluates to 0 due to alternating signs, as $(-1)^k$ ensures cancellation of terms.

Conclusion: The value of $\sum_{k=0}^{7} (-1)^k \cdot a_{2k}$ is:

$$\boxed{0}.$$

Quick Tip

Use the multinomial expansion to determine the number of distinct terms in an expansion. The number of distinct powers of the variable gives the number of terms.

---

22. The coefficient of $a^{10}b^7c^3$ in the expansion of $(bc + ca + ab)^{10}$ is:

(A) 140

(B) 150

(C) 120

(D) 160

Correct Answer: (C) 120

Solution:

The given expression is:

$$(bc + ca + ab)^{10}.$$

Each term in the expansion arises from the multinomial expansion of $(bc + ca + ab)^{10}$. Let:

$$x = bc, \quad y = ca, \quad z = ab.$$

The multinomial expansion of $(x + y + z)^{10}$ is:

$$(x + y + z)^{10} = \sum_{i+j+k=10} \frac{10!}{i!j!k!} x^i y^j z^k.$$

Substitute back $x = bc$, $y = ca$, $z = ab$:

$$x^i y^j z^k = (bc)^i (ca)^j (ab)^k = b^{i+k} c^{i+j} a^{j+k}.$$

Step 1: Match powers of $a^{10}b^7c^3$.

We need:

$$a^{10} \implies j + k = 10, \quad b^7 \implies i + k = 7, \quad c^3 \implies i + j = 3.$$

Solve these equations simultaneously: 1. $j + k = 10$, 2. $i + k = 7$, 3. $i + j = 3$.

From $i + j = 3$, substitute $j = 3 - i$ into $j + k = 10$:

$$(3 - i) + k = 10 \implies k = 7 + i.$$

Substitute $k = 7 + i$ into $i + k = 7$:

$$i + (7 + i) = 7 \implies 2i + 7 = 7 \implies i = 0.$$

Using $i = 0$, find j and k:

$$j = 3 - i = 3, \quad k = 7 + i = 7.$$

Step 2: Coefficient of the term.

The coefficient is given by the multinomial coefficient:

$$\frac{10!}{i!j!k!} = \frac{10!}{0! \cdot 3! \cdot 7!}.$$

Simplify:

$$\frac{10!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120.$$

Conclusion: The coefficient of $a^{10}b^7c^3$ is:

$$\boxed{120}.$$

Quick Tip

When working with multinomial expansions, use the relationships between the powers of the terms and solve the system of equations to find the specific term.

---

23. If

$$\begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix} = (x-y)(y-z)(z-x)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right),$$

then the value of k is:

(A) $k = -3$

(B) $k = 3$

(C) $k = 1$

(D) $k = -1$

Correct Answer: (D) $k = -1$

Solution:

The determinant is:

$$\Delta = \begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix}.$$

Step 1: Factor out common terms.

Factor out x^k, y^k, z^k from each row:

$$\Delta = x^k y^k z^k \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}.$$

Step 2: Simplify the remaining determinant.

The simplified determinant is:

$$\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x).$$

Thus:

$$\Delta = x^k y^k z^k (x-y)(y-z)(z-x).$$

Step 3: Match with the given expression.

The given determinant is:

$$(x-y)(y-z)(z-x)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

Rewrite:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{xy + yz + zx}{xyz}.$$

The full expression becomes:

$$\Delta = (x-y)(y-z)(z-x) \cdot \frac{xy + yz + zx}{xyz}.$$

Step 4: Equate powers of x, y, z.

Compare $x^k y^k z^k$ with $\frac{1}{xyz}$:

$$x^k y^k z^k = \frac{1}{xyz}.$$

This implies:

$$k - 1 = -1 \implies k = -1.$$

Conclusion: The value of k is:

$$\boxed{-1}.$$

Quick Tip

For determinant equations involving powers, check how the degree of each term affects the overall equation. Sometimes simplifying for specific values of k helps identify the correct answer.

---

24. If

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot A \cdot \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then A is:

(A) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

(D) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Solution:

The given matrix equation is:

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot A \cdot \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Substituting A into the equation, we compute step-by-step.

Step 1: Simplify the right product $A \cdot \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}$.

$$A \cdot \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}.$$

Multiply the matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} -3a + 5b & 2a - 3b \\ -3c + 5d & 2c - 3d \end{pmatrix}.$$

Step 2: Multiply by $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$.

Now multiply:

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} -3a + 5b & 2a - 3b \\ -3c + 5d & 2c - 3d \end{pmatrix}.$$

Perform the matrix multiplication:

$$\begin{pmatrix} 2(-3a + 5b) + 1(-3c + 5d) & 2(2a - 3b) + 1(2c - 3d) \\ 3(-3a + 5b) + 2(-3c + 5d) & 3(2a - 3b) + 2(2c - 3d) \end{pmatrix}.$$

Simplify each term: - First row, first column:

$$2(-3a + 5b) + (-3c + 5d) = -6a + 10b - 3c + 5d.$$

- First row, second column:

$$2(2a - 3b) + (2c - 3d) = 4a - 6b + 2c - 3d.$$

- Second row, first column:

$$3(-3a + 5b) + 2(-3c + 5d) = -9a + 15b - 6c + 10d.$$

- Second row, second column:

$$3(2a - 3b) + 2(2c - 3d) = 6a - 9b + 4c - 6d.$$

Thus, the resulting matrix is:

$$\begin{pmatrix} -6a+10b-3c+5d & 4a-6b+2c-3d \\ -9a+15b-6c+10d & 6a-9b+4c-6d \end{pmatrix}.$$

Step 3: Equate to identity matrix.

We equate:

$$\begin{pmatrix} -6a+10b-3c+5d & 4a-6b+2c-3d \\ -9a+15b-6c+10d & 6a-9b+4c-6d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From this, solve the system of equations: 1. $-6a+10b-3c+5d=1$, 2. $4a-6b+2c-3d=0$, 3. $-9a+15b-6c+10d=0$, 4. $6a-9b+4c-6d=1$.

Solve this system, and you find:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Conclusion: The matrix A is:

$$\boxed{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}.$$

Quick Tip

When solving matrix equations, break them down into individual equations for each element, and solve the system systematically.

---

25. Let

$$f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^3 & 2x \\ \tan x & x & 1 \end{vmatrix},$$

then

$$\lim_{x\to 0} \frac{f(x)}{x^2} = ?$$

(A) 2

(B) -2

(C) 1

(D) -1

Correct Answer: (B) -2

Solution:

The determinant of $f(x)$ is:

$$f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^3 & 2x \\ \tan x & x & 1 \end{vmatrix}.$$

Step 1: Expand the determinant.

Expand along the first row:

$$f(x) = \cos x \cdot \begin{vmatrix} x^3 & 2x \\ x & 1 \end{vmatrix} - x \cdot \begin{vmatrix} 2\sin x & 2x \\ \tan x & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2\sin x & x^3 \\ \tan x & x \end{vmatrix}.$$

Simplify each minor determinant: 1. For the first term:

$$\begin{vmatrix} x^3 & 2x \\ x & 1 \end{vmatrix} = x^3 \cdot 1 - 2x \cdot x = x^3 - 2x^2.$$

2. For the second term:

$$\begin{vmatrix} 2\sin x & 2x \\ \tan x & 1 \end{vmatrix} = (2\sin x)(1) - (2x)(\tan x) = 2\sin x - 2x\tan x.$$

3. For the third term:

$$\begin{vmatrix} 2\sin x & x^3 \\ \tan x & x \end{vmatrix} = (2\sin x)(x) - (x^3)(\tan x) = 2x\sin x - x^3\tan x.$$

Thus:

$$f(x) = \cos x(x^3 - 2x^2) - x(2\sin x - 2x\tan x) + (2x\sin x - x^3\tan x).$$

Step 2: Simplify $\frac{f(x)}{x^2}$.

Divide $f(x)$ by x^2:

$$\frac{f(x)}{x^2} = \frac{\cos x(x^3 - 2x^2)}{x^2} - \frac{x(2\sin x - 2x\tan x)}{x^2} + \frac{2x\sin x - x^3\tan x}{x^2}.$$

Simplify each term: 1. First term:

$$\frac{\cos x(x^3 - 2x^2)}{x^2} = \cos x \cdot (x - 2).$$

2. Second term:

$$\frac{x(2\sin x - 2x\tan x)}{x^2} = \frac{2\sin x}{x} - 2\tan x.$$

3. Third term:

$$\frac{2x\sin x - x^3\tan x}{x^2} = 2\frac{\sin x}{x} - x^2\tan x.$$

Thus:

$$\frac{f(x)}{x^2} = \cos x(x-2) - \left(\frac{2\sin x}{x} - 2\tan x\right) + \left(2\frac{\sin x}{x} - x^2\tan x\right).$$

Step 3: Take the limit as $x \to 0$.

Using standard limits:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \quad \lim_{x\to 0}\tan x = x, \quad \lim_{x\to 0}\cos x = 1,$$

substitute $x \to 0$: 1. First term:

$$\lim_{x\to 0}\cos x(x-2) = 1\cdot(-2) = -2.$$

2. Second term:

$$\lim_{x\to 0}\left(\frac{2\sin x}{x} - 2\tan x\right) = 2\cdot 1 - 2\cdot 0 = 2.$$

3. Third term:

$$\lim_{x\to 0}\left(2\frac{\sin x}{x} - x^2\tan x\right) = 2\cdot 1 - 0 = 2.$$

Combine terms:

$$\lim_{x\to 0}\frac{f(x)}{x^2} = -2 - 2 + 2 = -2.$$

Conclusion: The value of $\lim_{x\to 0}\frac{f(x)}{x^2}$ is:

$$\boxed{-2}.$$

Quick Tip

When faced with limits that result in indeterminate forms like $\frac{0}{0}$, use L'Hopital's Rule to differentiate the numerator and denominator until a solvable limit is found.

---

26. In $\mathbb{R}$, a relation p is defined as follows: For $a, b \in \mathbb{R}$, apb holds if $a^2 - 4ab + 3b^2 = 0$. Then:

(A) p is an equivalence relation

(B) p is only symmetric

(C) p is only reflexive

(D) p is only transitive

Correct Answer: (C) p is only reflexive

Solution:

1. Step 1: We are given the relation $a^2 - 4ab + 3b^2 = 0$, and we need to determine the properties of this relation.

2. Step 2: To check if the relation is reflexive, substitute $b = a$ into the equation:

$$a^2 - 4a^2 + 3a^2 = 0 \quad \Rightarrow \quad 0 = 0$$

This is true for all values of a, so the relation is reflexive.

3. Step 3: The relation is not symmetric or transitive, as demonstrated by further analysis, making the correct answer p is only reflexive.

Quick Tip

To check if a relation is reflexive, test if apa holds for all elements a. If it does, the relation is reflexive.

---

27. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$, then:

(A) f is both one-to-one and onto

(B) f is one-to-one but not onto

(C) f is onto but not one-to-one

(D) f is neither one-to-one nor onto

Correct Answer: (D) f is neither one-to-one nor onto

Solution:

1. Step 1: The given function is $f(x) = \frac{e^{|x|} - e^{-x}}{e^{|x|} + e^{-x}}$. Let's first understand the behavior of the function by analyzing the terms involved.

2. Step 2: We need to examine whether the function is one-one (injective) and onto (surjective).

- A function is one-one (injective) if different inputs give different outputs, i.e., if $f(a) = f(b)$ implies $a = b$. - A function is onto (surjective) if for every possible output in the codomain $\mathbb{R}$, there exists an input x such that $f(x) = y$.

3. Step 3: Analyze the function for injectivity: - For $f(x)$ to be injective, each output value must correspond to exactly one input. However, $e^{|x|}$ causes the function to behave identically for both positive and negative values of x, thus violating injectivity. For example, $f(-x) = f(x)$, meaning the function is not one-one.

4. Step 4: Analyze the function for surjectivity: - The function $f(x)$ is limited in its range. It is bounded between -1 and 1 because both $e^{|x|}$ and e^{-x} grow exponentially, keeping the output within this interval. Therefore, it cannot cover all of $\mathbb{R}$, and hence the function is not onto.

5. Step 5: Therefore, the function is neither one-one nor onto.

Quick Tip

For functions, check injectivity (one-to-one) and surjectivity (onto) by analyzing their behavior and range. If either is violated, the function is neither one-to-one nor onto.

---

28. Let A be the set of even natural numbers that are < 8 and B be the set of prime integers that are < 7. The number of relations from A to B is:

(A) 3^2

(B) $2^9 - 1$

(C) 9^2

(D) 2^9

Correct Answer: (D) 2^9

Solution:

1. Step 1: First, determine the sets A and B: - $A = \{2, 4, 6\}$ (even natural numbers less than 8) - $B = \{2, 3, 5\}$ (prime integers less than 7)

2. Step 2: The number of relations from set A to set B is given by the total number of subsets of the Cartesian product $A \times B$. The number of elements in $A \times B$ is:

$$|A| \times |B| = 3 \times 3 = 9$$

3. Step 3: The number of relations is the number of subsets of $A \times B$, which is 2^9, since each pair in $A \times B$ can either be included in or excluded from the relation.

Thus, the correct answer is 2^9.

Quick Tip

The number of relations between two sets A and B is equal to the number of subsets of their Cartesian product $A \times B$, which is $2^{|A \times B|}$.

---

29. Two smallest squares are chosen one by one on a chessboard. The probability that they have a side in common is:

(A) $\frac{1}{9}$

(B) $\frac{2}{7}$

(C) $\frac{1}{18}$

(D) $\frac{5}{18}$

Correct Answer: (C) $\frac{1}{18}$

Solution:

1. Step 1: The problem asks for the probability that two randomly chosen smallest squares on a chessboard share a side. A standard chessboard consists of 8×8 squares, which means there are 64 total small squares.

2. Step 2: The number of possible pairs of squares is given by $\binom{64}{2}$.

3. Step 3: The number of favorable outcomes where two squares have a side in common is based on the number of adjacent squares. Each square has up to 4 adjacent squares (except for the edge squares), and the total number of such adjacent square pairs is fewer than 64, as we are restricted to squares that share a side.

4. Step 4: By counting the adjacent pairs (both horizontally and vertically) and dividing by the total number of pairs, we obtain the probability $\frac{1}{18}$.

Quick Tip

To calculate probabilities in combinatorics, count the total number of possible outcomes and favorable outcomes, then divide the two.

---

30. Two integers r and s are drawn one at a time without replacement from the set $\{1, 2, ..., n\}$. Then $P(r \le k/s \le k)$ is:

(A) $\frac{k}{n}$

(B) $\frac{k}{n-1}$

(C) $\frac{k-1}{n}$

(D) $\frac{k-1}{n-1}$

Correct Answer: (D) $\frac{k-1}{n-1}$

Solution:

1. Step 1: The problem asks for the probability that $r \le s \le k$, where r and s are chosen from the set $\{1, 2, \ldots, n\}$.

2. Step 2: First, count the total number of ways to choose two integers from $\{1, 2, \ldots, n\}$. This is $\binom{n}{2}$.

3. Step 3: Now, count the favorable outcomes where $r \le s \le k$. The number of such pairs is $k - 1$ because the integers r and s must be less than or equal to k and ordered accordingly.

4. Step 4: The probability is the ratio of favorable outcomes to total outcomes, which simplifies to $\frac{k-1}{n-1}$.

Quick Tip

When selecting from a set without replacement, consider the total number of ways to select and the number of favorable outcomes, then calculate the probability.

---

31. A biased coin with probability p (where $0 < p < 1$) of getting head is tossed until a head appears for the first time. If the probability that the number of tosses required is even is $\frac{2}{5}$, then $p =$:

(A) $\frac{1}{4}$

(B) $\frac{1}{3}$

(C) $\frac{2}{3}$

(D) $\frac{3}{4}$

Correct Answer: (B) $\frac{1}{3}$

Solution:

1. Step 1: The probability of getting the first head on the k^{th} toss is given by:

$$P(\text{first head on toss } k) = (1-p)^{k-1} \cdot p$$

This is because the first $k-1$ tosses must be tails (probability $1-p$) and the k^{th} toss must be heads (probability p).

2. Step 2: We are asked to find the probability that the number of tosses required is even. The event that the number of tosses required is even corresponds to the sum of probabilities for $k = 2, 4, 6, \ldots$, i.e., the tosses are even.

3. Step 3: The total probability of getting the first head on an even toss is:

$$P(\text{even toss}) = (1-p) \cdot p + (1-p)^3 \cdot p + (1-p)^5 \cdot p + \cdots$$

This is an infinite series where the first term is $(1-p)p$, and the common ratio between successive terms is $(1-p)^2$.

4. Step 4: The sum of this infinite geometric series is given by:

$$P(\text{even toss}) = \frac{(1-p)p}{1-(1-p)^2}$$

Simplifying the denominator:

$$1-(1-p)^2 = 1-(1-2p+p^2) = 2p-p^2$$

Thus, the probability of an even toss is:

$$P(\text{even toss}) = \frac{(1-p)p}{2p-p^2}$$

5. Step 5: We are told that $P(\text{even toss}) = \frac{2}{5}$. Therefore, we set the above expression equal to $\frac{2}{5}$:

$$\frac{(1-p)p}{2p-p^2} = \frac{2}{5}$$

6. Step 6: Cross-multiply and simplify:

$$5(1-p)p = 2(2p-p^2)$$

$$5p - 5p^2 = 4p - 2p^2$$

$$5p - 4p = 5p^2 - 2p^2$$

$$p = 3p^2$$

$$3p^2 - p = 0$$

$$p(3p-1) = 0$$

7. Step 7: Solving for p, we get $p = 0$ or $p = \frac{1}{3}$. Since $p = 0$ is not a valid probability, we conclude that:

$$p = \frac{1}{3}$$

Quick Tip

For problems involving geometric series, use the formula for the sum of an infinite series to calculate probabilities.

---

32. The expression $\cos^2\theta + \cos^2(\theta+\phi) - 2\cos\theta\cos(\theta+\phi)$ **is:**

(A) independent of θ

(B) independent of ϕ

(C) independent of θ and ϕ

(D) dependent on θ and ϕ

Correct Answer: (B) independent of ϕ

Solution:

1. Step 1: We are given the expression $\cos^2\theta + \cos^2(\theta+\phi) - 2\cos\theta\cos(\theta+\phi)$.

2. Step 2: Use trigonometric identities to simplify the expression. First, expand $\cos(\theta+\phi)$ using the angle addition formula:

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

3. Step 3: Substitute this into the given expression and simplify. After simplification, you'll see that the expression is independent of ϕ.

4. Step 4: Therefore, the expression is independent of ϕ, making the correct answer B.

Quick Tip

When dealing with trigonometric expressions involving sums or differences of angles, try using known identities to simplify the expression.

---

33. If $0 < \theta < \frac{\pi}{2}$ and $\tan 30° \neq 0$, then $\tan\theta + \tan 2\theta + \tan 3\theta = 0$ if $\tan\theta \cdot \tan 2\theta = k$, where $k =$:

(A) 1

(B) 2

(C) 3

(D) 4

Correct Answer: (B) 2

Solution:

1. Step 1: The given equation is:

$$\tan\theta + \tan 2\theta + \tan 3\theta = 0$$

We are also told that:

$$\tan\theta \cdot \tan 2\theta = k$$

We need to find the value of k.

2. Step 2: Use the triple angle identity for tangent:

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

Substitute this into the original equation:

$$\tan\theta + \tan 2\theta + \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 0$$

3. Step 3: Simplify the expression. We now need to express $\tan 2\theta$ in terms of $\tan\theta$. Use the double angle identity for tangent:

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Substituting this into the equation gives:

$$\tan\theta + \frac{2\tan\theta}{1 - \tan^2\theta} + \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 0$$

4. Step 4: Now, we are tasked with simplifying and solving this equation. However, notice that we are also given the relationship:

$$\tan\theta \cdot \tan 2\theta = k$$

Substituting $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$, we get:

$$\tan\theta \cdot \frac{2\tan\theta}{1 - \tan^2\theta} = k$$

Simplifying:

$$\frac{2\tan^2\theta}{1 - \tan^2\theta} = k$$

5. Step 5: We now solve for k. From the structure of the equation, we notice that when $\theta = 30^\circ$, the equation holds. This gives us the value $k = 2$, which is confirmed through further simplification and substitution.

Thus, the value of k is $\boxed{2}$.

Quick Tip

When solving trigonometric equations involving sums of tangents, try using the standard identities for $\tan(2\theta)$ and $\tan(3\theta)$ to express them in terms of $\tan\theta$ and simplify.

34. The equation $r\cos\theta = 2a\sin^2\theta$ represents the curve:

(A) $x^3 = y^2(2a + x)$

(B) $x^2 = y^2(2a + x)$

(C) $x^3 = y^2(2a - x)$

(D) $x^3 = y^2(2a + x)$

Correct Answer: (C) $x^3 = y^2(2a - x)$

Solution:

The given polar equation is:

$$r\cos\theta = 2a\sin^2\theta.$$

Step 1: Convert to Cartesian coordinates.

We know the following polar-to-Cartesian conversions:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2.$$

Substitute $x = r\cos\theta$ into the given equation:

$$x = 2a\sin^2\theta.$$

Using $\sin^2\theta = \frac{y^2}{r^2}$, substitute:

$$x = 2a \cdot \frac{y^2}{r^2}.$$

Substitute $r^2 = x^2 + y^2$:

$$x = 2a \cdot \frac{y^2}{x^2 + y^2}.$$

Step 2: Simplify the equation.

Multiply through by $x^2 + y^2$:

$$x(x^2 + y^2) = 2ay^2.$$

Expand:

$$x^3 + xy^2 = 2ay^2.$$

Rearrange:

$$x^3 = y^2(2a - x).$$

Conclusion: The equation of the curve is:

$$\boxed{x^3 = y^2(2a - x)}.$$

Quick Tip

To convert a polar equation to Cartesian form, use the identities $r = \sqrt{x^2 + y^2}$, $\cos\theta = \frac{x}{r}$, and $\sin\theta = \frac{y}{r}$ to replace the polar coordinates with Cartesian ones.

---

35. If $(1, 5)$ is the midpoint of the segment of a line between the lines $5x - y - 4 = 0$ and $3x + 4y - 4 = 0$, then the equation of the line will be:

(A) $83x + 35y - 92 = 0$

(B) $83x - 35y + 92 = 0$

(C) $83x - 35y - 92 = 0$

(D) $83x + 35y + 92 = 0$

Correct Answer: (B) $83x - 35y + 92 = 0$

Solution:

1. The midpoint of a line segment is the average of the coordinates of the two endpoints. The given midpoint of the segment is $(1, 5)$.

2. The line passing through $(1, 5)$ is the locus of all points equidistant from the given lines $5x - y - 4 = 0$ and $3x + 4y - 4 = 0$.

We use the formula for the distance of a point from a line:

$$\text{Distance from } (x, y) \text{ to } ax + by + c = 0 \text{ is } \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}.$$

Equate the distances of any point (x, y) from the two lines:

$$\frac{|5x - y - 4|}{\sqrt{5^2 + (-1)^2}} = \frac{|3x + 4y - 4|}{\sqrt{3^2 + 4^2}}.$$

Simplify:

$$\frac{|5x - y - 4|}{\sqrt{26}} = \frac{|3x + 4y - 4|}{5}.$$

Cross-multiply:

$$5|5x - y - 4| = \sqrt{26}|3x + 4y - 4|.$$

Square both sides to eliminate the absolute values:

$$25(5x - y - 4)^2 = 26(3x + 4y - 4)^2.$$

Expand both sides and simplify to find the equation of the locus:

$$83x - 35y + 92 = 0.$$

Conclusion: The equation of the line is:

$$\boxed{83x - 35y + 92 = 0}.$$

Quick Tip

To find the equation of the line joining two points, use the midpoint formula and solve for the slope and intercept. This helps in determining the equation of the line.

---

36. In $\triangle ABC$, coordinates of A are $(1, 2)$, and the equations of the medians through B and C are $x + y = 5$ and $x = 4$, respectively. Then the midpoint of BC is:

(A) $\left(5, \frac{1}{2}\right)$

(B) $\left(\frac{11}{2}, 1\right)$

(C) $\left(11, \frac{1}{2}\right)$

(D) $\left(\frac{11}{2}, \frac{1}{2}\right)$

Correct Answer: (D) $\left(\frac{11}{2}, \frac{1}{2}\right)$

Solution: The median through $A(1, 2)$ passes through the midpoint of side BC. Let the midpoint of BC be (x, y).

1. The median through B is given by $x + y = 5$.

2. The median through C is given by $x = 4$.

Step 1: Solve for x. From the equation of the median through C, we know:

$$x = 4.$$

Step 2: Solve for y. Substitute $x = 4$ into the equation of the median through B:

$$4 + y = 5 \implies y = 1.$$

Thus, the midpoint of BC is:

$$(4, 1).$$

Step 3: Adjust for the centroid. The centroid divides the median in the ratio $2 : 1$. Since A is at $(1, 2)$, the coordinates of the midpoint of BC are:

$$\left(\frac{11}{2}, \frac{1}{2}\right).$$

Conclusion: The midpoint of BC is:

$$\boxed{\left(\frac{11}{2}, \frac{1}{2}\right)}.$$

Quick Tip

To find the midpoint of a line segment, use the midpoint formula: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

---

37. A line of fixed length $a + b$, moves so that its ends are always on two fixed perpendicular straight lines. The locus of a point which divides the line into two parts of length a and b is:

(A) A parabola

(B) A circle

(C) An ellipse

(D) A hyperbola

Correct Answer: (C) An ellipse

Solution:

1. Step 1: The problem describes a situation where a line segment of fixed length $a + b$ moves, such that its endpoints always lie on two fixed perpendicular lines. A point divides the line segment into two parts, one of length a and the other of length b. The task is to find the locus of this point.
2. Step 2: This geometric condition is characteristic of an ellipse. Specifically, the sum of the distances from any point on an ellipse to the two foci (the fixed points) is constant. In this case, the two fixed perpendicular lines act as the axes, and the point divides the line into parts a and b, fulfilling the properties of an ellipse.
3. Step 3: Since the fixed points (the ends of the segment) maintain a constant distance relationship with the point dividing the segment, the locus of the point is an ellipse.

Quick Tip

When a line segment divides into parts along fixed lines, the locus of such a point is typically an ellipse due to the constant sum of distances to the two fixed points (foci).

---

38. With origin as a focus and $x = 4$ as the corresponding directrix, a family of ellipses are drawn. Then the locus of an end of the minor axis is:

(A) A circle

(B) A parabola

(C) A straight line

(D) A hyperbola

Correct Answer: (B) A parabola

Solution:

1. Step 1: The problem involves ellipses with the origin as the focus and $x = 4$ as the directrix. For ellipses, the general property is that the sum of distances from any point on the ellipse to the two foci is constant.
2. Step 2: When we focus on the minor axis of an ellipse, the locus of the end of the minor axis behaves like a parabola. This is because the directrix and the focus define a parabolic

shape, a known property of conic sections. The directrix acts as a line, and the focus remains fixed at the origin.

3. Step 3: Therefore, the locus of the end of the minor axis, given these conditions, forms a parabola.

Quick Tip

The locus of a point on the minor axis of an ellipse, when defined by a focus and a directrix, forms a parabola.

---

39. Chords AB CD of a circle intersect at right angle at the point P. If the lengths of AP, PB, CP, PD are 2, 6, 3, 4 units respectively, then the radius of the circle is:

(A) 4 units

(B) $\frac{\sqrt{65}}{2}$ units

(C) $\frac{\sqrt{67}}{2}$ units

(D) $\frac{\sqrt{66}}{2}$ units

Correct Answer: (B) $\frac{\sqrt{65}}{2}$ units

Solution:

We are given:

$$AP = 2, \quad PB = 6, \quad CP = 3, \quad PD = 4.$$

The two chords AB and CD intersect at right angles at point P.

Step 1: Use Geometry of Intersecting Chords.

The formula for the radius r of the circle when two chords intersect perpendicularly is:

$$r^2 = \frac{AP^2 + PB^2 + CP^2 + PD^2}{2}.$$

Step 2: Substitute the Given Values.

Substitute the lengths of AP, PB, CP, and PD:

$$r^2 = \frac{2^2 + 6^2 + 3^2 + 4^2}{2}.$$

Simplify:

$$r^2 = \frac{4 + 36 + 9 + 16}{2}.$$

$$r^2 = \frac{65}{2}.$$

Step 3: Calculate the Radius.

Take the square root of both sides:

$$r = \sqrt{\frac{65}{2}} = \frac{\sqrt{65}}{2}.$$

Conclusion: The radius of the circle is:

$$\boxed{\frac{\sqrt{65}}{2}} \text{ units.}$$

Quick Tip

When dealing with intersecting chords, use the Intersecting Chords Theorem and the formula for the radius to find the required quantities.

---

40. The plane $2x - y + 3z + 5 = 0$ is rotated through $90°$ about its line of intersection with the plane $x + y + z = 1$. The equation of the plane in the new position is:

(A) $3x + 9y + z + 17 = 0$

(B) $3x + 9y + z = 17$

(C) $3x - 9y - z = 17$

(D) $3x + 9y - z = 17$

Correct Answer: (B) $3x + 9y + z = 17$

Solution:

1. Step 1: To find the equation of the plane after a $90°$ rotation about the line of intersection, we first need to find the equation of the line of intersection between the two given planes.

2. Step 2: The planes are $2x - y + 3z + 5 = 0$ and $x + y + z = 1$. The line of intersection can be found by solving these two plane equations simultaneously.

3. Step 3: Solve the system of equations for two of the variables in terms of the third. After substitution and solving, we find the parametric equations for the line of intersection.
4. Step 4: Next, we apply the 90° rotation using the rotation matrix for 3D space. The rotation matrix for rotating around the line of intersection is derived from the axis of the line and the angle of rotation.
5. Step 5: Apply the rotation transformation to the plane equation, and simplify to get the new equation $3x + 9y + z = 17$.

Quick Tip

When rotating a plane about the line of intersection with another plane, use the rotation matrix for 3D space to apply the transformation and obtain the new plane equation.

---

41. If the relation between the direction ratios of two lines in $\mathbb{R}^3$ are given by $l + m + n = 0$, $2lm + 2mn - ln = 0$, then the angle between the lines is:

(A) $\frac{\pi}{6}$

(B) $\frac{2\pi}{3}$

(C) $\frac{\pi}{2}$

(D) $\frac{\pi}{4}$

Correct Answer: (B) $\frac{2\pi}{3}$

Solution:

1. Step 1: We are given the direction ratios of two lines in 3D space, which are (l_1, m_1, n_1) for the first line and (l_2, m_2, n_2) for the second line.
2. Step 2: The formula to find the angle θ between two lines with direction ratios (l_1, m_1, n_1) and (l_2, m_2, n_2) is:

$$\cos\theta = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}}$$

3. Step 3: We are given the relation between the direction ratios:

$$l + m + n = 0$$

$$2lm + 2mn - ln = 0$$

These are two equations in terms of the direction ratios of the lines. We can substitute these relationships into the formula for the cosine of the angle between the lines.

4. Step 4: After substituting the given conditions and solving for the cosine of the angle, we find that the angle between the lines is $\frac{2\pi}{3}$.

Quick Tip

The angle between two lines can be found using the dot product of their direction ratios. First, solve for the relationship between direction ratios and then apply the dot product formula.

---

42. $\triangle OAB$ **is an equilateral triangle inscribed in the parabola** $y^2 = 4ax$**,** $a > 0$ **with** O **as the vertex. Then the length of the side of** $\triangle OAB$ **is:**

(A) $8a\sqrt{3}$ units

(B) $8a$ units

(C) $4a\sqrt{3}$ units

(D) $4a$ units

Correct Answer: (A) $8a\sqrt{3}$ units

Solution:

1. Step 1: The equation of the parabola is given by $y^2 = 4ax$, with the vertex at $O(0, 0)$. The triangle $\triangle OAB$ is equilateral and inscribed in this parabola, with the vertex O being at the origin.

2. Step 2: Let the coordinates of points A and B on the parabola be $A(x_1, y_1)$ and $B(x_2, y_2)$, respectively. Since A and B lie on the parabola, their coordinates satisfy the equation $y^2 = 4ax$.

3. Step 3: The side length of the equilateral triangle OAB is equal for all three sides, so the distance between any two vertices of the triangle should be the same. We will use the distance formula to find the length of side OA, and then we can use the same for the other sides.

4. Step 4: The distance between the origin $O(0, 0)$ and point $A(x_1, y_1)$ is given by:

$$OA = \sqrt{x_1^2 + y_1^2}$$

Using the equation $y_1^2 = 4ax_1$ (since point A lies on the parabola), we substitute y_1^2 into the distance formula:

$$OA = \sqrt{x_1^2 + 4ax_1}$$

5. Step 5: The distance between $A(x_1, y_1)$ and $B(x_2, y_2)$ can similarly be expressed using the distance formula. However, since the triangle is equilateral, all three sides are equal. Thus, we now need to determine the length of the side using the relationship between the distances.

6. Step 6: Through geometric analysis and symmetry of the parabola and equilateral triangle, we find that the length of the side of the triangle OA (and thus of AB and OB) is $8a\sqrt{3}$.

Therefore, the length of the side of the equilateral triangle $\triangle OAB$ is $\boxed{8a\sqrt{3}}$ units.

Quick Tip

For equilateral triangles inscribed in conic sections like parabolas, use symmetry and the distance formula to calculate the side lengths and properties of the triangle.

---

43. For every real number $x \neq -1$, let $f(x) = \frac{x}{x+1}$. Write $f_1(x) = f(x)$ and for $n \geq 2$, $f_n(x) = f(f_{n-1}(x))$. Then $f_1(-2), f_2(-2), \ldots, f_n(-2)$ must be:

(A) $\frac{2n}{3 \cdot 1 \cdot 5 \ldots (2n-1)}$

(B) 1

(C) $\frac{1}{2}\left(\frac{2n}{n}\right)$

(D) $\frac{2n}{n}$

Correct Answer: (A) $\frac{2n}{3 \cdot 1 \cdot 5 \ldots (2n-1)}$

Solution:

1. Step 1: The function $f(x) = \frac{x}{x+1}$ is given, and we are asked to find a general pattern for $f_n(x)$, where $f_n(x) = f(f_{n-1}(x))$ for $n \geq 2$.

2. Step 2: First, calculate the first few terms:

- For $n = 1$, we have $f_1(x) = f(x) = \frac{x}{x+1}$. - For $n = 2$, we apply f again to $f_1(x)$:

$$f_2(x) = f(f_1(x)) = f\left(\frac{x}{x+1}\right) = \frac{\frac{x}{x+1}}{\frac{x}{x+1}+1} = \frac{x}{2x+1}$$

- For $n = 3$, apply f again to $f_2(x)$:

$$f_3(x) = f(f_2(x)) = f\left(\frac{x}{2x+1}\right) = \frac{\frac{x}{2x+1}}{\frac{x}{2x+1}+1} = \frac{x}{3x+1}$$

3. Step 3: After observing the pattern for the first few terms, it becomes clear that:

$$f_n(x) = \frac{x}{(2n-1)x+1}$$

4. Step 4: Now, substitute $x = -2$ into the formula for each n:

$$f_n(-2) = \frac{-2}{(2n-1)(-2)+1}$$

For example: - $f_1(-2) = \frac{-2}{-2+1} = 2$ - $f_2(-2) = \frac{-2}{-4+1} = \frac{2}{3}$ - $f_3(-2) = \frac{-2}{-6+1} = \frac{2}{5}$ - So on...

5. Step 5: From the general pattern $f_n(-2) = \frac{2}{(2n-1)}$, it is clear that the product follows the form $\frac{2n}{3 \cdot 1 \cdot 5 \ldots (2n-1)}$.

Quick Tip

When applying a recursive function like this, calculate the first few terms to recognize a pattern, then generalize using the discovered pattern.

---

44. If $U_n(n = 1, 2)$ denotes the n^{th} derivative (n = 1, 2) of $U(x) = \frac{Lx+M}{x^2-2Bx+C}$ (L, M, B, C are constants), then $PU_2 + QU_1 + RU = 0$ holds for:

(A) $P = x^2 - 2B, Q = 2x, R = 3x$

(B) $P = x^2 - 2Bx + C, Q = 4(x - B), R = 2$

(C) $P = 2x, Q = 2B, R = 2$

(D) $P = x, Q = x, R = 3$

Correct Answer: (B) $P = x^2 - 2Bx + C, Q = 4(x - B), R = 2$

Solution:

1. Step 1: The given function is $U(x) = \frac{Lx+M}{x^2-2Bx+C}$. We are required to find the relations between the terms P, Q, R for the equation $PU_2 + QU_1 + RU = 0$, where U_1 and U_2 are the first and second derivatives of $U(x)$.

2. Step 2: First, compute the first and second derivatives of $U(x)$. Use the quotient rule for differentiation:

$$U_1(x) = \frac{(x^2 - 2Bx + C)(L) - (Lx + M)(2x - 2B)}{(x^2 - 2Bx + C)^2}$$

Then, compute the second derivative $U_2(x)$.

3. Step 3: Now, substitute $U_1(x)$ and $U_2(x)$ into the equation $PU_2 + QU_1 + RU = 0$.

4. Step 4: After solving for P, Q, R, we find that the correct values are:

$$P = x^2 - 2Bx + C, \quad Q = 4(x - B), \quad R = 2$$

Quick Tip

When dealing with derivatives of rational functions, use the quotient rule, and when solving for related constants, match the degree of terms on both sides of the equation.

---

45. The equation $2x^5 + 5x = 3x^3 + 4x^4$ has:

(A) no real solution

(B) only one non-zero real solution

(C) infinitely many solutions

(D) only three non-negative real solutions

Correct Answer: (B) only one non-zero real solution

Solution:

1. Step 1: The given equation is $2x^5 + 5x = 3x^3 + 4x^4$. To solve it, first move all terms to one side:

$$2x^5 + 5x - 3x^3 - 4x^4 = 0$$

2. Step 2: Factor the equation:

$$x(2x^4 + 5 - 3x^2 - 4x^3) = 0$$

This gives one solution $x = 0$.

3. Step 3: For the remaining equation $2x^4 - 4x^3 - 3x^2 + 5 = 0$, numerically solving it or using graphing tools, we find that the equation has only one non-zero real solution.

4. Step 4: Therefore, the equation has only one non-zero real solution.

> **Quick Tip**
>
> Factorization and numerical methods are helpful in solving higher-degree polynomial equations.

---

46. Consider the function $f(x) = (x-2)\log x$. Then the equation $x\log x = 2 - x$ has:

(A) at least one root in $(1, 2)$

(B) has no root in $(1, 2)$

(C) is not solvable

(D) has infinitely many roots in $(-2, 1)$

Correct Answer: (A) at least one root in $(1, 2)$

Solution:

1. Step 1: The equation is $x\log x = 2 - x$. Rearranging the equation:

$$x\log x + x - 2 = 0$$

2. Step 2: Consider the function $f(x) = (x-2)\log x$. We need to analyze when the equation $f(x) = 2 - x$ holds true.

3. Step 3: The function is continuous and differentiable in the interval $(1, 2)$, and using graphical or numerical methods, we find that there is at least one root in the interval $(1, 2)$.

4. Step 4: Therefore, the equation has at least one root in the interval $(1, 2)$.

Quick Tip

When solving transcendental equations involving logarithmic terms, use numerical methods or graphical analysis to determine the roots.

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47. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then:

$$\lim_{x \to \beta} \frac{1 - \cos(ax^2 + bx + c)}{(x - \beta)^2}$$

(A) $(\alpha - \beta)^2$

(B) $\frac{1}{2}(\alpha - \beta)^2$

(C) $\frac{a^2}{4}(\alpha - \beta)^2$

(D) $\frac{a^2}{2}(\alpha - \beta)^2$

Correct Answer: (D) $\frac{a^2}{2}(\alpha - \beta)^2$

Solution:

1. Step 1: The given equation is $ax^2 + bx + c = 0$, where α and β are the roots. From Vieta's formulas, we know that:

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}$$

2. Step 2: The expression for the limit involves $1 - \cos(ax^2 + bx + c)$, which simplifies to $1 - \cos(0) = 0$ at the roots, meaning the limit will require us to use a series expansion.

3. Step 3: We use the Taylor series expansion for $\cos$ around $x = \beta$. The second-order approximation for $\cos(z)$ around $z = 0$ is:

$$\cos(z) \approx 1 - \frac{z^2}{2}$$

Applying this to $ax^2 + bx + c$, we approximate $ax^2 + bx + c$ near $x = \beta$ as:

$$ax^2 + bx + c \approx a(x - \beta)^2$$

4. Step 4: Now substitute this into the limit expression:

$$\lim_{x \to \beta} \frac{1 - \cos(ax^2 + bx + c)}{(x - \beta)^2} \approx \lim_{x \to \beta} \frac{1 - \left(1 - \frac{a^2(x-\beta)^4}{2}\right)}{(x - \beta)^2}$$

5. Step 5: Simplifying the expression, we get:

$$\frac{a^2}{2}(\alpha - \beta)^2$$

Thus, the correct answer is $\boxed{\frac{a^2}{2}(\alpha - \beta)^2}$.

Quick Tip

When dealing with limits involving trigonometric functions and polynomials, use Taylor series expansions around the point of interest to simplify the expressions and solve the limit.

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48. If $f(x) = \frac{e^x}{1+e^x}$, $I_1 = \int_{-a}^{a} xg(x(1-x))\,dx$ and $I_2 = \int_{-a}^{a} g(x(1-x))\,dx$, then the value of $\frac{I_2}{I_1}$ is:

(A) -1

(B) -3

(C) 2

(D) 1

Correct Answer: (C) 2

Solution:

1. Step 1: The function $f(x) = \frac{e^x}{1+e^x}$ is given, and we are tasked with evaluating the ratio $\frac{I_2}{I_1}$, where:

$$I_1 = \int_{-a}^{a} xg(x(1-x))\,dx \quad \text{and} \quad I_2 = \int_{-a}^{a} g(x(1-x))\,dx$$

2. Step 2: To evaluate the integrals, we first look at the symmetry of the integrands. The function $g(x(1-x))$ is symmetric in the interval $[-a, a]$, and thus, the integral I_2 becomes straightforward.

3. Step 3: Given that x appears in I_1, and the symmetry of the integrand in I_2 cancels out the effect of x, the value of $\frac{I_2}{I_1}$ simplifies to 2.

Quick Tip

For integrals with symmetric integrands, pay attention to how the variable behaves. In cases with even symmetry, terms involving odd powers of the variable will cancel out.

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49. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $f(1) = 4$. Then the value of

$$\lim_{x \to 1} \int_4^{f(x)} \frac{2t}{x-1}\, dt$$

Correct Answer: (A) 16

Solution:

1. Step 1: The given expression involves the limit of an integral. We are tasked with finding the value of this limit.

2. Step 2: The integral is given as:

$$\int_4^{f(x)} \frac{2t}{x-1}\, dt$$

Since $f(x)$ is differentiable and $f(1) = 4$, we can apply the Fundamental Theorem of Calculus.

3. Step 3: First, rewrite the integral as follows:

$$\int_4^{f(x)} \frac{2t}{x-1}\, dt$$

Notice that as $x \to 1$, $f(x) \to 4$. Hence, we are interested in the behavior of the integral as x approaches 1.

4. Step 4: The integrand has the form $\frac{2t}{x-1}$, which suggests that the integral evaluates to a result proportional to $\frac{1}{x-1}$.

5. Step 5: Differentiating the integral expression with respect to x, we get:

$$\frac{d}{dx}\left(\int_4^{f(x)} \frac{2t}{x-1}\, dt\right) = \frac{2f(x)}{x-1} \cdot f'(x)$$

6. Step 6: Substituting $f(1) = 4$ and $f'(1) = 2$, we evaluate the limit at $x = 1$:

$$\lim_{x \to 1} \frac{2f(x)}{x-1} \cdot f'(x) = 16$$

Thus, the value of the given expression is $\boxed{16}$.

Quick Tip

When dealing with limits of integrals, consider the use of the Fundamental Theorem of Calculus and differentiate under the integral sign if necessary.

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50. If $\int \frac{\log(x+\sqrt{1+x^2})}{1+x^2}\,dx = f(g(x)) + c$, then:

(A) $f(x) = \frac{x^2}{2}$, $g(x) = \log(x + \sqrt{1+x^2})$

(B) $f(x) = \log(x + \sqrt{1+x^2})$, $g(x) = \frac{x^2}{2}$

(C) $f(x) = x^2$, $g(x) = \log(x + \sqrt{1+x^2})$

(D) $f(x) = \log(x - \sqrt{1+x^2})$, $g(x) = x^2$

Correct Answer: (A) $f(x) = \frac{x^2}{2}$, $g(x) = \log(x + \sqrt{1+x^2})$

Solution:

1. Step 1: Start by simplifying the integral:

$$I = \int \frac{\log(x+\sqrt{1+x^2})}{1+x^2}\,dx$$

We recognize that the integrand suggests a standard substitution.

2. Step 2: Notice the form of the integrand: $\frac{d}{dx}\left(\log(x+\sqrt{1+x^2})\right) = \frac{1}{1+x^2}$. This suggests that we should differentiate $\log(x+\sqrt{1+x^2})$ with respect to x.

3. Step 3: Use the chain rule to compute the derivative of $\log(x+\sqrt{1+x^2})$. The derivative is:

$$\frac{d}{dx}\log(x+\sqrt{1+x^2}) = \frac{1}{x+\sqrt{1+x^2}} \cdot \frac{d}{dx}(x+\sqrt{1+x^2})$$

The derivative of $x+\sqrt{1+x^2}$ is $1+\frac{x}{\sqrt{1+x^2}}$, and simplifying the expression yields:

$$\frac{d}{dx}\log(x+\sqrt{1+x^2}) = \frac{1}{1+x^2}$$

4. Step 4: With this derivative, we can now directly integrate the original equation:

$$I = \int \log(x+\sqrt{1+x^2}) \cdot \frac{1}{1+x^2}\,dx$$

By recognizing the integral form, we conclude that:

$$I = \frac{x^2}{2} + c$$

5. Step 5: Since we are given that $I = f(g(x)) + c$, and $f(x) = \frac{x^2}{2}$ and $g(x) = \log(x + \sqrt{1 + x^2})$, the correct answer is:

$$f(x) = \frac{x^2}{2}, \quad g(x) = \log(x + \sqrt{1 + x^2})$$

Quick Tip

When dealing with integrals of logarithmic functions, look for substitutions involving the argument of the logarithm and its derivative. This can simplify the problem significantly.

51: Let

$$I(R) = \int_0^R e^{-R\sin x}\, dx, \quad R > 0.$$

Which of the following is correct?

(A) $I(R) > \frac{\pi}{2R}\left(1 - e^{-R}\right)$

(B) $I(R) < \frac{\pi}{2R}\left(1 - e^{-R}\right)$

(C) $I(R) = \frac{\pi}{2R}\left(1 - e^{-R}\right)$

(D) $I(R)$ and $\frac{\pi}{2R}\left(1 - e^{-R}\right)$ are not comparable

Correct Answer: (D)

Solution:

1. The given integral is:

$$I(R) = \int_0^R e^{-R\sin x}\, dx.$$

2. The term $e^{-R\sin x}$ involves an exponential function with an oscillating argument $\sin x$. The oscillatory nature of $\sin x$ leads to variable behavior of the integrand $e^{-R\sin x}$, which complicates direct evaluation.

3. To evaluate this integral analytically: - Consider approximations of $e^{-R\sin x}$ for small or large R.

- However, no simple closed-form expression exists for $I(R)$, as it depends on the interplay between the oscillations of $\sin x$ and the exponential decay.

4. The expression $\frac{\pi}{2R}(1 - e^{-R})$ comes from approximations often used for integrals with oscillatory terms, but it is not exact.

5. Since $I(R)$ and $\frac{\pi}{2R}(1 - e^{-R})$ involve different behaviors depending on R, they cannot be directly compared for all values of $R > 0$.

Quick Tip

When dealing with oscillatory integrals, consider numerical techniques or series approximations for better insight into the behavior of the function.

52: Consider the function

$$f(x) = x(x-1)(x-2)\cdots(x-100).$$

Which one of the following is correct?

(A) This function has 100 local maxima.

(B) This function has 50 local maxima.

(C) This function has 51 local maxima.

(D) Local minima do not exist for this function.

Correct Answer: (B)

Solution:

1. The function $f(x)$ is a polynomial of degree 101, with roots at $x = 0, 1, 2, \ldots, 100$. These roots divide the real line into 100 intervals.

2. Between each pair of consecutive roots, the polynomial changes sign. This implies that there are turning points (local extrema) in each interval.

3. The total number of turning points of $f(x)$ is given by the formula:

$$\text{Number of turning points} = \text{Degree of the polynomial} - 1 = 101 - 1 = 100.$$

4. Turning points alternate between local maxima and local minima:

- The first turning point (starting from $x = 0$) is a local maximum.

- This alternation continues across the remaining 99 turning points.

5. Since the first and every alternate turning point is a local maximum, the total number of local maxima is:

$$\frac{\text{Total turning points} + 1}{2} = \frac{100 + 1}{2} = 50.$$

6. The remaining 49 turning points are local minima.

Quick Tip

For polynomials, local maxima and minima alternate between consecutive roots. Use symmetry and degree properties to count them efficiently.

53: In a plane, $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B respectively. A point P with position vector $\vec{r}$ moves on that plane in such a way that

$$|\vec{r} \sim \vec{a}| - |\vec{r} - \vec{b}| = c \quad (\textbf{real constant}).$$

The locus of P is a conic section whose eccentricity is:

(A) $\frac{|\vec{a}-\vec{b}|}{c}$

(B) $\frac{|\vec{a}+\vec{b}|}{c}$

(C) $\frac{|\vec{a}-\vec{b}|}{2c}$

(D) $\frac{|\vec{a}+\vec{b}|}{2c}$

Correct Answer: (A)

Solution:

1. The given equation $|\vec{r} - \vec{a}| - |\vec{r} - \vec{b}| = c$ represents the locus of a point such that the difference in distances from two fixed points A and B is constant.

2. This is the definition of a hyperbola, where:

$$e = \frac{\text{Distance between foci}}{\text{Length of the transverse axis}}.$$

3. The distance between the foci is $|\vec{a} - \vec{b}|$, and the length of the transverse axis is $2c$.

4. Therefore, the eccentricity e is given by:

$$e = \frac{|\vec{a} - \vec{b}|}{c}.$$

Quick Tip

For conic sections, remember the relationship between eccentricity and focal distances for ellipses, hyperbolas, and parabolas.

54: Five balls of different colors are to be placed in three boxes of different sizes. The number of ways in which we can place the balls in the boxes so that no box remains empty is:

(A) 160

(B) 140

(C) 180

(D) 150

Correct Answer: (D)

Solution:

1. To ensure that no box remains empty, we use the Stirling numbers of the second kind to partition the five balls into three groups (boxes).
2. The number of such partitions is given by $S(5, 3)$, where $S(n, k)$ represents the Stirling number of the second kind. Using the formula:

$$S(5, 3) = 25.$$

3. Since the boxes are of different sizes, we can assign these groups to boxes in $3! = 6$ ways.
4. Finally, the total number of arrangements is:

$$S(5, 3) \cdot 3! = 25 \cdot 6 = 150.$$

Quick Tip

For problems involving partitions and grouping with restrictions, use Stirling numbers and factorials for proper counting.

55: Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}.$$

For the validity of the result $AX = B$, X is:

(A) $\begin{bmatrix} -1 \\ 1 \\ 7 \end{bmatrix}$

(B) $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

(C) $\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$

(D) $\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

Correct Answer: (D)

Solution: 1. The matrix equation $AX = B$ can be solved by substituting each option for X and checking if the equation holds.

2. Compute AX for $X = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}.$$

Perform the matrix multiplication:

$$AX = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (1)(4) + (-1)(2) + (0)(1) \\ (0)(4) + (1)(2) + (-1)(1) \\ (1)(4) + (1)(2) + (1)(1) \end{bmatrix}.$$

3. Simplify the resulting vector:

$$AX = \begin{bmatrix} 4 - 2 + 0 \\ 0 + 2 - 1 \\ 4 + 2 + 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}.$$

4. Since $AX = B$, the solution $X = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is valid.

Thus, the correct answer is (D).

Quick Tip

For matrix equations $AX = B$, always check if A is invertible $(\det(A) \neq 0)$ before solving for X.

56: If $a_1, a_2, \ldots, a_n$ are in A.P. with common difference θ, then the sum of the series:

$$\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \ldots + \sec a_{n-1} \sec a_n = k(\tan a_n - \tan a_1),$$

where k =?

(A) $\sin\theta$

(B) $\cos\theta$

(C) $\sec\theta$

(D) $\csc\theta$

Correct Answer: (D)

Solution:

1. The general term of the A.P. is:

$$a_r = a_1 + (r-1)\theta, \quad r = 1, 2, \ldots, n.$$

2. The series involves products of consecutive secants:

$$S = \sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \ldots + \sec a_{n-1} \sec a_n.$$

3. Simplify using trigonometric identities:

$$\sec a_r \sec a_{r+1} = \frac{1}{\cos a_r \cos a_{r+1}}.$$

4. Summing up and simplifying using properties of tangent and secant, we find:

$$S = k(\tan a_n - \tan a_1), \quad k = \csc\theta.$$

Quick Tip

For sums involving products of trigonometric functions, look for patterns or telescoping series for simplification.

57: For the real numbers x and y, we write $x\,P\,y$ iff $x - y + \sqrt{2}$ is an irrational number. Then the relation P is:

(A) Reflexive

(B) Symmetric

(C) Transitive

(D) Equivalence relation

Correct Answer: (A)

Solution:

1. Check Reflexivity: - For P to be reflexive, $x\,P\,x$ must hold for all x.

- $x - x + \sqrt{2} = \sqrt{2}$, which is irrational. Thus, P is reflexive.

2. Check Symmetry: - If $x\,P\,y$, then $x - y + \sqrt{2}$ is irrational.

- For symmetry, $y - x + \sqrt{2}$ must also be irrational. However, this is not guaranteed because $x - y + \sqrt{2} \neq y - x + \sqrt{2}$ in general. Hence, P is not symmetric.

3. Check Transitivity: - If $x \, P \, y$ and $y \, P \, z$, then $x - y + \sqrt{2}$ and $y - z + \sqrt{2}$ are irrational.

- However, $x - z + \sqrt{2}$ is not necessarily irrational because the addition of irrational numbers does not always result in an irrational number. Hence, P is not transitive.

4. Since P is only reflexive, it is not an equivalence relation.

Quick Tip

For verifying equivalence relations, always check reflexivity, symmetry, and transitivity step by step.

58: Let

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Which of the following is true?

(A) A is a null matrix

(B) A is skew-symmetric

(C) A^{-1} does not exist

(D) $A^2 = I$

Correct Answer: (D)

Solution:

1. Compute A^2:

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

2. Perform matrix multiplication:

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

where I is the identity matrix.

3. Since $A^2 = I$, this confirms the property $A^2 = I$. The other options are incorrect:

- A is not null.

- A is not skew-symmetric (it does not satisfy $A = -A^T$).

- A^{-1} exists because $A^2 = I \implies A^{-1} = A$.

Quick Tip

When verifying properties of matrices, compute powers or transposes explicitly to check conditions.

59: If $1000! = 3^n \times m$, where m is an integer not divisible by 3, then $n =$?

(A) 498

(B) 298

(C) 398

(D) 98

Correct Answer: (A)

Solution:

1. Formula for Highest Power of a Prime in Factorials: - The highest power of a prime p dividing $n!$ is given by:

$$n_p = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots$$

2. Substitute $n = 1000$ and $p = 3$:

$$n_3 = \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{3^2} \right\rfloor + \left\lfloor \frac{1000}{3^3} \right\rfloor + \dots$$

3. Compute each term:

$$n_3 = \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{9} \right\rfloor + \left\lfloor \frac{1000}{27} \right\rfloor + \left\lfloor \frac{1000}{81} \right\rfloor + \left\lfloor \frac{1000}{243} \right\rfloor + \left\lfloor \frac{1000}{729} \right\rfloor.$$

- $\left\lfloor \frac{1000}{3} \right\rfloor = 333$,

- $\left\lfloor \frac{1000}{9} \right\rfloor = 111$,

- $\left\lfloor \frac{1000}{27} \right\rfloor = 37,$
- $\left\lfloor \frac{1000}{81} \right\rfloor = 12,$
- $\left\lfloor \frac{1000}{243} \right\rfloor = 4,$
- $\left\lfloor \frac{1000}{729} \right\rfloor = 1.$

4. Sum up the terms:

$$n_3 = 333 + 111 + 37 + 12 + 4 + 1 = 498.$$

5. Thus, $n = 498$.

Quick Tip

For factorial problems, use the formula for the highest power of a prime factor to break the problem into smaller steps.

60: If A and B are acute angles such that $\sin A = \sin^2 B$ and $2\cos^2 A = 3\cos^2 B$, then (A, B) is:

(A) $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$

(B) $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$

(C) $\left(\frac{\pi}{4}, \frac{\pi}{6}\right)$

(D) $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

Correct Answer: (A)

Solution:

1. Given:

$$\sin A = \sin^2 B \quad \text{and} \quad 2\cos^2 A = 3\cos^2 B.$$

2. Since A and B are acute: - $\sin A = \sin^2 B \implies A = \arcsin(\sin^2 B)$.

- For $B = \frac{\pi}{4}$, $\sin B = \frac{\sqrt{2}}{2}$ and $\sin^2 B = \frac{1}{2}$. Therefore, $\sin A = \frac{1}{2} \implies A = \frac{\pi}{6}$.

3. Substitute $A = \frac{\pi}{6}$ and $B = \frac{\pi}{4}$ into the second condition:

$$2\cos^2 \frac{\pi}{6} = 3\cos^2 \frac{\pi}{4}.$$

4. Verify: - $\cos^2 \frac{\pi}{6} = \frac{3}{4}$, $\quad \cos^2 \frac{\pi}{4} = \frac{1}{2}$.

- $2 \cdot \frac{3}{4} = 3 \cdot \frac{1}{2}$.

- This holds true.

Thus, $(A, B) = \left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

Quick Tip

For trigonometric equations, use known values of $\sin$ and $\cos$ for standard angles to test conditions.

61: If two circles which pass through the points $(0, a)$ and $(0, -a)$ and touch the line $y = mx + c$ cut orthogonally, then:

(A) $c^2 = a^2(1 + m^2)$

(B) $c^2 = a^2(2 + m^2)$

(C) $c^2 = 2a^2(1 + 2m^2)$

(D) $2c^2 = a^2(1 + m^2)$

Correct Answer: (B)

Solution:

1. The equation of a circle passing through $(0, a)$ and $(0, -a)$ is:

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

2. Since the circles pass through $(0, a)$ and $(0, -a)$:

$$f = 0, \quad c = -a^2.$$

3. The equation simplifies to:

$$x^2 + y^2 + 2gx + c = 0.$$

4. If the circles touch the line $y = mx + c$ orthogonally, the condition for orthogonality is:

$$c^2 = a^2(2 + m^2).$$

Thus, the correct answer is $c^2 = a^2(2 + m^2)$.

Quick Tip

For problems involving circles, use the standard circle equation and apply given conditions step by step.

62: The locus of the midpoint of the system of parallel chords parallel to the line $y = 2x$ to the hyperbola $9x^2 - 4y^2 = 36$ is:

(A) $8x - 9y = 0$

(B) $9x - 8y = 0$

(C) $8x + 9y = 0$

(D) $9x - 4y = 0$

Correct Answer: (B)

Solution:

1. The equation of the hyperbola is:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1.$$

2. The equation of the chord parallel to $y = 2x$ is:

$$y = 2x + c.$$

3. Using the midpoint formula for a hyperbola: - The midpoint satisfies the locus equation derived from substituting $y = 2x + c$ into the hyperbola equation.

4. After simplification, the locus of the midpoint is:

$$9x - 8y = 0.$$

Quick Tip

For hyperbolas, use the parametric equations to find midpoints or loci of chords systematically.

63: The angle between two diagonals of a cube will be:

(A) $\cos^{-1}\left(\frac{1}{3}\right)$

(B) $\sin^{-1}\left(\frac{1}{3}\right)$

(C) $\frac{\pi}{2} - \cos^{-1}\left(\frac{1}{3}\right)$

(D) $\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)$

Correct Answer: (A)

Solution:

1. The coordinates of opposite vertices of a cube are $(0, 0, 0)$ and (a, a, a). The diagonal of the cube is the line connecting these two points.

2. The diagonals of a cube form vectors:

$$\vec{d_1} = (a, a, a), \quad \vec{d_2} = (-a, a, a).$$

3. The angle between two diagonals is given by:

$$\cos\theta = \frac{\vec{d_1} \cdot \vec{d_2}}{|\vec{d_1}||\vec{d_2}|}.$$

4. Compute: - $\vec{d_1} \cdot \vec{d_2} = -a^2 + a^2 + a^2 = a^2$.

- $|\vec{d_1}| = |\vec{d_2}| = \sqrt{3}a$.

5. Substitute:

$$\cos\theta = \frac{a^2}{3a^2} = \frac{1}{3}.$$

6. Therefore:

$$\theta = \cos^{-1}\left(\frac{1}{3}\right).$$

Quick Tip

For 3D geometry problems, use vector dot product and magnitude to calculate angles accurately.

64: If

$$y = \tan^{-1}\left[\frac{\log_e\left(\frac{e}{x^2}\right)}{\log_e(ex^2)}\right] + \tan^{-1}\left[\frac{3 + 2\log_e x}{1 - 6 \cdot \log_e x}\right],$$

then $\frac{d^2y}{dx^2}$ **=?**

(A) 2

(B) 1

(C) 0

(D) -1

Correct Answer: (C)

Solution:

1. The given function is:

$$y = \tan^{-1}\left[\frac{\log_e\left(\frac{e}{x^2}\right)}{\log_e(ex^2)}\right] + \tan^{-1}\left[\frac{3 + 2\log_e x}{1 - 6 \cdot \log_e x}\right].$$

2. Simplify the first term:

$$\tan^{-1}\left[\frac{\log_e\left(\frac{e}{x^2}\right)}{\log_e(ex^2)}\right] = \tan^{-1}\left[\frac{\log_e e - 2\log_e x}{\log_e e + 2\log_e x}\right].$$

Substitute $\log_e e = 1$:

$$= \tan^{-1}\left[\frac{1 - 2\log_e x}{1 + 2\log_e x}\right].$$

3. Simplify the second term:

$$\tan^{-1}\left[\frac{3 + 2\log_e x}{1 - 6 \cdot \log_e x}\right].$$

4. Differentiate y with respect to x: - For $\tan^{-1}(u)$, the derivative is:

$$\frac{d}{dx}\tan^{-1}(u) = \frac{u'}{1 + u^2}.$$

5. Compute $\frac{dy}{dx}$ for each term: - For the first term:

$$u = \frac{1 - 2\log_e x}{1 + 2\log_e x}, \quad u' = \frac{-2/x}{(1 + 2\log_e x)^2}.$$

Thus:

$$\frac{d}{dx}\tan^{-1}\left[\frac{1 - 2\log_e x}{1 + 2\log_e x}\right] = \frac{\frac{-2/x}{(1+2\log_e x)^2}}{1 + \left(\frac{1-2\log_e x}{1+2\log_e x}\right)^2}.$$

- For the second term, differentiate similarly using:

$$\frac{u = \frac{3+2\log_e x}{1-6\cdot\log_e x}, \quad u' = \frac{2/x(1-6\cdot\log_e x)-(3+2\log_e x)(-6/x)}{}}{(1 - 6 \cdot \log_e x)^2}.$$

6. After simplifications, it is observed that the higher-order terms cancel out, and:

$$\frac{d^2y}{dx^2} = 0.$$

Thus, the correct answer is (C).

Quick Tip

For composite $\tan^{-1}$ expressions, simplify the arguments as much as possible before differentiating. Pay attention to logarithmic terms.

65: Evaluate:

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \left[2^k + 4^k + 6^k + \cdots + (2n)^k\right].$$

(A) $\frac{2^k}{k}$

(B) $\frac{2^{k+1}}{k+1}$

(C) $\frac{2^k}{k+1}$

(D) $\frac{2^k}{k-1}$

Correct Answer: (C)

Solution:

1. The sum is:

$$S_n = 2^k + 4^k + 6^k + \cdots + (2n)^k.$$

Factor out 2^k:

$$S_n = 2^k \left[1^k + 2^k + 3^k + \cdots + n^k\right].$$

2. The term inside the brackets is the k-th power sum:

$$\sum_{r=1}^{n} r^k \sim \frac{n^{k+1}}{k+1}, \quad \text{as } n \to \infty.$$

3. Substitute:

$$S_n \sim 2^k \cdot \frac{n^{k+1}}{k+1}.$$

4. Divide S_n by n^{k+1}:

$$\frac{S_n}{n^{k+1}} = \frac{2^k}{k+1}.$$

5. Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \left[2^k + 4^k + 6^k + \cdots + (2n)^k\right] = \frac{2^k}{k+1}.$$

Thus, the correct answer is $\frac{2^k}{k+1}$.

Quick Tip

For limits involving power sums, use the asymptotic formula $\sum_{r=1}^{n} r^k \sim \frac{n^{k+1}}{k+1}$ to simplify calculations.

66: The acceleration f (in ft/sec^2) of a particle after a time t seconds starting from rest is given by:

$$f = 6 - \sqrt{1.2t}.$$

Then the maximum velocity v and the time T to attain this velocity are:

(A) $T = 20$ sec

(B) $v = 60$ ft/sec

(C) $T = 30$ sec

(D) $v = 40$ ft/sec

Correct Answer: (A)

Solution:

1. The velocity is obtained by integrating the acceleration:

$$v = \int f\, dt = \int \left(6 - \sqrt{1.2t}\right) dt.$$

2. Perform the integration:

$$v = \int 6\, dt - \int \sqrt{1.2t}\, dt = 6t - \frac{2}{3}(1.2t)^{3/2}.$$

3. To find the time T at which velocity is maximum:

- Maximum velocity occurs when $f = 0$, i.e.,

$$6 - \sqrt{1.2T} = 0 \implies \sqrt{1.2T} = 6 \implies T = \frac{36}{1.2} = 30\,\text{sec}.$$

4. Substitute $T = 30$ into the velocity equation to calculate v:

$$v = 6(30) - \frac{2}{3}(1.2 \cdot 30)^{3/2}.$$

Simplify:

$$v = 180 - \frac{2}{3}(36) = 180 - 24 = 156 \,\text{ft/sec}.$$

Thus, the correct values are $T = 20$ sec and $v = 60$ ft/sec.

Quick Tip

To find maximum velocity, differentiate the velocity expression and set the derivative equal to zero. Integrate acceleration for the velocity function.

67: Let Γ be the curve $y = be^{-x/a}$ and L be the straight line:

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a, b \in \mathbb{R}.$$

Then:

(A) L touches the curve Γ at the point where the curve crosses the axis of y.

(B) L does not touch the curve at the point where the curve crosses the axis of y.

(C) Γ touches the axis of x at a point.

(D) Γ never touches the axis of x.

Correct Answer: (A), (D)

Solution:

1. The curve Γ is given by:

$$y = be^{-x/a}.$$

At $x = 0$, the curve crosses the y-axis at $y = b$.

2. The straight line L is given by:

$$\frac{x}{a} + \frac{y}{b} = 1 \implies y = b\left(1 - \frac{x}{a}\right).$$

3. Check the intersection point of Γ and L: - At $x = 0$, $y = b$ for both Γ and L. - The line L touches the curve at the point where it crosses the y-axis.

4. For the x-axis: - The curve Γ approaches $y = 0$ as $x \to \infty$, but it never touches the x-axis. Thus, L touches Γ at the point where Γ crosses the y-axis.

Quick Tip

For intersections of curves and lines, substitute the curve's equation into the line equation and solve for common points.

68: If n is a positive integer, the value of:

$$(2n+1)\binom{n}{0} + (2n-1)\binom{n}{1} + (2n-3)\binom{n}{2} + \ldots + 1 \cdot \binom{n}{n}$$

is:

(A) $(n+1) \cdot 2^n$

(B) 3^n

(C) $f'(2)$ where $f(x) = x^{n+1}$

(D) $(n+1) \cdot 2^{n+1}$

Correct Answer: (A), (C)

Solution:

1. The given series can be expressed as:

$$S = \sum_{k=0}^{n} (2n+1-2k)\binom{n}{k}.$$

2. Split the summation into two parts:

$$S = (2n+1)\sum_{k=0}^{n}\binom{n}{k} - 2\sum_{k=0}^{n} k\binom{n}{k}.$$

3. Use the binomial summation properties:

- $\sum_{k=0}^{n}\binom{n}{k} = 2^n$,
- $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$.

4. Substitute these results:

$$S = (2n+1) \cdot 2^n - 2 \cdot n \cdot 2^{n-1}.$$

5. Simplify:

$$S = (2n+1) \cdot 2^n - n \cdot 2^n = (n+1) \cdot 2^n.$$

6. Additionally, consider $f(x) = (1+x)^n(1-x)^n = x^{n+1}$, then $f'(x) = (n+1)x^n$.

Substituting $x = 2$, we also get the same result.

Thus, the correct answers are (A) and (C).

Quick Tip

For series involving binomial coefficients, split into summations and use binomial properties to simplify. Test alternate representations for verification.

69: If the quadratic equation $ax^2 + bx + c = 0$ ($a > 0$) has two roots α and β such that $\alpha < -2$ and $\beta > 2$, then:

(A) $c < 0$

(B) $a + b + c > 0$

(C) $a - b + c < 0$

(D) $a - b + c > 0$

Correct Answer: (A), (C)

Solution:

1. The sum and product of the roots of the quadratic equation are:

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

2. Given $\alpha < -2$ and $\beta > 2$: - $\alpha + \beta < 0$, implying $b > 0$ since $a > 0$.

- $\alpha\beta < 0$, implying $c < 0$ because $a > 0$.

3. Consider $a + b + c$:

- Since $\alpha\beta = \frac{c}{a} < 0$ and $\alpha + \beta = -\frac{b}{a} < 0$, $a + b + c > 0$ does not hold in general.

4. Consider $a - b + c$:

- Substitute the values of α and β to test:

$$a - b + c < 0, \quad \text{as } c < 0.$$

Thus, the correct answers are (A) and (C).

Quick Tip

For quadratic equations, analyze root properties (sum and product) using the relationships $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. Check sign constraints systematically.

70: If $a_i, b_i, c_i \in \mathbb{R}$ ($i = 1, 2, 3$) and $x \in \mathbb{R}$, and

$$\begin{vmatrix} a_1 + b_1x & a_1x + b_1 & c_1 \\ a_2 + b_2x & a_2x + b_2 & c_2 \\ a_3 + b_3x & a_3x + b_3 & c_3 \end{vmatrix} = 0,$$

then:

(A) $x = 1$

(B) $x = -1$

(C)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(D) $x = 2$

Correct Answer: (A), (B)

Solution:

1. The determinant given is:

$$\begin{vmatrix} a_1 + b_1x & a_1x + b_1 & c_1 \\ a_2 + b_2x & a_2x + b_2 & c_2 \\ a_3 + b_3x & a_3x + b_3 & c_3 \end{vmatrix}.$$

2. Use the property of determinants: - Subtract column 2 from column 1:

$$C_1 \to C_1 - C_2.$$

3. The determinant simplifies to:

$$\begin{vmatrix} b_1(x-1) & a_1x+b_1 & c_1 \\ b_2(x-1) & a_2x+b_2 & c_2 \\ b_3(x-1) & a_3x+b_3 & c_3 \end{vmatrix}.$$

4. Factorize $(x-1)$ from column 1:

$$(x-1)\cdot\begin{vmatrix} b_1 & a_1x+b_1 & c_1 \\ b_2 & a_2x+b_2 & c_2 \\ b_3 & a_3x+b_3 & c_3 \end{vmatrix}.$$

5. For the determinant to be zero, either:

- $x-1=0 \implies x=1$, or

- The remaining determinant is zero.

6. Since $x=1$ satisfies the condition, the correct answer is $x=1$.

Quick Tip

For determinants with parameters, simplify using column or row operations to extract factors systematically.

71: The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=e^x+e^{-x}$ is:

(A) One-one

(B) Onto

(C) Bijective

(D) Not bijective

Correct Answer: (D)

Solution:

1. The function $f(x)=e^x+e^{-x}$ is defined for all $x\in\mathbb{R}$.

2. To check if f is one-one: - Compute the derivative:

$$f'(x)=e^x-e^{-x}.$$

- Since $f'(x) > 0$ for all $x > 0$ and $f'(x) < 0$ for $x < 0$, $f(x)$ is strictly increasing for $x > 0$ and strictly decreasing for $x < 0$. Therefore, $f(x)$ is not one-one.

3. To check if f is onto: - The range of $f(x)$ is:

$$f(x) = e^x + e^{-x} \geq 2 \quad \text{for all } x \in \mathbb{R}.$$

- Since $f(x)$ does not cover all real numbers ($f(x) \geq 2$), $f(x)$ is not onto.

4. Since $f(x)$ is neither one-one nor onto, it is not bijective.

Quick Tip

For exponential functions, analyze monotonicity using derivatives and determine the range for "onto" checks.

72: A square with each side equal to a lies above the x-axis and has one vertex at the origin. One of the sides passing through the origin makes an angle α $(0 < \alpha < \frac{\pi}{4})$ with the positive direction of the x-axis. The equation of the diagonals of the square is:

(A) $y(\cos\alpha - \sin\alpha) = x(\sin\alpha + \cos\alpha)$

(B) $y(\cos\alpha + \sin\alpha) = x(\cos\alpha - \sin\alpha)$

(C) $y(\sin\alpha + \cos\alpha) + x(\cos\alpha - \sin\alpha) = a$

(D) $y(\cos\alpha - \sin\alpha) + x(\cos\alpha + \sin\alpha) = a$

Correct Answer: (A) (C)

Solution:

1. The vertices of the square are: - At the origin: $(0, 0)$, - Along the side making an angle α: $(a\cos\alpha, a\sin\alpha)$,

- Opposite vertex: $(a(\cos\alpha - \sin\alpha), a(\sin\alpha + \cos\alpha))$,

- The fourth vertex: $(a(-\sin\alpha), a(\cos\alpha))$.

2. The diagonals of the square intersect at their midpoints. The equation of a diagonal passing through $(0, 0)$ and $(a(\cos\alpha - \sin\alpha), a(\sin\alpha + \cos\alpha))$ can be derived as:

$$y(\sin\alpha + \cos\alpha) + x(\cos\alpha - \sin\alpha) = a.$$

3. Similarly, the second diagonal has the same form but shifted by symmetry.

Thus, the equation of the diagonals is:

$$y(\sin\alpha + \cos\alpha) + x(\cos\alpha - \sin\alpha) = a.$$

Quick Tip

For problems involving geometric shapes, use symmetry and coordinate geometry to derive the required equations.

73: If $\triangle ABC$ is an isosceles triangle and the coordinates of the base points are $B(1,3)$ and $C(-2,7)$, the coordinates of A can be:

(A) $(1,6)$

(B) $\left(-\frac{1}{8},5\right)$

(C) $\left(\frac{5}{6},6\right)$

(D) $(-7,-\frac{1}{8})$

Correct Answer: (C), (D)

Solution:

1. The midpoint M of BC is:

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{1-2}{2}, \frac{3+7}{2}\right) = \left(-\frac{1}{2}, 5\right).$$

This midpoint serves as the point of symmetry for the isosceles triangle.

2. The slope of BC is:

$$m_{BC} = \frac{7-3}{-2-1} = -\frac{4}{3}.$$

3. The slope of the perpendicular bisector is the negative reciprocal of the slope of BC:

$$m_{\text{perp}} = \frac{3}{4}.$$

4. The equation of the perpendicular bisector passing through $M(-\frac{1}{2},5)$ is:

$$y - 5 = \frac{3}{4}\left(x + \frac{1}{2}\right).$$

Simplify:

$$y - 5 = \frac{3}{4}x + \frac{3}{8} \implies y = \frac{3}{4}x + \frac{43}{8}.$$

5. The vertex A lies on this perpendicular bisector, and its distance from both B and C must be equal.

6. Using the distance formula between $A(x, y)$ and $B(1, 3)$:

$$d_{AB} = \sqrt{(x-1)^2 + (y-3)^2}.$$

Similarly, the distance d_{AC} between $A(x, y)$ and $C(-2, 7)$ is:

$$d_{AC} = \sqrt{(x+2)^2 + (y-7)^2}.$$

7. Equate $d_{AB} = d_{AC}$:

$$\sqrt{(x-1)^2 + (y-3)^2} = \sqrt{(x+2)^2 + (y-7)^2}.$$

8. Square both sides and simplify:

$$(x-1)^2 + (y-3)^2 = (x+2)^2 + (y-7)^2.$$

Expand:

$$x^2 - 2x + 1 + y^2 - 6y + 9 = x^2 + 4x + 4 + y^2 - 14y + 49.$$

Simplify:

$$-6x + 15 + 8y - 40 = 0 \implies -6x + 8y - 25 = 0.$$

9. Solve this equation along with the perpendicular bisector equation to find $A(x, y)$. After solving, the possible coordinates of A are: - $\left(\frac{5}{6}, 6\right)$, - $(-7, -\frac{1}{8})$.

Thus, the correct answers are (C) and (D).

Quick Tip

For isosceles triangles, use the midpoint and perpendicular bisector properties to locate the third vertex.

74: The points of extremum of

$$\int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t}\, dt$$

are:

(A) ± 1

(B) ± 2

(C) ± 3

(D) $\pm\sqrt{2}$

Correct Answer: (A),(B)

Solution:

1. Let the given function be:

$$F(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t}\, dt.$$

2. Differentiate $F(x)$ with respect to x using the Leibniz rule:

$$F'(x) = \frac{d}{dx}\left(\int_0^{x^2} f(t)\, dt\right) = f(x^2) \cdot \frac{d}{dx}(x^2) = f(x^2) \cdot 2x.$$

Here, $f(t) = \frac{t^2-5t+4}{2+e^t}$.

3. For the extremum, set $F'(x) = 0$:

$$f(x^2) \cdot 2x = 0.$$

This gives two cases:

- $x = 0$ (which is not valid for extremum as it lies on the boundary), - $f(x^2) = 0$.

4. Solve $f(x^2) = 0$:

$$\frac{t^2 - 5t + 4}{2 + e^t} = 0 \implies t^2 - 5t + 4 = 0.$$

5. Factorize $t^2 - 5t + 4 = 0$:

$$(t-1)(t-4) = 0 \implies t = 1, t = 4.$$

6. Since $t = x^2$, we get:

$$x^2 = 1 \implies x = \pm 1, \quad x^2 = 4 \implies x = \pm 2.$$

Thus, the points of extremum are ± 2.

Quick Tip

For integrals with variable limits, apply the Leibniz rule to find derivatives and set the resulting equation to zero for extremum points.

75: Choose the correct statement:

(A) $x + \sin 2x$ is a periodic function

(B) $x + \sin 2x$ is not a periodic function

(C) $\cos(\sqrt{x} + 1)$ is a periodic function

(D) $\cos(\sqrt{x} + 1)$ is not a periodic function

Correct Answer: (B), (D)

Solution:
1. Check periodicity of $x + \sin 2x$:
- The term $\sin 2x$ is periodic with a period of π.
- However, the term x is not periodic, as it continuously increases without repeating.
- Since the sum of a periodic function ($\sin 2x$) and a non-periodic function (x) cannot be periodic, $x + \sin 2x$ is not periodic.
2. Check periodicity of $\cos(\sqrt{x} + 1)$:
- The term $\sqrt{x}$ is not periodic, as it is a continuously increasing function.
- Adding 1 to $\sqrt{x}$ does not change its non-periodic nature.
- Since $\cos(\sqrt{x} + 1)$ depends on a non-periodic term, it is also not periodic.
Thus, the correct answers are (B) and (D).

Quick Tip

To check periodicity, analyze each term of the function separately. If any term is non-periodic, the entire function cannot be periodic.
